EXERCISE 8.1 PAGE NO: 166

Expand each of the expressions in Exercises 1 to 5.

1. $(1 – 2x)^5$

Solution:

From binomial theorem expansion we can write as

$(1 – 2x)^5$

$= {}^5C_0 (1)^5 - {}^5C_1 (1)^4 (2x) + {}^5C_2 (1)^3 (2x)^2 - {}^5C_3 (1)^2 (2x)^3 + {}^5C_4 (1)^1 (2x)^4 - {}^5C_5 (2x)^5$

$= 1 – 5 (2x) + 10 (4x)^2 – 10 (8x^3) + 5 (16 x^4) – (32 x^5)$

$= 1 – 10x + 40x^2 – 80x^3 – 32x^5$

2. $\left(\frac{2}{x}-\frac{x}{2}\right)^5$

Solution:

From binomial theorem, given equation can be expanded as

$$\left(\frac{2}{x}-\frac{x}{2}\right)^5 = {}^5C_0\left(\frac{2}{x}\right)^3 - {}^5C_1\left(\frac{2}{x}\right)^4\left(\frac{x}{2}\right) + {}^5C_2\left(\frac{2}{x}\right)^3\left(\frac{x}{2}\right)^2$$

$$-{}^3C_3\left(\frac{2}{x}\right)^2\left(\frac{x}{2}\right)^3 + {}^3C_4\left(\frac{2}{x}\right)\left(\frac{x}{2}\right)^4 - {}^3C_5\left(\frac{x}{2}\right)^5$$

$$=\frac{32}{x^5}-5\left(\frac{16}{x^4}\right)\left(\frac{x}{2}\right)+10\left(\frac{8}{x^3}\right)\left(\frac{x^2}{4}\right)-10\left(\frac{4}{x^2}\right)+5\left(\frac{2}{x}\right)\left(\frac{x^4}{16}\right)-\frac{x^5}{32}$$

$$=\frac{32}{x^5}-\frac{40}{x^3}+\frac{20}{x}-5x+\frac{5}{8}x^3-\frac{x^3}{32}$$

3. $(2x – 3)^6$

Solution:

From binomial theorem, given equation can be expanded as

$$(2x-3)^6 = {}^6C_0(2x)^6 - {}^6C_1(2x)^5(3) + {}^6C_1(2x)^4(3)^2 - {}^4C_3(2x)^3(3)^3$$

$$= 64x^6 - 6\left(32x^5\right)(3) + 15\left(16x^4\right)(9) - 20\left(8x^3\right)(27)$$

$$+ 15\left(4x^2\right)(81) - 6(2x)(243) + 729$$

$$= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729$$

4. $\left(\frac{x}{3}+\frac{1}{x}\right)^{5}$

Solution:
From binomial theorem, given equation can be expanded as

$$\left(\frac{x}{3}+\frac{1}{x}\right)^{5}={}^{5}C_{0}\left(\frac{x}{3}\right)^{5}+{}^{3}C_{1}\left(\frac{x}{3}\right)^{4}\left(\frac{1}{x}\right)+{}^{3}C_{2}\left(\frac{x}{3}\right)^{3}\left(\frac{1}{x}\right)^{2}$$

$$=\frac{x^{5}}{243}+5\left(\frac{x^{4}}{81}\right)\left(\frac{1}{x}\right)+10\left(\frac{x^{3}}{27}\right)\left(\frac{1}{x^{2}}\right)+10\left(\frac{x^{2}}{9}\right)\left(\frac{1}{x^{3}}\right)+5\left(\frac{x}{3}\right)\left(\frac{1}{x^{4}}\right)+\frac{1}{x^{5}}$$

$$=\frac{x^{5}}{243}+\frac{5x^{3}}{81}+\frac{10x}{27}+\frac{10}{9x}+\frac{5}{3x^{3}}+\frac{1}{x^{3}}$$

5. $\left(x+\frac{1}{x}\right)^{6}$

Solution:
From binomial theorem, given equation can be expanded as

$$\left(x+\frac{1}{x}\right)^{6}={}^{6}C_{0}(x)^{6}+{}^{6}C_{1}(x)'\left(\frac{1}{x}\right)+{}^{6}C_{2}(x)^{4}\left(\frac{1}{x}\right)^{2}$$

$$+{}^{6}C_{3}(x)^{3}\left(\frac{1}{x}\right)^{3}+{}^{6}C_{4}(x)^{2}\left(\frac{1}{x}\right)^{4}+{}^{6}C_{3}(x)\left(\frac{1}{x}\right)^{5}+{}^{6}C_{6}\left(\frac{1}{x}\right)^{6}$$

$$=x^{4}+6(x)^{3}\left(\frac{1}{x}\right)+15(x)^{4}\left(\frac{1}{x^{2}}\right)+20(x)^{3}\left(\frac{1}{x^{3}}\right)+15(x)^{2}\left(\frac{1}{x^{4}}\right)+6(x)\left(\frac{1}{x^{5}}\right)+\frac{1}{x^{6}}$$

$$=x^{6}+6x^{4}+15x^{2}+20+\frac{15}{x^{2}}+\frac{6}{x^{4}}+\frac{1}{x^{6}}$$

6. $(96)^3$

Solution:
Given $(96)^3$
96 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.
The given question can be written as 96 = 100 – 4
$(96)^3 = (100 – 4)^3$
$= {}^3C_0\,(100)^3 - {}^3C_1\,(100)^2\,(4) - {}^3C_2\,(100)\,(4)^2 - {}^3C_3\,(4)^3$
$= (100)^3 – 3\,(100)^2\,(4) + 3\,(100)\,(4)^2 – (4)^3$
= 1000000 – 120000 + 4800 – 64

= 884736

7. $(102)^5$

Solution:

Given $(102)^5$

102 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 102 = 100 + 2

$(102)^5 = (100 + 2)^5$

$= {}^5C_0 (100)^5 + {}^5C_1 (100)^4 (2) + {}^5C_2 (100)^3 (2)^2 + {}^5C_3 (100)^2 (2)^3 + {}^5C_4 (100) (2)^4 + {}^5C_5 (2)^5$

$= (100)^5 + 5 (100)^4 (2) + 10 (100)^3 (2)^2 + 5 (100) (2)^3 + 5 (100) (2)^4 + (2)^5$

= 1000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32

= 11040808032

8. $(101)^4$

Solution:

Given $(101)^4$

101 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 101 = 100 + 1

$(101)^4 = (100 + 1)^4$

$= {}^4C_0 (100)^4 + {}^4C_1 (100)^3 (1) + {}^4C_2 (100)^2 (1)^2 + {}^4C_3 (100) (1)^2 + {}^4C_4 (1)^4$

$= (100)^4 + 4 (100)^3 + 6 (100)^2 + 4 (100) + (1)^4$

= 100000000 + 400000 + 60000 + 400 + 1

= 1040604001

9. $(99)^5$

Solution:

Given $(99)^5$

99 can be written as the sum or difference of two numbers then binomial theorem can be applied.

The given question can be written as 99 = 100 -1

$(99)^5 = (100 – 1)^5$

$= {}^5C_0 (100)^5 - {}^5C_1 (100)^4 (1) + {}^5C_2 (100)^3 (1)^2 - {}^5C_3 (100)^2 (1)^3 + {}^5C_4 (100) (1)^4 - {}^5C_5 (1)^5$

$= (100)^5 - 5\,(100)^4 + 10\,(100)^3 - 10\,(100)^2 + 5\,(100) - 1$
$= 1000000000 - 5000000000 + 10000000 - 100000 + 500 - 1$
$= 9509900499$

10. Using Binomial Theorem, indicate which number is larger $(1.1)^{10000}$ or 1000.

Solution:
By splitting the given 1.1 and then applying binomial theorem, the first few terms of $(1.1)^{10000}$ can be obtained as

$(1.1)^{10000} = (1 + 0.1)^{10000}$
$= (1 + 0.1)^{10000}\, C_1\,(1.1)$ + other positive terms
$= 1 + 10000 \times 1.1$ + other positive terms
$= 1 + 11000$ + other positive terms
> 1000
$(1.1)^{10000} > 1000$

11. Find $(a + b)^4 - (a - b)^4$. Hence, evaluate

$$\left(\sqrt{3}+\sqrt{2}\right)^4 - \left(\sqrt{3}-\sqrt{2}\right)^4.$$

Solution:
Using binomial theorem the expression $(a + b)^4$ and $(a - b)^4$, can be expanded

$(a + b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 + {}^4C_3\, a b^3 + {}^4C_4\, b^4$
$(a - b)^4 = {}^4C_0\, a^4 - {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 - {}^4C_3\, a b^3 + {}^4C_4\, b^4$
Now $(a + b)^4 - (a - b)^4 = {}^4C_0\, a^4 + {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 + {}^4C_3\, a b^3 + {}^4C_4\, b^4 - [{}^4C_0\, a^4 - {}^4C_1\, a^3 b + {}^4C_2\, a^2 b^2 - {}^4C_3\, a b^3 + {}^4C_4\, b^4]$
$= 2\,({}^4C_1\, a^3 b + {}^4C_3\, a b^3)$
$= 2\,(4a^3 b + 4ab^3)$
$= 8ab\,(a^2 + b^2)$
Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ we get
$(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8\,(\sqrt{3})\,(\sqrt{2})\,\{(\sqrt{3})^2 + (\sqrt{2})^2\}$
$= 8\,(\sqrt{6})\,(3 + 2)$
$= 40\sqrt{6}$

12. Find $(x + 1)^6 + (x - 1)^6$. Hence or otherwise evaluate

$$\left(\sqrt{2}+1\right)^6 + \left(\sqrt{2}-1\right)^6$$

Solution:

Using binomial theorem the expressions, $(x + 1)^6$ and $(x – 1)^6$ can be expressed as

$(x + 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6$

$(x – 1)^6 = {}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6$

Now, $(x + 1)^6 - (x – 1)^6 = {}^6C_0\, x^6 + {}^6C_1\, x^5 + {}^6C_2\, x^4 + {}^6C_3\, x^3 + {}^6C_4\, x^2 + {}^6C_5\, x + {}^6C_6 – [{}^6C_0\, x^6 - {}^6C_1\, x^5 + {}^6C_2\, x^4 - {}^6C_3\, x^3 + {}^6C_4\, x^2 - {}^6C_5\, x + {}^6C_6]$

$= 2\, [{}^6C_0\, x^6 + {}^6C_2\, x^4 + {}^6C_4\, x^2 + {}^6C_6]$

$= 2\, [x^6 + 15x^4 + 15x^2 + 1]$

Now by substituting $x = \sqrt{2}$ we get

$(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6 = 2\, [(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1]$

$= 2\, (8 + 15 \times 4 + 15 \times 2 + 1)$

$= 2\, (8 + 60 + 30 + 1)$

$= 2\, (99)$

$= 198$

13. Show that $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is a positive integer.

Solution:

In order to show that $9^{n+1} – 8n – 9$ is divisible by 64, it has to be show that $9^{n+1} – 8n – 9 = 64\, k$, where k is some natural number

Using binomial theorem,

$(1 + a)^m = {}^mC_0 + {}^mC_1\, a + {}^mC_2\, a^2 + \dots + {}^mC_m\, a^m$

For a = 8 and m = n + 1 we get

$(1 + 8)^{n+1} = {}^{n+1}C_0 + {}^{n+1}C_1\, (8) + {}^{n+1}C_2\, (8)^2 + \dots + {}^{n+1}C_{n+1}\, (8)^{n+1}$

$9^{n+1} = 1 + (n + 1)\, 8 + 8^2\, [{}^{n+1}C_2 + {}^{n+1}C_3\, (8) + \dots + {}^{n+1}C_{n+1}\, (8)^{n-1}]$

$9^{n+1} = 9 + 8n + 64\, [{}^{n+1}C_2 + {}^{n+1}C_3\, (8) + \dots + {}^{n+1}C_{n+1}\, (8)^{n-1}]$

$9^{n+1} – 8n – 9 = 64\, k$

Where $k = [{}^{n+1}C_2 + {}^{n+1}C_3\, (8) + \dots + {}^{n+1}C_{n+1}\, (8)^{n-1}]$ is a natural number

Thus, $9^{n+1} – 8n – 9$ is divisible by 64, whenever n is positive integer.

Hence the proof

14. Prove that

$$\sum_{r=0}^{n} 3^r\, {}^nC_r = 4^n$$

Solution:

By Binomial Theorem

$$\sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^{r} = (a+b)^{n}$$

On right side we need 4^n so we will put the values as,

Putting b = 3 & a = 1 in the above equation, we get

$$\sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} (3)^{r} = (1+3)^{n}$$

$$\sum_{r=0}^{n} \binom{n}{r} (1)(3)^{r} = (4)^{n}$$

$$\sum_{r=0}^{n} \binom{n}{r} (3)^{r} = (4)^{n}$$

Hence Proved.

EXERCISE 8.2

Find the coefficient of

1. x^5 in $(x + 3)^8$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here x^5 is the T_{r+1} term so a= x, b = 3 and n =8

$T_{r+1} = {}^8C_r\, x^{8-r}\, 3^r$………….. (i)

For finding out x^5

We have to equate $x^5= x^{8-r}$

⇒ r= 3

Putting value of r in (i) we get

$$T_{3+1} = {}^8C_3\, x^{8-3}\, 3^3$$

$$T_4 = \frac{8!}{3!\,5!} \times x^5 \times 27$$

$= 1512\, x^5$

Hence the coefficient of x^5= 1512

2. a^5b^7 in $(a – 2b)^{12}$.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a = a, b = -2b & n =12

Substituting the values, we get

$T_{r+1} = {}^{12}C_r\, a^{12-r}\, (-2b)^r$………. (i)

To find a^5

We equate $a^{12-r} =a^5$

r = 7

Putting r = 7 in (i)

$T_8 = {}^{12}C_7\, a^5\, (-2b)^7$

$$T_8 = \frac{12!}{7!\,5!} \times a^5 \times (-2)^7\, b^7$$

$= -101376\, a^5\, b^7$

Hence the coefficient of a^5b^7= -101376

Write the general term in the expansion of
3. $(x^2 - y)^6$

Solution:
The general term T_{r+1} in the binomial expansion is given by
$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$........ (i)
Here $a = x^2$, $n = 6$ and $b = -y$
Putting values in (i)
$T_{r+1} = {}^6C_r\, x^{2(6-r)}\, (-1)^r\, y^r$

$$= \frac{6!}{r!\,(6-r)!} \times x^{12-2r} \times (-1)^r \times y^r$$

$$= -1^r\, \frac{6!}{r!\,(6-r)!} \times x^{12-2r} \times y^r$$

$= -1^r\, {}^6c_r\, .x^{12-2r}.\, y^r$

4. $(x^2 - y\,x)^{12}$, $x \neq 0$.

Solution:
The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here $n = 12$, $a= x^2$ and $b = -y\,x$
Substituting the values we get
$T_{n+1} = {}^{12}C_r \times x^{2(12-r)}\, (-1)^r\, y^r\, x^r$

$$= \frac{12!}{r!(12-r)!} \times x^{24-2r} -1^r\, y^r\, x^r$$

$$= -1^r\, \frac{12!}{r!(12-r)!}\, x^{24-r}\, y^r$$

$= -1^r\, {}^{12}c_r\, .x^{24-2r}.\, y^r$

5. Find the 4th term in the expansion of $(x - 2y)^{12}$.

Solution:
The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
Here $a= x$, $n =12$, $r= 3$ and $b = -2y$
By substituting the values we get
$T_4 = {}^{12}C_3\, x^9\, (-2y)^3$

$= \frac{12!}{3!9!} \times x^9 \times -8 \times y^3$

$= -\frac{12 \times 11 \times 10 \times 8}{3 \times 2 \times 1} \times x^9 y^3$

$= -1760\, x^9 y^3$

6. Find the 13th term in the expansion of

$$\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0$$

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a=9x, $b = -\frac{1}{3\sqrt{x}}$ n =18 and r = 12

Putting values

$$T_{13} = \frac{18!}{12!\,6!}\, 9x^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}$$

$$= \frac{(18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12!)}{12! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 3^{12} \times x^6 \times \frac{1}{x^6} \times \frac{1}{3^{12}}$$

= 18564

Find the middle terms in the expansions of

7. $\left(3 - \frac{x^3}{6}\right)^7$

Solution:

Here n = 7 so there would be two middle terms given by

$$\left(\frac{n+1}{2}^{\text{th}}\right) \text{term} = 4^{\text{ th}} \text{ and } \left(\frac{n+1}{2} + 1\right)^{\text{th}} \text{ term} = 5^{\text{th}}$$

We have

$a = 3, n = 7$ and $b = -\frac{x^3}{6}$

For T_4, r= 3

The term will be

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$T_4 = \frac{7!}{3!} 3^4 \left(-\frac{x^3}{6}\right)^3$$

$$= -\frac{7 \times 6 \times 5 \times 4}{3 \times 2 \times 1} \times 3^4 \times \frac{x^9}{2^3 3^3}$$

$$= -\frac{105}{8} x^9$$

For T_5 term, r = 4

The term T_{r+1} in the binomial expansion is given by

$T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

$$T_5 = \frac{7!}{4!\,3!} 3^3 \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4!\,3!} \times \frac{3^3}{2^4 3^4} \times x^3 = \frac{35\, x^{12}}{48}$$

8. $\left(\frac{x}{3} + 9y\right)^{10}$

Solution:

Here n is even so the middle term will be given by $\left(\frac{n+1}{2}\right)^{\text{th}}$ term $= 6^{\text{th}}$ term

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Now $a = \frac{x}{3}$, $b = 9y$, $n = 10$ and $r = 5$

Substituting the values

$$T_6 = \frac{10!}{5!\,5!} \times \left(\frac{x}{3}\right)^5 \times (9y)^5$$

$$= \frac{10\times9\times8\times7\times6\times5!}{5!\times5\times4\times3\times2\times1} \times \frac{x^5}{3^5} \times 3^{10} \times y^5$$

$= 61236\, x^5y^5$

9. In the expansion of $(1 + a)^{m+n}$, prove that coefficients of a^m and a^n are equal.

Solution:

We know that the general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here n= m+n, a = 1 and b= a

Substituting the values in the general form

$T_{r+1} = {}^{m+n}C_r\, 1^{m+n-r}\, a^r$

$= {}^{m+n}C_r\, a^r$………… (i)

Now we have that the general term for the expression is,

$T_{r+1} = {}^{m+n}C_r\, a^r$

Now, For coefficient of a^m

$T_{m+1} = {}^{m+n}C_m\, a^m$

Hence, for coefficient of a^m, value of r = m

So, the coefficient is ${}^{m+n}C_m$

Similarly, Coefficient of a^n is ${}^{m+n}C_n$

${}^{m+n}C_m = \frac{(m+n)!}{m!n!}$

And also, ${}^{m+n}C_n = \frac{(m+n)!}{m!n!}$

The coefficient of a^m and a^n are same that is $\frac{(m+n)!}{m!n!}$

10. The coefficients of the $(r – 1)^{th}$, r^{th} and $(r + 1)^{th}$ terms in the expansion of $(x + 1)^n$ are in the ratio 1 : 3 : 5. Find n and r.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here the binomial is $(1+x)^n$ with a = 1 , b = x and n = n

The $(r+1)^{th}$ term is given by

$T_{(r+1)} = {}^nC_r\, 1^{n-r}\, x^r$

$T_{(r+1)} = {}^nC_r\, x^r$

The coefficient of $(r+1)^{th}$ term is ${}^nC_r$

The r^{th} term is given by $(r-1)^{th}$ term

$T_{(r+1-1)} = {}^nC_{r-1}\, x^{r-1}$

$T_r = {}^nC_{r-1}\, x^{r-1}$

∴ the coefficient of r^{th} term is ${}^nC_{r-1}$

For $(r-1)^{th}$ term we will take $(r-2)^{th}$ term

$T_{r-2+1} = {}^nC_{r-2}\, x^{r-2}$

$T_{r-1} = {}^nC_{r-2}\, x^{r-2}$

∴ the coefficient of $(r-1)^{th}$ term is ${}^nC_{r-2}$

Given that the coefficient of $(r-1)^{th}$, r^{th} and $r+1^{th}$ term are in ratio 1:3:5

Therefore,

$$\frac{\text{the coefficient of } r-1^{\text{th}} \text{ term}}{\text{coefficient of } r^{\text{th}} \text{ term}} = \frac{1}{3}$$

$$\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n!}{(r-2)!(n-r+2)!}}{\frac{n!}{(r-1)!(n-r+1)!}} = \frac{1}{3}$$

On rearranging we get

$$\frac{n!}{(r-2)!(n-r+2)!} \times \frac{(r-1)!(n-r+1)!}{n!} = \frac{1}{3}$$

By multiplying

$$\Rightarrow \frac{(r-1)(r-2)!(n-r+1)!}{(r-2)!(n-r+2)!} = \frac{1}{3}$$

$$\Rightarrow \frac{(r-1)(n-r+1)!}{(n-r+2)(n-r+1)!} = \frac{1}{3}$$

On simplifying we get

$$\Rightarrow \frac{(r-1)}{(n-r+2)} = \frac{1}{3}$$

⇒ 3r - 3 = n – r + 2

⇒ n - 4r + 5 =0…………1

Also

$$\frac{\text{the coefficient of } r^{th} \text{ term}}{\text{coefficient of } r+1^{th} \text{ term}} = \frac{3}{5}$$

$$\Rightarrow \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{3}{5}$$

On rearranging we get

$$\Rightarrow \frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{3}{5}$$

By multiplying

$$\Rightarrow \frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)!} = \frac{3}{5}$$

$$\Rightarrow \frac{r(n-r)!}{(n-r+1)!} = \frac{3}{5}$$

$$\Rightarrow \frac{r(n-r)!}{(n-r+1)(n-r)!} = \frac{3}{5}$$

On simplifying we get

$$\Rightarrow \frac{r}{(n-r+1)} = \frac{3}{5}$$

Also

$$\frac{\text{the coefficient of } r^{th} \text{ term}}{\text{coefficient of } r+1^{th} \text{ term}} = \frac{3}{5}$$

$$\Rightarrow \frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{3}{5}$$

On rearranging we get

⇒ 5r = 3n - 3r + 3

$\Rightarrow$ 8r – 3n - 3 =0…………2
We have 1 and 2 as
n - 4r ± 5 =0………..1
8r – 3n - 3 =0…………….2
Multiplying equation 1 by number 2
2n -8r +10 =0………………3
Adding equation 2 and 3
2n -8r +10 =0
-3n – 8r - 3 =0
$\Rightarrow$ -n = -7
n =7 and r = 3

11. Prove that the coefficient of x^n in the expansion of $(1 + x)^{2n}$ is twice the coefficient of x^n in the expansion of $(1 + x)^{2n-1}$.

Solution:
The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$
The general term for binomial $(1+x)^{2n}$ is
$T_{r+1} = {}^{2n}C_r\, x^r$ …………………..1
To find the coefficient of x^n
r = n
$T_{n+1} = {}^{2n}C_n\, x^n$
The coefficient of $x^n = {}^{2n}C_n$
The general term for binomial $(1+x)^{2n-1}$ is
$T_{r+1} = {}^{2n-1}C_r\, x^r$
To find the coefficient of x^n
Putting n = r
$T_{r+1} = {}^{2n-1}C_r\, x^n$
The coefficient of $x^n = {}^{2n-1}C_n$
We have to prove
Coefficient of x^n in $(1+x)^{2n}$ = 2 coefficient of x^n in $(1+x)^{2n-1}$
Consider LHS = ${}^{2n}C_n$

$$= \frac{2n!}{n!\,(2n-n)!}$$

$$= \frac{2n!}{n!\,(n)!}$$

Again consider RHS = $2 \times {}^{2n-1}C_n$

$$= 2 \times \frac{(2n-1)!}{n!\,(2n-1-n)!}$$

$$= 2 \times \frac{(2n-1)!}{n!\,(n-1)!}$$

Now multiplying and dividing by n we get

$$= 2 \times \frac{(2n-1)!}{n!\,(n-1)!} \times \frac{n}{n}$$

$$= \frac{2n(2n-1)!}{n!\,n(n-1)!}$$

$$= \frac{2n!}{n!\,n!}$$

From above equations LHS = RHS

Hence the proof.

12. Find a positive value of m for which the coefficient of x^2 in the expansion $(1 + x)^m$ is 6.

Solution:

The general term T_{r+1} in the binomial expansion is given by $T_{r+1} = {}^nC_r\, a^{n-r}\, b^r$

Here a = 1, b = x and n = m

Putting the value

$T_{r+1} = {}^mC_r\, 1^{m-r}\, x^r$

$= {}^mC_r\, x^r$

We need coefficient of x^2

∴ putting r = 2

$T_{2+1} = {}^mC_2\, x^2$

The coefficient of $x^2 = {}^mC_2$

Given that coefficient of $x^2 = {}^mC_2 = 6$

$\Rightarrow \frac{m!}{2!(m-2)!} = 6$

$\Rightarrow \frac{m(m-1)(m-2)!}{2\times1\times(m-2)!} = 6$

$\Rightarrow m\,(m - 1) = 12$

$\Rightarrow m^2 - m - 12 = 0$

$\Rightarrow m^2 - 4m + 3m - 12 = 0$

$\Rightarrow m\,(m - 4) + 3\,(m - 4) = 0$

$\Rightarrow (m+3)\,(m - 4) = 0$

$\Rightarrow m = -3, 4$

We need positive value of m so m = 4

MISCELLANEOUS EXERCISE

1. Find a, b and n in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Solution:

We know that $(r + 1)^{th}$ term, (T_{r+1}), in the binomial expansion of $(a + b)^n$ is given by

$T_{r+1} = {}^nC_r\, a^{n-t}\, b^r$

The first three terms of the expansion are given as 729, 7290 and 30375 respectively.

Then we have,

$T_1 = {}^nC_0\, a^{n-0}\, b^0 = a^n = 729$….. 1

$T_2 = {}^nC_1\, a^{n-1}\, b^1 = na^{n-1}\, b = 7290$…. 2

$T_3 = {}^nC_2\, a^{n-2}\, b^2 = n\,(n -1)/2\, a^{n-2}\, b^2 = 30375$……3

Dividing 2 by 1 we get

$$\mathrm{na^{n-1}ba^{n}} = \frac{7290}{729}$$

n b a = 10 ….. 4

Dividing 3 by 2 we get

$$n(n-1)a^{n-2}b^2 2na^{n-1}b = \frac{30375}{7290}$$

$$\Rightarrow (n-1)b2a = \frac{30375}{7290}$$

$$\Rightarrow (n-1)ba = \frac{30375 \times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow \mathrm{nba} - \frac{\mathrm{b}}{\mathrm{a}} = \frac{25}{3}$$

$$\Rightarrow 10 - \mathrm{ba} = \frac{25}{3}$$

$$\Rightarrow \mathrm{ba} = 10 - \frac{25}{3} = \frac{5}{3} \qquad \text{……. 5}$$

From 4 and 5 we have

n. 5/3 = 10

n = 6

Substituting n = 6 in 1 we get

$a^6 = 729$

a = 3

From 5 we have, b/3 = 5/3

b = 5

Thus a = 3, b = 5 and n = 76

2. Find a if the coefficients of x^2 and x^3 in the expansion of $(3 + a x)^9$ are equal.

Solution:

We know that general term of expansion $(a + b)^n$ is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

For $(3+ax)^9$

Putting a = 3, b = a x & n = 9

General term of $(3+ax)^9$ is

$$T_{r+1} = \binom{9}{r} 3^{n-r} (ax)^r$$

$$T_{r+1} = \binom{9}{r} 3^{n-r} a^r x^r$$

Since we need to find the coefficients of x^2 and x^3, therefore

For r = 2

$$T_{2+1} = \binom{9}{2} 3^{n-2} a^2 x^2$$

Thus, the coefficient of x^2 = $\binom{9}{2} 3^{n-2} a^2$

For r = 3

$$T_{3+1} = \binom{9}{3} 3^{n-3} a^3 x^3$$

Thus, the coefficient of x^3 = $\binom{9}{3} 3^{n-3} a^3$

Given that coefficient of x^2 = Coefficient of x^3

$$\Rightarrow \binom{9}{2} 3^{n-2} a^2 = \binom{9}{3} 3^{n-3} a^3$$

$$\Rightarrow \frac{9!}{2!\,(9-2)!} \times 3^{n-2}a^2 = \frac{9!}{3!\,(9-3)!} \times 3^{n-3}a^3$$

$$\Rightarrow \frac{3^{n-2}a^2}{3^{n-3}a^3} = \frac{2!\,(9-2)!}{3!\,(9-3)!}$$

$$\Rightarrow \frac{3^{(n-2)-(n-3)}}{a} = \frac{2!\,7!}{3!\,6!}$$

$$\Rightarrow \frac{3}{a} = \frac{7}{3}$$

∴ a = 9/7

Hence, a = 9/7

3. Find the coefficient of x^5 in the product $(1 + 2x)^6 (1 – x)^7$ using binomial theorem.

Solution:

$(1 + 2x)^6 = {}^6C_0 + {}^6C_1 (2x) + {}^6C_2 (2x)^2 + {}^6C_3 (2x)^3 + {}^6C_4 (2x)^4 + {}^6C_5 (2x)^5 + {}^6C_6 (2x)^6$
$= 1 + 6 (2x) + 15 (2x)^2 + 20 (2x)^3 + 15 (2x)^4 + 6 (2x)^5 + (2x)^6$
$= 1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6$
$(1 – x)^7 = {}^7C_0 - {}^7C_1 (x) + {}^7C_2 (x)^2 - {}^7C_3 (x)^3 + {}^7C_4 (x)^4 - {}^7C_5 (x)^5 + {}^7C_6 (x)^6 - {}^7C_7 (x)^7$
$= 1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7$
$(1 + 2x)^6 (1 – x)^7 = (1 + 12 x + 60x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64x^6) (1 – 7x + 21x^2 – 35x^3 + 35x^4 – 21x^5 + 7x^6 – x^7)$
192 – 21 = 171
Thus, the coefficient of x^5 in the expression $(1+2x)^6(1-x)7$ is 171.

4. If a and b are distinct integers, prove that a – b is a factor of $a^n – b^n$, whenever n is a positive integer. [Hint write $a^n = (a – b + b)^n$ and expand]

Solution:

In order to prove that (a – b) is a factor of $(a^n – b^n)$, it has to be proved that
$a^n - b^n = k (a – b)$ where k is some natural number.
a can be written as a = a – b + b
$a^n = (a – b + b)^n = [(a – b) + b]^n$
$= {}^nC_0 (a – b)^n + {}^nC_1 (a – b)^{n-1} b + …… + {}^nC_n b^n$
$a^n – b^n = (a – b) [(a –b)^{n-1} + {}^nC_1 (a – b)^{n-1} b + …… + {}^nC_n b^n]$

$a^n - b^n = (a - b)\, k$
Where $k = [(a - b)^{n-1} + {}^nC_1 (a - b)^{n-1}\, b + \ldots\ldots + {}^nC_n\, b^n]$ is a natural number
This shows that $(a - b)$ is a factor of $(a^n - b^n)$, where n is positive integer.

5. Evaluate

$$(\sqrt{3}+\sqrt{2})^6 - (\sqrt{3}-\sqrt{2})^6$$

Solution:

Using binomial theorem the expression $(a + b)^6$ and $(a - b)^6$, can be expanded
$(a + b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5 b + {}^6C_2\, a^4 b^2 + {}^6C_3\, a^3 b^3 + {}^6C_4\, a^2 b^4 + {}^6C_5\, a b^5 + {}^6C_6\, b^6$
$(a - b)^6 = {}^6C_0\, a^6 - {}^6C_1\, a^5 b + {}^6C_2\, a^4 b^2 - {}^6C_3\, a^3 b^3 + {}^6C_4\, a^2 b^4 - {}^6C_5\, a b^5 + {}^6C_6\, b^6$
Now $(a + b)^6 - (a - b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5 b + {}^6C_2\, a^4 b^2 + {}^6C_3\, a^3 b^3 + {}^6C_4\, a^2 b^4 + {}^6C_5\, a b^5 + {}^6C_6\, b^6$
$- [{}^6C_0\, a^6 - {}^6C_1\, a^5 b + {}^6C_2\, a^4 b^2 - {}^6C_3\, a^3 b^3 + {}^6C_4\, a^2 b^4 - {}^6C_5\, a b^5 + {}^6C_6\, b^6]$
Now by substituting $a = \sqrt{3}$ and $b = \sqrt{2}$ we get
$(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2\,[6\,(\sqrt{3})^5\,(\sqrt{2}) + 20\,(\sqrt{3})^3\,(\sqrt{2})^3 + 6\,(\sqrt{3})\,(\sqrt{2})^5]$
$= 2\,[54(\sqrt{6}) + 120\,(\sqrt{6}) + 24\sqrt{6}]$
$= 2\,(\sqrt{6})\,(198)$
$= 396\sqrt{6}$

6. Find the value of

$$\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$$

Solution:

Firstly the expression $(x + y)^4 + (x - y)^4$ is simplified by using binomial theorem

$$(x+y)^4 = {}^4C_0x^4 + {}^4C_1x^3y + {}^4C_2x^2y^2 + {}^+C_3xy^3 + {}^4C_4y^4$$

$$= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(x-y)^4 = {}^4C_0x^4 - {}^4C_1x^3y + {}^4C_2x^2y^2 - {}^4C_3xy^3 + {}^4C_4y^4$$

$$= x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

$$\therefore (x+y)^4 + (x-y)^4 = 2\left(x^4 + 6x^2y^2 + y^4\right)$$

Putting $x = a^2$ and $y = \sqrt{a^2-1}$, we obtain

$$\left(a^2+\sqrt{a^2-1}\right)^4+\left(a^2-\sqrt{a^2-1}\right)^4$$
$$=2\left[(a^2)^4+6(a^2)^2\left(\sqrt{a^2-1}\right)^2+\left(\sqrt{a^2-1}\right)^4\right]$$
$$=2\left[a^8+6a^4\left(a^2-1\right)+\left(a^2-1\right)^2\right]$$
$$=2\left[a^8+6a^6-6a^4+a^4-2a^2+1\right]$$
$$=2\left[a^8+6a^6-5a^4-2a^2+1\right]$$
$$=2a^8+12a^6-10a^4-4a^2+2$$

7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution:
0.99 can be written as
0.99 = 1 – 0.01
Now by applying binomial theorem we get
$(o.\ 99)^5 = (1 - 0.01)^5$
$= {}^5C_0\,(1)^5 - {}^5C_1\,(1)^4\,(0.01) + {}^5C_2\,(1)^3\,(0.01)^2$
$= 1 - 5\,(0.01) + 10\,(0.01)^2$
$= 1 - 0.05 + 0.001$
$= 0.951$

8. Find n, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $\left(\sqrt[4]{2}+\frac{1}{\sqrt[4]{3}}\right)^n$ is √6: 1

Solution:
In the expansion $(a + b)^n$, if n is even then the middle term is $(n/2 + 1)^{th}$ term

$${}^nC_4(\sqrt[4]{2})^{n-1}\left(\frac{1}{\sqrt[4]{3}}\right)^4={}^nC_4\frac{(\sqrt[4]{2})^n}{(\sqrt[4]{2})^4}\cdot\frac{1}{3}={}^nC_4\frac{(\sqrt[4]{2})^n}{2}\cdot\frac{1}{3}=\frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^n$$

$${}^nC_{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4}={}^nC_{n-1}\cdot 2\cdot\frac{(\sqrt[4]{3})^4}{(\sqrt[4]{3})^n}={}^nC_{n-1}\cdot 2\cdot\frac{3}{(\sqrt[4]{3})^n}=\frac{6n!}{(n-4)!4!}\cdot\frac{1}{(\sqrt[4]{3})^n}$$

$$\frac{n!}{6.4!(n-4)!}(\sqrt[4]{2})^n : \frac{6n!}{(n-4)!!4!} \cdot \frac{1}{(\sqrt[4]{3})^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} : \frac{6}{(\sqrt[4]{3})^n} = \sqrt{6} : 1$$

$$\Rightarrow \frac{(\sqrt[4]{2})^n}{6} \times \frac{(\sqrt[4]{3})^n}{6} = \sqrt{6}$$

$$\Rightarrow (\sqrt[4]{6})^n = 36\sqrt{6}$$

$$\Rightarrow 6^{\frac{n}{4}} = 6^{\frac{5}{2}}$$

$$\Rightarrow \frac{n}{4} = \frac{5}{2}$$

$$\Rightarrow n = 4 \times \frac{5}{2} = 10$$

Thus the value of n = 10

9. Expand using Binomial Theorem

$$\left(1+\frac{x}{2}-\frac{2}{x}\right)^4, x \neq 0$$

Solution:

Using binomial theorem the given expression can be expanded as

$$\left[\left(1+\frac{x}{2}\right)-\frac{2}{x}\right]^4$$

$$= {}^4C_0\left(1+\frac{x}{2}\right)^4 - {}^4C_1\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + {}^4C_2\left(1+\frac{x}{2}\right)^2\left(\frac{2}{x}\right)^2 - {}^4C_3\left(1+\frac{x}{2}\right)\left(\frac{2}{x}\right)^3 + {}^4C_4\left(\frac{2}{x}\right)^4$$

$$= \left(1+\frac{x}{2}\right)^4 - 4\left(1+\frac{x}{2}\right)^3\left(\frac{2}{x}\right) + 6\left(1+x+\frac{x^2}{4}\right)\left(\frac{4}{x^2}\right) - 4\left(1+\frac{x}{2}\right)\left(\frac{8}{x^3}\right) + \frac{16}{x^4}$$

$$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{24}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} - \frac{16}{x^2} + \frac{16}{x^4}$$

$$= \left(1+\frac{x}{2}\right)^4 - \frac{8}{x}\left(1+\frac{x}{2}\right)^3 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4} \qquad \ldots(1)$$

Again by using binomial theorem to expand the above terms we get

$$\left(1+\frac{x}{2}\right)^4 = {}^4C_0(1)^4 + {}^4C_1(1)^3\left(\frac{x}{2}\right) + {}^4C_2(1)^2\left(\frac{x}{2}\right)^2 + {}^4C_3(1)^1\left(\frac{x}{2}\right)^3 + {}^4C_4\left(\frac{x}{2}\right)^4$$

$$= 1 + 4\times\frac{x}{2} + 6\times\frac{x^2}{4} + 4\times\frac{x^3}{8} + \frac{x^4}{16}$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} \qquad ...(2)$$

$$\left(1+\frac{x}{2}\right)^3 = {}^3C_0(1)^3 + {}^3C_1(1)^2\left(\frac{x}{2}\right) + {}^3C_2(1)\left(\frac{x}{2}\right)^2 + {}^3C_3\left(\frac{x}{2}\right)^3$$

$$= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \qquad ...(3)$$

From equation 1, 2 and 3 we get

$$\left[\left(1+\frac{x}{2}\right) - \frac{2}{x}\right]^4$$

$$= 1 + 2x + \frac{3x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x}\left(1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}\right) + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= 1 + 2x + \frac{3}{2}x^2 + \frac{x^3}{2} + \frac{x^4}{16} - \frac{8}{x} - 12 - 6x - x^2 + \frac{8}{x^2} + \frac{24}{x} + 6 - \frac{32}{x^3} + \frac{16}{x^4}$$

$$= \frac{16}{x} + \frac{8}{x^2} - \frac{32}{x^3} + \frac{16}{x^4} - 4x + \frac{x^2}{2} + \frac{x^3}{2} + \frac{x^4}{16} - 5$$

10. Find the expansion of $(3x^2 – 2ax + 3a^2)^3$ using binomial theorem.

Solution:

We know that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

Putting $a = 3x^2$ & $b = -a (2x-3a)$, we get

$[3x^2 + (-a (2x-3a))]^3$

$= (3x^2)^3+3(3x^2)^2(-a (2x-3a)) + 3(3x^2) (-a (2x-3a))^2 + (-a (2x-3a))^3$

$= 27x^6 - 27ax^4 (2x-3a) + 9a^2x^2 (2x-3a)^2 - a^3(2x-3a)^3$

$= 27x^6 - 54ax^5 + 81a^2x^4 + 9a^2x^2 (4x^2-12ax+9a^2) - a^3 [(2x)^3 - (3a)^3 - 3(2x)^2(3a) + 3(2x)(3a)^2]$

$= 27x^6 - 54ax^5 + 81a^2x^4 + 36a^2x^4 - 108a^3x^3 + 81a^4x^2 - 8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x$

$= 27x^6 - 54ax^5+ 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$

Thus, $(3x^2 – 2ax + 3a^2)^3$

$= 27x^6 - 54ax^5+ 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6$