EXERCISE 8.1

Expand each of the expressions in Exercises 1 to 5.

1. $(1 - 2x)^5$

Solution:

From binomial theorem expansion we can write as

$(1 - 2x)^5$

$= {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (2x) + {}^5C_2 (1)^3 (2x)^2 - {}^5C_3 (1)^2 (2x)^3 + {}^5C_4 (1)^1 (2x)^4 - {}^5C_5 (2x)^5$

$= 1 - 5 (2x) + 10 (4x)^2 - 10 (8x^3) + 5 (16 x^4) - (32 x^5)$

$= 1 - 10x + 40x^2 - 80x^3 - 32x^5$

2. $\left(\frac{2}{x} - \frac{x}{2}\right)^5$

Solution:

From binomial theorem, given equation can be expanded as

$$\left(\frac{2}{x} - \frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^3 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$$

$$- {}^3C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^3C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^3C_5\left(\frac{x}{2}\right)^5$$

$$= \frac{32}{x^5} - 5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right) + 10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right) - 10\left(\frac{4}{x^2}\right) + 5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right) - \frac{x^5}{32}$$

$$= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^3}{32}$$

3. $(2x - 3)^6$

Solution:

From binomial theorem, given equation can be expanded as

$$(2x - 3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_1(2x)^4(3)^2 - {}^4C_3(2x)^3(3)^3$$

$$= 64x^6 - 6\left(32x^5\right)(3) + 15\left(16x^4\right)(9) - 20\left(8x^3\right)(27)$$

$$+ 15\left(4x^2\right)(81) - 6(2x)(243) + 729$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^5$

Solution:

From binomial theorem, given equation can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^5 = {}^5C_0\left(\frac{x}{3}\right)^5 + {}^3C_1\left(\frac{x}{3}\right)^4\left(\frac{1}{x}\right) + {}^3C_2\left(\frac{x}{3}\right)^3\left(\frac{1}{x}\right)^2$$
$$= \frac{x^5}{243} + 5\left(\frac{x^4}{81}\right)\left(\frac{1}{x}\right) + 10\left(\frac{x^3}{27}\right)\left(\frac{1}{x^2}\right) + 10\left(\frac{x^2}{9}\right)\left(\frac{1}{x^3}\right) + 5\left(\frac{x}{3}\right)\left(\frac{1}{x^4}\right) + \frac{1}{x^5}$$
$$= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^3}$$

5. $\left(x+\frac{1}{x}\right)^6$

Solution:

From binomial theorem, given equation can be expanded as

$$\left(x+\frac{1}{x}\right)^6 = {}^6C_0(x)^6 + {}^6C_1(x)'\left(\frac{1}{x}\right) + {}^6C_2(x)^4\left(\frac{1}{x}\right)^2$$
$$+ {}^6C_3(x)^3\left(\frac{1}{x}\right)^3 + {}^6C_4(x)^2\left(\frac{1}{x}\right)^4 + {}^6C_3(x)\left(\frac{1}{x}\right)^5 + {}^6C_6\left(\frac{1}{x}\right)^6$$
$$= x^4 + 6(x)^3\left(\frac{1}{x}\right) + 15(x)^4\left(\frac{1}{x^2}\right) + 20(x)^3\left(\frac{1}{x^3}\right) + 15(x)^2\left(\frac{1}{x^4}\right) + 6(x)\left(\frac{1}{x^5}\right) + \frac{1}{x^6}$$
$$= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}$$

6. $(96)^3$

Solution:

Given $(96)^3$

96 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 96 = 100 – 4

$(96)^3 = (100 – 4)^3$

$= {}^3C_0 (100)^3 - {}^3C_1 (100)^2 (4) - {}^3C_2 (100) (4)^2 - {}^3C_3 (4)^3$

$= (100)^3 – 3 (100)^2 (4) + 3 (100) (4)^2 – (4)^3$

$= 1000000 – 120000 + 4800 – 64$

= 884736

7. $(102)^5$

Solution:

Given $(102)^5$

102 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 102 = 100 + 2

$(102)^5 = (100 + 2)^5$

$= {}^5C_0\,(100)^5 + {}^5C_1\,(100)^4\,(2) + {}^5C_2\,(100)^3\,(2)^2 + {}^5C_3\,(100)^2\,(2)^3 + {}^5C_4\,(100)\,(2)^4 + {}^5C_5\,(2)^5$

$= (100)^5 + 5\,(100)^4\,(2) + 10\,(100)^3\,(2)^2 + 5\,(100)\,(2)^3 + 5\,(100)\,(2)^4 + (2)^5$

= 1000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32

= 11040808032

8. $(101)^4$

Solution:

Given $(101)^4$

101 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 101 = 100 + 1

$(101)^4 = (100 + 1)^4$

$= {}^4C_0\,(100)^4 + {}^4C_1\,(100)^3\,(1) + {}^4C_2\,(100)^2\,(1)^2 + {}^4C_3\,(100)\,(1)^2 + {}^4C_4\,(1)^4$

$= (100)^4 + 4\,(100)^3 + 6\,(100)^2 + 4\,(100) + (1)^4$

= 100000000 + 400000 + 60000 + 400 + 1

= 1040604001

9. $(99)^5$

Solution:

Given $(99)^5$

99 can be written as the sum or difference of two numbers then binomial theorem can be applied.

The given question can be written as 99 = 100 -1

$(99)^5 = (100 – 1)^5$

$= {}^5C_0\,(100)^5 - {}^5C_1\,(100)^4\,(1) + {}^5C_2\,(100)^3\,(1)^2 - {}^5C_3\,(100)^2\,(1)^3 + {}^5C_4\,(100)\,(1)^4 - {}^5C_5\,(1)^5$

$= (100)^5 - 5\,(100)^4 + 10\,(100)^3 – 10\,(100)^2 + 5\,(100) - 1$
$= 1000000000 - 5000000000 + 10000000 - 100000 + 500 - 1$
$= 9509900499$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:
By splitting the given 1.1 and then applying binomial theorem, the first few terms of $(1.1)^{10000}$ can be obtained as
$(1.1)^{10000} = (1 + 0.1)^{10000}$
$= (1 + 0.1)^{10000}\,C_1\,(1.1)$ + other positive terms
$= 1 + 10000 \times 1.1$ + other positive terms
$= 1 + 11000$ + other positive terms
> 1000
$(1.1)^{10000} > 1000$

11. Find $(a + b)^4 – (a – b)^4$. Hence, evaluate
$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4$.

Solution:
Using binomial theorem the expression $(a + b)^4$ and $(a – b)^4$, can be expanded
$(a + b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 + {}^4C_3\, a b^3 + {}^4C_4\, b^4$
$(a – b)^4 = {}^4C_0\, a^4 - {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 - {}^4C_3\, a b^3 + {}^4C_4\, b^4$
Now $(a + b)^4 - (a – b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 + {}^4C_3\, a b^3 + {}^4C_4\, b^4 – [{}^4C_0\, a^4 - {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 - {}^4C_3\, a b^3 + {}^4C_4\, b^4]$
$= 2\,({}^4C_1\, a^3 b + {}^4C_3\, a b^3)$
$= 2\,(4a^3 b + 4ab^3)$
$= 8ab\,(a^2 + b^2)$
Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ we get
$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8\,(\sqrt{3})\,(\sqrt{2})\,\{(\sqrt{3})^2 + (\sqrt{2})^2\}$
$= 8\,(\sqrt{6})\,(3 + 2)$
$= 40\sqrt{6}$

12. Find $(x + 1)^6 + (x – 1)^6$. Hence or otherwise evaluate
$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$

Solution:

Using binomial theorem the expressions, $(x + 1)^6$ and $(x – 1)^6$ can be expressed as

$(x + 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6$

$(x – 1)^6 = {}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6$

Now, $(x + 1)^6 - (x – 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6 – [{}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6]$

$= 2\, [{}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6]$

$= 2\, [x^6 + 15x^4 + 15x^2 + 1]$

Now by substituting $x = \sqrt{2}$ we get

$(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6 = 2\, [(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1]$

$= 2\, (8 + 15 \times 4 + 15 \times 2 + 1)$

$= 2\, (8 + 60 + 30 + 1)$

$= 2\, (99)$

$= 198$

13. Show that $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is a positive integer.

Solution:

In order to show that $9^{n+1} – 8n – 9$ is divisible by 64, it has to be show that $9^{n+1} – 8n – 9 = 64\, k$, where k is some natural number

Using binomial theorem,

$(1 + a)^m = {}^mC_0 + {}^mC_1\, a + {}^mC_2\, a^2 + \ldots. + {}^mC_m\, a^m$

For a = 8 and m = n + 1 we get

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\, (8) + {}^{n+1}C_2\, (8)^2 + \ldots. + {}^{n+1}C_{n+1}\, (8)^{n+1}$

$9^{n+1} = 1 + (n + 1)\, 8 + 8^2\, [{}^{n+1}C_2 + {}^{n+1}C_3\, (8) + \ldots. + {}^{n+1}C_{n+1}\, (8)^{n-1}]$

$9^{n+1} = 9 + 8n + 64\, [{}^{n+1}C_2 + {}^{n+1}C_3\, (8) + \ldots. + {}^{n+1}C_{n+1}\, (8)^{n-1}]$

$9^{n+1} – 8n – 9 = 64\, k$

Where $k = [{}^{n+1}C_2 + {}^{n+1}C_3\, (8) + \ldots. + {}^{n+1}C_{n+1}\, (8)^{n-1}]$ is a natural number

Thus, $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is positive integer.

Hence the proof

14. Prove that

$$\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Solution:

By Binomial Theorem

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r} = (a+b)^{n}$$

On right side we need 4^n so we will put the values as,

Putting b = 3 & a = 1 in the above equation, we get

$$\sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} (3)^{r} = (1+3)^{n}$$

$$\sum_{r=0}^{n} \binom{n}{r} (1)(3)^{r} = (4)^{n}$$

$$\sum_{r=0}^{n} \binom{n}{r} (3)^{r} = (4)^{n}$$

Hence Proved.