MISCELLANEOUS EXERCISE

1. Find a, b and n in the expansion of (a + b)^n if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Solution:

We know that (r + 1)th term, (T_{r+1}), in the binomial expansion of (a + b)^n is given by

$T_{r+1} = {}^nC_r\, a^{n-t}\, b^r$

The first three terms of the expansion are given as 729, 7290 and 30375 respectively.

Then we have,

$T_1 = {}^nC_0\, a^{n-0}\, b^0 = a^n = 729$….. 1

$T_2 = {}^nC_1\, a^{n-1}\, b^1 = na^{n-1}\, b = 7290$…. 2

$T_3 = {}^nC_2\, a^{n-2}\, b^2 = n\,(n-1)/2\; a^{n-2}\, b^2 = 30375$……3

Dividing 2 by 1 we get

$$\mathrm{na^{n-1}ba^n} = \frac{7290}{729}$$

n b a = 10 ….. 4

Dividing 3 by 2 we get

$$n(n-1)a^{n-2}b^2 2na^{n-1}b = \frac{30375}{7290}$$

$$\Rightarrow (n-1)b2a = \frac{30375}{7290}$$

$$\Rightarrow (n-1)ba = \frac{30375\times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \mathrm{nba} - \frac{\mathrm{b}}{\mathrm{a}} = \frac{25}{3}$$

$$\Rightarrow 10 - \mathrm{ba} = \frac{25}{3}$$

$$\Rightarrow \mathrm{ba} = 10 - \frac{25}{3} = \frac{5}{3} \qquad \text{....... 5}$$

From 4 and 5 we have

n. 5/3 = 10

n = 6

Substituting n = 6 in 1 we get

$a^6 = 729$

a = 3

From 5 we have, b/3 = 5/3

b = 5

Thus a = 3, b = 5 and n = 76

2. Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + a x)^9$ are equal.

Solution:

We know that general term of expansion $(a + b)^n$ is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

For $(3+ax)^9$

Putting a = 3, b = a x & n = 9

General term of $(3+ax)^9$ is

$$T_{r+1} = \binom{9}{r} 3^{n-r} (ax)^r$$

$$T_{r+1} = \binom{9}{r} 3^{n-r} a^r x^r$$

Since we need to find the coefficients of x^2 and x^3, therefore

For r = 2

$$T_{2+1} = \binom{9}{2} 3^{n-2} a^2 x^2$$

Thus, the coefficient of x^2 = $\binom{9}{2} 3^{n-2} a^2$

For r = 3

$$T_{3+1} = \binom{9}{3} 3^{n-3} a^3 x^3$$

Thus, the coefficient of x^3 = $\binom{9}{3} 3^{n-3} a^3$

Given that coefficient of x^2 = Coefficient of x^3

$$\Rightarrow \binom{9}{2} 3^{n-2} a^2 = \binom{9}{3} 3^{n-3} a^3$$

$$\Rightarrow \frac{9!}{2!(9-2)!} \times 3^{n-2} a^2 = \frac{9!}{3!(9-3)!} \times 3^{n-3} a^3$$

$$\Rightarrow \frac{3^{n-2} a^2}{3^{n-3} a^3} = \frac{2!(9-2)!}{3!(9-3)!}$$

$$\Rightarrow \frac{3^{(n-2)-(n-3)}}{a} = \frac{2!\,7!}{3!\,6!}$$

$$\Rightarrow \frac{3}{a} = \frac{7}{3}$$

$\therefore$ a = 9/7

Hence, a = 9/7

3. Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 – x)^7$ using binomial theorem.

Solution:

$(1 + 2x)^6 = {}^6C_0 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$
$= 1 + 6 (2x) + 15 (2x)^2 + 20 (2x)^3 + 15 (2x)^4 + 6 (2x)^5 + (2x)^6$
$= 1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6$
$(1 – x)^7 = {}^7C_0 - {}^7C_1 (x) + {}^7C_2 (x)^2 - {}^7C_3 (x)^3 + {}^7C_4 (x)^4 - {}^7C_5 (x)^5 + {}^7C_6 (x)^6 - {}^7C_7 (x)^7$
$= 1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7$
$(1 + 2x)^6 (1 – x)^7 = (1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6) (1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7)$
192 – 21 = 171
Thus, the coefficient of x^5 in the expression $(1+2x)^6(1-x)7$ is 171.

4. If a and b are distinct integers, prove that a – b is a factor of $a^n – b^n$, whenever n is a positive integer. [Hint write $a^n = (a – b + b)^n$ and expand]

Solution:

In order to prove that (a – b) is a factor of $(a^n – b^n)$, it has to be proved that
$a^n - b^n = k (a – b)$ where k is some natural number.
a can be written as a = a – b + b
$a^n = (a – b + b)^n = [(a – b) + b]^n$
$= {}^nC_0 (a – b)^n + {}^nC_1 (a – b)^{n-1} b + \ldots\ldots + {}^nC_n b^n$
$a^n – b^n = (a – b) [(a –b)^{n-1} + {}^nC_1 (a – b)^{n-1} b + \ldots\ldots + {}^nC_n b^n]$

$a^n - b^n = (a - b) k$

Where $k = [(a - b)^{n-1} + {}^nC_1 (a - b)^{n-1} b + \ldots\ldots + {}^nC_n b^n]$ is a natural number

This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is positive integer.

5. Evaluate

$$(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$$

Solution:

Using binomial theorem the expression $(a + b)^6$ and $(a - b)^6$, can be expanded

$(a + b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 + {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 + {}^6C_5\, a\, b^5 + {}^6C_6\, b^6$

$(a - b)^6 = {}^6C_0\, a^6 - {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 - {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 - {}^6C_5\, a\, b^5 + {}^6C_6\, b^6$

Now $(a + b)^6 - (a - b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 + {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 + {}^6C_5\, a\, b^5 + {}^6C_6\, b^6$
$- [{}^6C_0\, a^6 - {}^6C_1\, a^5\, b + {}^6C_2\, a^4\, b^2 - {}^6C_3\, a^3\, b^3 + {}^6C_4\, a^2\, b^4 - {}^6C_5\, a\, b^5 + {}^6C_6\, b^6]$

Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ we get

$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2\, [6\, (\sqrt{3})^5\, (\sqrt{2}) + 20\, (\sqrt{3})^3\, (\sqrt{2})^3 + 6\, (\sqrt{3})\, (\sqrt{2})^5]$

$= 2\, [54(\sqrt{6}) + 120\, (\sqrt{6}) + 24\, \sqrt{6}]$

$= 2\, (\sqrt{6})\, (198)$

$= 396\, \sqrt{6}$

6. Find the value of

$$\left(a^2+\sqrt{a^2-1}\right)^4 + \left(a^2-\sqrt{a^2-1}\right)^4$$

Solution:

Firstly the expression $(x + y)^4 + (x - y)^4$ is simplified by using binomial theorem

$$(x+y)^4 = {}^4C_0 x^4 + {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 + {}^+C_3 x y^3 + {}^4C_4 y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x-y)^4 = {}^4C_0 x^4 - {}^4C_1 x^3 y + {}^4C_2 x^2 y^2 - {}^4C_3 x y^3 + {}^4C_4 y^4$$

$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$$
$$=2\left[(a^2)^4+6(a^2)^2\left(\sqrt{a^2-1}\right)^2+\left(\sqrt{a^2-1}\right)^4\right]$$
$$=2\left[a^8+6a^4\left(a^2-1\right)+\left(a^2-1\right)^2\right]$$
$$=2\left[a^8+6a^6-6a^4+a^4-2a^2+1\right]$$
$$=2\left[a^8+6a^6-5a^4-2a^2+1\right]$$
$$=2a^8+12a^6-10a^4-4a^2+2$$

7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution:
0.99 can be written as
0.99 = 1 – 0.01
Now by applying binomial theorem we get
$(o.\ 99)^5 = (1 - 0.01)^5$
$= {}^5C_0\,(1)^5 - {}^5C_1\,(1)^4\,(0.01) + {}^5C_2\,(1)^3\,(0.01)^2$
$= 1 - 5\,(0.01) + 10\,(0.01)^2$
$= 1 - 0.05 + 0.001$
$= 0.951$

8. Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is √6: 1

Solution:
In the expansion $(a + b)^n$, if n is even then the middle term is $(n/2 + 1)^{th}$ term

$${}^nC_4(\sqrt[4]{2})^{n-1}\left(\frac{1}{\sqrt[4]{3}}\right)^4={}^nC_4\frac{(\sqrt[4]{2})^n}{(\sqrt[4]{2})^4}\cdot\frac{1}{3}={}^nC_4\frac{(\sqrt[4]{2})^n}{2}\cdot\frac{1}{3}=\frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^n$$

$${}^nC_{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}={}^nC_{n-1}\cdot 2\cdot\frac{(\sqrt[4]{3})^4}{(\sqrt[4]{3})^n}={}^nC_{n-1}\cdot 2\cdot\frac{3}{(\sqrt[4]{3})^n}=\frac{6n!}{(n-4)!4!}\cdot\frac{1}{(\sqrt[4]{3})^n}$$

$$\frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^n : \frac{6n!}{(n-4)!!4!} \cdot \frac{1}{(\sqrt[4]{3})^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} : \frac{6}{(\sqrt[4]{3})^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} \times \frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$$

$$\Rightarrow (\sqrt[4]{6})^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4 \times \frac{5}{2} = 10$$

Thus the value of n = 10

9. Expand using Binomial Theorem

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$$

Solution:

Using binomial theorem the given expression can be expanded as

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$= \left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \qquad ...(1)$$

Again by using binomial theorem to expand the above terms we get

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$= 1 + 4\times\frac{x}{2} + 6\times\frac{x^2}{4} + 4\times\frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \qquad \ldots(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \qquad \ldots(3)$$

From equation 1, 2 and 3 we get

$$\left[\left(1+\frac{x}{2}\right) - \frac{2}{x}\right]^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

10. Find the expansion of $(3x^2 – 2ax + 3a^2)^3$ using binomial theorem.

Solution:

We know that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Putting $a = 3x^2$ & $b = -a\,(2x-3a)$, we get

$[3x^2 + (-a\,(2x-3a))]^3$

$= (3x^2)^3+3(3x^2)^2(-a\,(2x-3a)) + 3(3x^2)\,(-a\,(2x-3a))^2 + (-a\,(2x-3a))^3$

$= 27x^6 - 27ax^4\,(2x-3a) + 9a^2x^2\,(2x-3a)^2 - a^3(2x-3a)^3$

$= 27x^6 - 54ax^5 + 81a^2x^4 + 9a^2x^2\,(4x^2-12ax+9a^2) - a^3\,[(2x)^3 - (3a)^3 - 3(2x)^2(3a) + 3(2x)(3a)^2]$

$= 27x^6 - 54ax^5 + 81a^2x^4 + 36a^2x^4 - 108a^3x^3 + 81a^4x^2 - 8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x$

$= 27x^6 - 54ax^5+ 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

Thus, $(3x^2 – 2ax + 3a^2)^3$

$= 27x^6 - 54ax^5+ 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$