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1. Three vertices of a parallelogram ABCD are A(3, – 1, 2), B (1, 2, – 4) and C (– 1, 1, 2). Find the coordinates of the fourth vertex.

Solution:

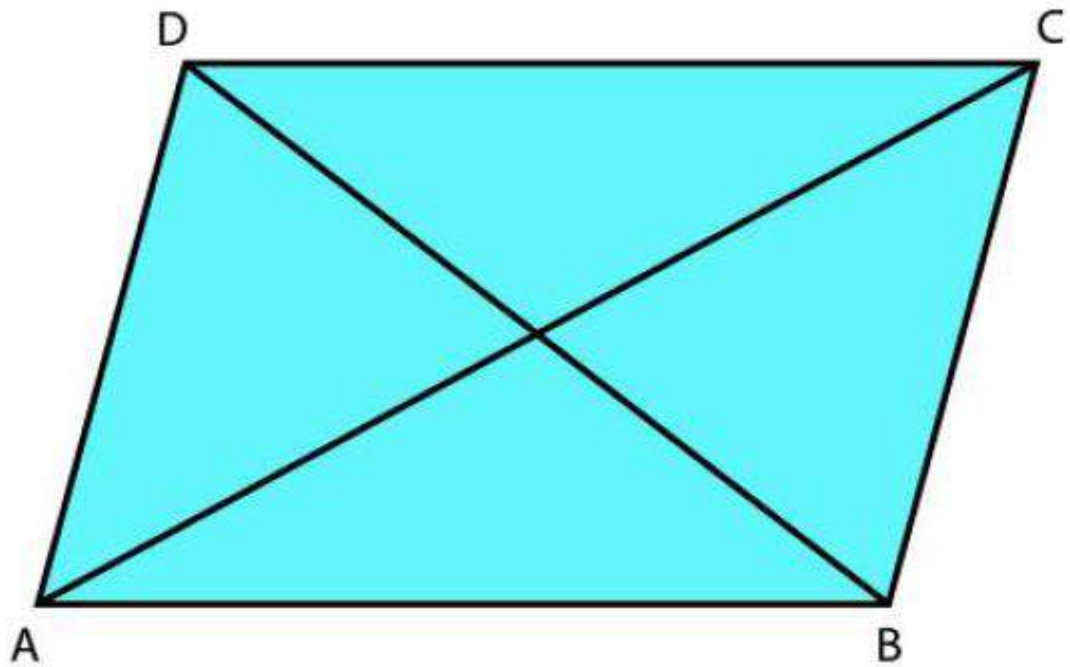
Given:

ABCD is a parallelogram, with vertices A (3, -1, 2), B (1, 2, -4), C (-1, 1, 2).

Where, $x_1 = 3$, $y_1 = -1$, $z_1 = 2$;

$x_2 = 1$, $y_2 = 2$, $z_2 = -4$;

$x_3 = -1$, $y_3 = 1$, $z_3 = 2$



Let the coordinates of the fourth vertex be D (x, y, z).

We also know that the diagonals of a parallelogram bisect each other, so the mid points of AC and BD are equal, i.e. Midpoint of AC = Midpoint of BD ……….(1)

Now, by Midpoint Formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $[(x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2]$

So we have,

Co-ordinates of the midpoint of AC:

$$= \left(\frac{3-1}{2}, \frac{-1+1}{2}, \frac{2+2}{2}\right)$$

= (2/2, 0/2, 4/2)

= (1, 0, 2)

Co-ordinates of the midpoint of BD:

$$= \left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right)$$

So, using (1), we have

$$\left(\frac{1+x}{2}, \frac{2+y}{2}, \frac{-4+z}{2}\right) = (1,0,2)$$

$$\frac{1+x}{2} = 1, \frac{2+y}{2} = 0, \frac{-4+z}{2} = 2$$

1 + x = 2, 2 + y = 0, -4 + z = 4

x = 1, y = -2, z = 8

Hence, the coordinates of the fourth vertex is D (1, -2, 8).

2. Find the lengths of the medians of the triangle with vertices A (0, 0, 6), B (0, 4, 0) and (6, 0, 0).

Solution:

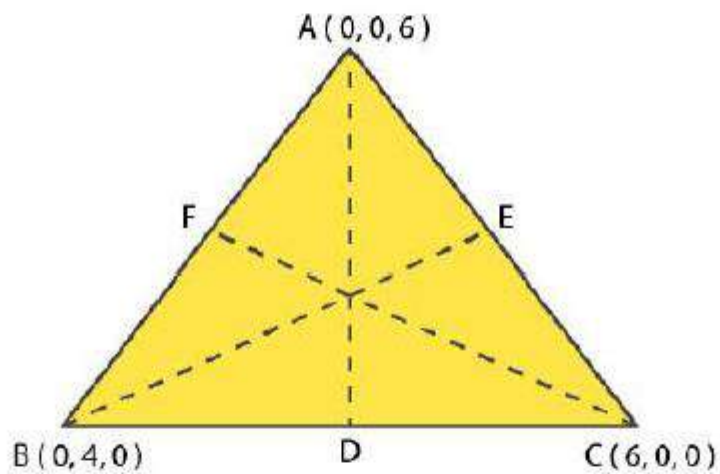
Given:

The vertices of the triangle are A (0, 0, 6), B (0, 4, 0) and C (6, 0, 0).

$x_1 = 0, y_1 = 0, z_1 = 6;$

$x_2 = 0, y_2 = 4, z_2 = 0;$

$x_3 = 6, y_3 = 0, z_3 = 0$



So, let the medians of this triangle be AD, BE and CF corresponding to the vertices A, B and C respectively.

D, E and F are the midpoints of the sides BC, AC and AB respectively.

By Midpoint Formula, we know that the coordinates of the mid-point of the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $[(x_1+x_2)/2, (y_1+y_2)/2, (z_1+z_2)/2]$

So we have,

The coordinates of D:

$$= \left(\frac{0+6}{2}, \frac{4+0}{2}, \frac{0+0}{2}\right) = \left(\frac{6}{2}, \frac{4}{2}, \frac{0}{2}\right)$$

$$= (3, 2, 0)$$

The coordinates of E:

$$= \left(\frac{0+6}{2}, \frac{0+0}{2}, \frac{6+0}{2}\right) = \left(\frac{6}{2}, \frac{0}{2}, \frac{6}{2}\right)$$

$$= (3, 0, 3)$$

And the coordinates of F:

$$= \left(\frac{0+0}{2}, \frac{0+4}{2}, \frac{6+0}{2}\right) = \left(\frac{0}{2}, \frac{4}{2}, \frac{6}{2}\right)$$

$$= (0, 2, 3)$$

By Distance Formula, we know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

So the lengths of the medians are:

$$AD = \sqrt{(3-0)^2 + (2-0)^2 + (0-6)^2} = \sqrt{3^2 + 2^2 + (-6)^2} = \sqrt{9+4+36}$$

$$= \sqrt{49} = 7$$

$$BE = \sqrt{(3-0)^2 + (0-4)^2 + (3-0)^2} = \sqrt{3^2 + (-4)^2 + 3^2} = \sqrt{9+16+9}$$

$$= \sqrt{34}$$

$$CF = \sqrt{(0-6)^2 + (2-0)^2 + (3-0)^2} = \sqrt{(-6)^2 + 2^2 + 3^2} = \sqrt{36+4+9}$$

$$= \sqrt{49} = 7$$

∴ The lengths of the medians of the given triangle are 7, √34 and 7.

3. If the origin is the centroid of the triangle PQR with vertices P (2a, 2, 6), Q (– 4, 3b, –10) and R(8, 14, 2c), then find the values of a, b and c.

Solution:

Given:

The vertices of the triangle are P (2a, 2, 6), Q (-4, 3b, -10) and R (8, 14, 2c).

Where,

$x_1 = 2a$, $y_1 = 2$, $z_1 = 6$;

$x_2 = -4$, $y_2 = 3b$, $z_2 = -10$;

$x_3 = 8$, $y_3 = 14$, $z_3 = 2c$

We know that the coordinates of the centroid of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), are $[(x_1+x_2+x_3)/3, (y_1+y_2+y_3)/3, (z_1+z_2+z_3)/3]$

So, the coordinates of the centroid of the triangle PQR are

$$\left(\frac{2a-4+8}{3},\frac{2+3b+14}{3},\frac{6-10+2c}{3}\right)=\left(\frac{2a+4}{3},\frac{3b+16}{3},\frac{2c-4}{3}\right)$$

Now, it is given that the origin (0, 0, 0) is the centroid.

So, we have $\left(\frac{2a+4}{3},\frac{3b+16}{3},\frac{2c-4}{3}\right)=(0,0,0)$

$$\frac{2a+4}{3}=0,\frac{3b+16}{3}=0,\frac{2c-4}{3}=0$$

2a + 4 = 0, 3b + 16 = 0, 2c – 4 = 0

a = -2, b = -16/3, c = 2

∴ The values of a, b and c are a = -2, b = -16/3, c = 2

4. Find the coordinates of a point on y-axis which are at a distance of 5√2 from the point P (3, –2, 5).

Solution:

Let the point on y-axis be A (0, y, 0).

Then, it is given that the distance between the points A (0, y, 0) and P (3, -2, 5) is 5√2.

Now, by using distance formula,

We know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

Distance of PQ = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

So, the distance between the points A (0, y, 0) and P (3, -2, 5) is given by

Distance of AP = $\sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$

= $\sqrt{[(3-0)^2 + (-2-y)^2 + (5-0)^2]}$

= $\sqrt{[3^2 + (-2-y)^2 + 5^2]}$

= $\sqrt{[(-2-y)^2 + 9 + 25]}$

5√2 = $\sqrt{[(-2-y)^2 + 34]}$

Squaring on both the sides, we get

$(-2 -y)^2 + 34 = 25 \times 2$

$(-2 -y)^2 = 50 - 34$

$4 + y^2 + (2 \times -2 \times -y) = 16$

$y^2 + 4y + 4 -16 = 0$

$y^2 + 4y - 12 = 0$

$y^2 + 6y - 2y - 12 = 0$

y (y + 6) – 2 (y + 6) = 0

(y + 6) (y - 2) = 0

y = -6, y = 2

∴ The points (0, 2, 0) and (0, -6, 0) are the required points on the y-axis.

5. A point R with x-coordinate 4 lies on the line segment joining the points P (2, –3, 4) and Q (8, 0, 10). Find the coordinates of the point R.
[Hint Suppose R divides PQ in the ratio k: 1. The coordinates of the point R are given by
$\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$]

Solution:
Given:
The coordinates of the points P (2, -3, 4) and Q (8, 0, 10).
$x_1 = 2, y_1 = -3, z_1 = 4;$
$x_2 = 8, y_2 = 0, z_2 = 10$
Let the coordinates of the required point be (4, y, z).
So now, let the point R (4, y, z) divides the line segment joining the points P (2, -3, 4) and Q (8, 0, 10) in the ratio k: 1.
By using Section Formula,
We know that the coordinates of the point R which divides the line segment joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) internally in the ratio m: n is given by:
$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$$
So, the coordinates of the point R are given by $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right)$
So, we have $\left(\frac{8k+2}{k+1}, \frac{-3}{k+1}, \frac{10k+4}{k+1}\right) = (4, y, z)$
$$\Rightarrow \frac{8k+2}{k+1} = 4$$
8k + 2 = 4 (k + 1)
8k + 2 = 4k + 4
8k – 4k = 4 – 2
4k = 2
k = 2/4
= 1/2
Now let us substitute the values, we get
$$\Rightarrow y = \frac{-3}{\frac{1}{2}+1} = \frac{-3}{\frac{3}{2}} = \frac{-3 \times 2}{3} = -2,$$
$$z = \frac{10\left(\frac{1}{2}\right) + 4}{\frac{1}{2}+1} = \frac{5+4}{\frac{3}{2}} = \frac{9 \times 2}{3} = 3 \times 2$$

= 6

∴ The coordinates of the required point are (4, -2, 6).

6. If A and B be the points (3, 4, 5) and (–1, 3, –7), respectively, find the equation of the set of points P such that $PA^2 + PB^2 = k^2$, where k is a constant.

Solution:

Given:

The points A (3, 4, 5) and B (-1, 3, -7)

$x_1 = 3, y_1 = 4, z_1 = 5;$

$x_2 = -1, y_2 = 3, z_2 = -7;$

$PA^2 + PB^2 = k^2$ ……….(1)

Let the point be P (x, y, z).

Now by using distance formula,

We know that the distance between two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) is given by

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

So,

$$PA = \sqrt{(3 - x)^2 + (4 - y)^2 + (5 - z)^2}$$

And

$$PB = \sqrt{(-1 - x)^2 + (3 - y)^2 + (-7 - z)^2}$$

Now, substituting these values in (1), we have

$[(3 - x)^2 + (4 - y)^2 + (5 - z)^2] + [(-1 - x)^2 + (3 - y)^2 + (-7 - z)^2] = k^2$

$[(9 + x^2 – 6x) + (16 + y^2 – 8y) + (25 + z^2 – 10z)] + [(1 + x^2 + 2x) + (9 + y^2 – 6y) + (49 + z^2 + 14z)] = k^2$

$9 + x^2 – 6x + 16 + y^2 – 8y + 25 + z^2 – 10z + 1 + x^2 + 2x + 9 + y^2 – 6y + 49 + z^2 + 14z = k^2$

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 = k^2$

$2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z = k^2 – 109$

$2 (x^2 + y^2 + z^2 – 2x – 7y + 2z) = k^2 – 109$

$(x^2 + y^2 + z^2 – 2x – 7y + 2z) = (k^2 – 109)/2$

Hence, the required equation is $(x^2 + y^2 + z^2 – 2x – 7y + 2z) = (k^2 – 109)/2$