EXERCISE 8.2 PAGE NO: 171

Find the coefficient of

1. x^5 in $(x + 3)^8$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here x^5 is the T_{r+1} term so a= x, b = 3 and n =8

$T_{r+1} = {}^8C_r\, x^{8-r}\, 3^r$.............. (i)

For finding out x^5

We have to equate $x^5 = x^{8-r}$

⇒ r= 3

Putting value of r in (i) we get

$T_{3+1} = {}^8C_3\, x^{8-3}\, 3^3$

$$T_4 = \frac{8!}{3!\,5!} \times x^5 \times 27$$

= 1512 x^5

Hence the coefficient of x^5= 1512

2. a^5b^7 in $(a – 2b)^{12}$.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a = a, b = -2b & n =12

Substituting the values, we get

$T_{r+1} = {}^{12}C_r\, a^{12-r}\, (-2b)^r$.......... (i)

To find a^5

We equate $a^{12-r} = a^5$

r = 7

Putting r = 7 in (i)

$T_8 = {}^{12}C_7\, a^5\, (-2b)^7$

$$T_8 = \frac{12!}{7!\,5!} \times a^5 \times (-2)^7\, b^7$$

= -101376 $a^5 b^7$

Hence the coefficient of a^5b^7= -101376

Write the general term in the expansion of
3. $(x^2 – y)^6$

Solution:
The general term T_{r+1} in the binomial expansion is given by
$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$........ (i)
Here $a = x^2$, $n = 6$ and $b = -y$
Putting values in (i)
$T_{r+1} = {}^6C_r\, x^{2(6-r)}\, (-1)^r\, y^r$

$$= \frac{6!}{r!\,(6-r)!} \times x^{12-2r} \times (-1)^r \times y^r$$

$$= -1^r \frac{6!}{r!\,(6-r)!} \times x^{12-2r} \times y^r$$

$= -1^r\, {}^6c_r\, .x^{12-2r}.\, y^r$

4. $(x^2 – y\,x)^{12}$, $x \neq 0$.

Solution:
The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here $n = 12$, $a= x^2$ and $b = -y\,x$
Substituting the values we get
$T_{n+1} = {}^{12}C_r \times x^{2(12-r)}\, (-1)^r\, y^r\, x^r$

$$= \frac{12!}{r!(12-r)!} \times x^{24-2r} -1^r\, y^r\, x^r$$

$$= -1^r \frac{12!}{r!(12-r)!} x^{24-r}\, y^r$$

$= -1^r\, {}^{12}c_r\, .x^{24-2r}.\, y^r$

5. Find the 4th term in the expansion of $(x – 2y)^{12}$.

Solution:
The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here $a= x$, $n =12$, $r= 3$ and $b = -2y$
By substituting the values we get
$T_4 = {}^{12}C_3\, x^9\, (-2y)^3$

$= \frac{12!}{3!9!} \times x^9 \times -8 \times y^3$

$= -\frac{12 \times 11 \times 10 \times 8}{3 \times 2 \times 1} \times x^9 y^3$

$= -1760\, x^9 y^3$

6. Find the 13th term in the expansion of

$$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a=9x, $b = -\frac{1}{3\sqrt{x}}$ n =18 and r = 12

Putting values

$T_{13} = \frac{18!}{12!\,6!}\, 9x^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}$

$= \frac{(18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12!)}{12! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 3^{12} \times x^6 \times \frac{1}{x^6} \times \frac{1}{3^{12}}$

= 18564

Find the middle terms in the expansions of

7. $\left(3 - \frac{x^3}{6}\right)^7$

Solution:

Here n = 7 so there would be two middle terms given by

$\left(\frac{n+1}{2}^{th}\right)$ term $= 4^{\text{ th}}$ and $\left(\frac{n+1}{2} + 1\right)^{\text{ th}}$ term $= 5^{th}$

We have

$a = 3, n = 7$ and $b = -\frac{x^3}{6}$

For T_4, r= 3

The term will be

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$T_4 = \frac{7!}{3!}\, 3^4 \left(-\frac{x^3}{6}\right)^3$$

$$= -\frac{7 \times 6 \times 5 \times 4}{3 \times 2 \times 1} \times 3^4 \times \frac{x^9}{2^3 3^3}$$

$$= -\frac{105}{8}\, x^9$$

For T_5 term, r = 4

The term T_{r+1} in the binomial expansion is given by

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$T_5 = \frac{7!}{4!\,3!}\, 3^3 \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4!\,3!} \times \frac{3^3}{2^4 3^4} \times x^3 = \frac{35\, x^{12}}{48}$$

8. $\left(\frac{x}{3} + 9y\right)^{10}$

Solution:

Here n is even so the middle term will be given by $\left(\frac{n+1}{2}\right)^{\text{th}}$ term $= 6^{\text{th}}$ term

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now $a = \frac{x}{3}$, $b = 9y$, $n = 10$ and $r = 5$

Substituting the values

$$T_6 = \frac{10!}{5!\,5!} \times \left(\frac{x}{3}\right)^5 \times (9y)^5$$

$$= \frac{10\times9\times8\times7\times6\times5!}{5!\times5\times4\times3\times2\times1} \times \frac{x^5}{3^5} \times 3^{10} \times y^5$$

$= 61236\, x^5y^5$

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution:

We know that the general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here n= m+n, a = 1 and b= a

Substituting the values in the general form

$T_{r+1} = {}^{m+n}C_r\, 1^{m+n-r}\, a^r$

$= {}^{m+n}C_r\, a^r$………… (i)

Now we have that the general term for the expression is,

$T_{r+1} = {}^{m+n}C_r\, a^r$

Now, For coefficient of a^m

$T_{m+1} = {}^{m+n}C_m\, a^m$

Hence, for coefficient of a^m, value of r = m

So, the coefficient is ${}^{m+n}C_m$

Similarly, Coefficient of a^n is ${}^{m+n}C_n$

${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$

And also, ${}^{m+n}C_n = \frac{(m+n)!}{m!n!}$

The coefficient of a^m and a^n are same that is $\frac{(m+n)!}{m!n!}$

10. The coefficients of the $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here the binomial is $(1+x)^n$ with a = 1 , b = x and n = n

The $(r+1)^{th}$ term is given by

$T_{(r+1)} = {}^nC_r\, 1^{n-r}\, x^r$

$T_{(r+1)} = {}^nC_r\, x^r$

The coefficient of $(r+1)^{th}$ term is ${}^nC_r$

The r^{th} term is given by $(r-1)^{th}$ term

$T_{(r+1-1)} = {}^nC_{r-1}\, x^{r-1}$

$T_r = {}^nC_{r-1}\, x^{r-1}$

∴ the coefficient of r^{th} term is ${}^nC_{r-1}$

For $(r-1)^{th}$ term we will take $(r-2)^{th}$ term

$T_{r-2+1} = {}^nC_{r-2}\, x^{r-2}$

$T_{r-1} = {}^nC_{r-2}\, x^{r-2}$

∴ the coefficient of $(r-1)^{th}$ term is ${}^nC_{r-2}$

Given that the coefficient of $(r-1)^{th}$, r^{th} and $r+1^{th}$ term are in ratio 1:3:5

Therefore,

$$\frac{\text{the coefficient of } r-1^{th} \text{ term}}{\text{coefficient of } r^{th} \text{ term}} = \frac{1}{3}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n!}{(r-2)!(n-r+2)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{1}{3}$$

On rearranging we get

$$\frac{n!}{(r-2)!\,(n-r+2)!} \times \frac{(r-1)!\,(n-r+1)!}{n!} = \frac{1}{3}$$

By multiplying

$$\Rightarrow \frac{(r-1)(r-2)!\,(n-r+1)!}{(r-2)!\,(n-r+2)!} = \frac{1}{3}$$

$$\Rightarrow \frac{(r-1)(n-r+1)!}{(n-r+2)(n-r+1)!} = \frac{1}{3}$$

On simplifying we get

$\Rightarrow \dfrac{(r-1)}{(n-r+2)} = \dfrac{1}{3}$

$\Rightarrow$ 3r - 3 = n – r + 2

$\Rightarrow$ n - 4r + 5 =0………..1

Also

$\dfrac{\text{the coefficient of } r^{th} \text{ term}}{\text{coefficient of } r+1^{th} \text{ term}} = \dfrac{3}{5}$

$\Rightarrow \dfrac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \dfrac{3}{5}$

On rearranging we get

$\Rightarrow \dfrac{n!}{(r-1)!(n-r+1)!} \times \dfrac{r!(n-r)!}{n!} = \dfrac{3}{5}$

By multiplying

$\Rightarrow \dfrac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)!} = \dfrac{3}{5}$

$\Rightarrow \dfrac{r(n-r)!}{(n-r+1)!} = \dfrac{3}{5}$

$\Rightarrow \dfrac{r(n-r)!}{(n-r+1)(n-r)!} = \dfrac{3}{5}$

On simplifying we get

$\Rightarrow \dfrac{r}{(n-r+1)} = \dfrac{3}{5}$

Also

$\dfrac{\text{the coefficient of } r^{th} \text{ term}}{\text{coefficient of } r+1^{th} \text{ term}} = \dfrac{3}{5}$

$\Rightarrow \dfrac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \dfrac{3}{5}$

On rearranging we get

$\Rightarrow$ 5r = 3n - 3r + 3

$\Rightarrow$ 8r – 3n - 3 =0…………2
We have 1 and 2 as
n - 4r ± 5 =0………..1
8r – 3n - 3 =0…………….2
Multiplying equation 1 by number 2
2n -8r +10 =0……………….3
Adding equation 2 and 3
2n -8r +10 =0
-3n – 8r - 3 =0
$\Rightarrow$ -n = -7
n =7 and r = 3

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Solution:
The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
The general term for binomial $(1+x)^{2n}$ is
$T_{r+1} = {}^{2n}C_r\, x^r$ …………………..1
To find the coefficient of x^n
r = n
$T_{n+1} = {}^{2n}C_n\, x^n$
The coefficient of $x^n = {}^{2n}C_n$
The general term for binomial $(1+x)^{2n-1}$ is
$T_{r+1} = {}^{2n-1}C_r\, x^r$
To find the coefficient of x^n
Putting n = r
$T_{r+1} = {}^{2n-1}C_r\, x^n$
The coefficient of $x^n = {}^{2n-1}C_n$
We have to prove
Coefficient of x^n in $(1+x)^{2n}$ = 2 coefficient of x^n in $(1+x)^{2n-1}$
Consider LHS = ${}^{2n}C_n$

$$= \frac{2n!}{n!\,(2n-n)!}$$

$$= \frac{2n!}{n!\,(n)!}$$

Again consider RHS = 2 × $^{2n-1}C_n$

$$= 2 \times \frac{(2n-1)!}{n!\,(2n-1-n)!}$$

$$= 2 \times \frac{(2n-1)!}{n!\,(n-1)!}$$

Now multiplying and dividing by n we get

$$= 2 \times \frac{(2n-1)!}{n!\,(n-1)!} \times \frac{n}{n}$$

$$= \frac{2n(2n-1)!}{n!\,n(n-1)!}$$

$$= \frac{2n!}{n!\,n!}$$

From above equations LHS = RHS

Hence the proof.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a = 1, b = x and n = m

Putting the value

$T_{r+1} = {}^mC_r\, 1^{m-r}\, x^r$

$= {}^mC_r\, x^r$

We need coefficient of x^2

∴ putting r = 2

$T_{2+1} = {}^mC_2\, x^2$

The coefficient of $x^2 = {}^mC_2$

Given that coefficient of $x^2 = {}^mC_2 = 6$

$\Rightarrow \frac{m!}{2!(m-2)!} = 6$

$\Rightarrow \frac{m(m-1)(m-2)!}{2\times1\times(m-2)!} = 6$

$\Rightarrow m\,(m - 1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m\,(m - 4) + 3\,(m - 4) = 0$

$\Rightarrow (m+3)\,(m - 4) = 0$

$\Rightarrow m = -3, 4$

We need positive value of m so m = 4