Expand each of the expressions in Exercises 1 to 5.

1. \((1 - 2x)^5\)

Solution:
From binomial theorem expansion we can write as
\[
(1 - 2x)^5 = 5C_0 (1)^5 - 5C_1 (1)^4 (2x) + 5C_2 (1)^3 (2x)^2 - 5C_3 (1)^2 (2x)^3 + 5C_4 (1)^1 (2x)^4 - 5C_5 (2x)^5
\]
\[
= 1 - 5(2x) + 10(4x^2) - 10(8x^3) + 5(16x^4) - 32x^5
\]
\[
= 1 - 10x + 40x^2 - 80x^3 - 32x^5
\]

2. \(\left(\frac{2}{x} - \frac{x}{2}\right)^5\)

Solution:
From binomial theorem, given equation can be expanded as
\[
\left(\frac{2}{x} - \frac{x}{2}\right)^5 = 5C_0 \left(\frac{2}{x}\right)^5 - 5C_1 \left(\frac{2}{x}\right)^4 (\frac{x}{2}) + 5C_2 \left(\frac{2}{x}\right)^3 (\frac{x}{2})^2 - 5C_3 \left(\frac{2}{x}\right)^2 (\frac{x}{2})^3 + 5C_4 \left(\frac{2}{x}\right) (\frac{x}{2})^4 - 5C_5 (\frac{x}{2})^5
\]
\[
= \frac{32}{x^5} - 5 \left(\frac{16}{x^2}\right) \left(\frac{x}{2}\right) + 10 \left(\frac{8}{x}\right) \left(\frac{x^2}{4}\right) - 10 \left(\frac{4}{x^3}\right) + 5 \left(\frac{2}{x}\right) \left(\frac{x^4}{16}\right) - \frac{x^5}{32}
\]
\[
= \frac{32}{x^5} - \frac{40}{x^3} + \frac{20}{x} - 5x + \frac{5}{8}x^3 - \frac{x^5}{32}
\]

3. \((2x - 3)^6\)

Solution:
From binomial theorem, given equation can be expanded as
\[
(2x - 3)^6 = 6C_0 (2x)^6 - 6C_1 (2x)^5 (3) + 6C_2 (2x)^4 (3)^2 - 4C_3 (2x)^3 (3)^3
\]
\[
= 64x^6 - 6 \cdot 32x^5 \cdot 3 + 15 \cdot 16x^4 \cdot 9 - 20 \cdot 8x^3 \cdot 27
\]
\[
+ 15 \cdot 4x^2 \cdot (81) - 6 \cdot 2x \cdot (243) + 729
\]
\[
= 64x^6 - 576x^5 + 2160x^4 - 4320x^3 + 4860x^2 - 2916x + 729
\]
4. \( \left( \frac{x}{3} + \frac{1}{x} \right)^5 \)

Solution:
From binomial theorem, given equation can be expanded as
\[
\left( \frac{x}{3} + \frac{1}{x} \right)^5 = C_0 \left( \frac{x}{3} \right)^5 + 5 C_1 \left( \frac{x}{3} \right)^4 \left( \frac{1}{x} \right) + 10 C_2 \left( \frac{x}{3} \right)^3 \left( \frac{1}{x} \right)^2 + 10 C_3 \left( \frac{x}{3} \right)^2 \left( \frac{1}{x} \right)^3 + 5 C_4 \left( \frac{x}{3} \right) \left( \frac{1}{x} \right)^4 + \left( \frac{1}{x} \right)^5
\]
\[
= \frac{x^5}{243} + 5 \left( \frac{x^4}{81} \right) \left( \frac{1}{x} \right) + 10 \left( \frac{x^3}{27} \right) \left( \frac{1}{x^2} \right) + 10 \left( \frac{x^2}{9} \right) \left( \frac{1}{x^3} \right) + 5 \left( \frac{x}{3} \right) \left( \frac{1}{x^4} \right) + \frac{1}{x^5}
\]
\[
= \frac{x^5}{243} + \frac{5x^3}{81} + \frac{10x}{27} + \frac{10}{9x} + \frac{5}{3x^3} + \frac{1}{x^5}
\]

5. \( (x + \frac{1}{x})^6 \)

Solution:
From binomial theorem, given equation can be expanded as
\[
(x + \frac{1}{x})^6 = C_0 (x)^6 + C_1 (x)^5 \left( \frac{1}{x} \right) + C_2 (x)^4 \left( \frac{1}{x} \right)^2 + C_3 (x)^3 \left( \frac{1}{x} \right)^3 + C_4 (x)^2 \left( \frac{1}{x} \right)^4 + C_5 (x)^1 \left( \frac{1}{x} \right)^5 + C_6 (\frac{1}{x})^6
\]
\[
= x^6 + 6(x)^5 \left( \frac{1}{x} \right) + 15(x)^4 \left( \frac{1}{x^2} \right) + 20(x)^3 \left( \frac{1}{x^3} \right) + 15(x)^2 \left( \frac{1}{x^4} \right) + 6(x) \left( \frac{1}{x^5} \right) + \frac{1}{x^6}
\]
\[
= x^6 + 6x^4 + 15x^2 + 20 + \frac{15}{x^2} + \frac{6}{x^4} + \frac{1}{x^6}
\]

6. \( (96)^3 \)

Solution:
Given \((96)^3\)

96 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as \(96 = 100 - 4\)

\((96)^3 = (100 - 4)^3\)

\[
= ^3C_0 (100)^3 - ^3C_1 (100)^2 (4) - ^3C_2 (100) (4)^2 - ^3C_3 (4)^3
\]
\[
= (100)^3 - 3 (100)^2 (4) + 3 (100) (4)^2 - (4)^3
\]
\[
= 1000000 - 120000 + 4800 - 64
\]

https://byjus.com
7. \((102)^5\)

**Solution:**
Given \((102)^5\)

102 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 102 = 100 + 2

\((102)^5 = (100 + 2)^5\)

\[= \binom{5}{0} (100)^5 + \binom{5}{1} (100)^4 (2) + \binom{5}{2} (100)^3 (2)^2 + \binom{5}{3} (100)^2 (2)^3 + \binom{5}{4} (100)(2)^4 + \binom{5}{5} (2)^5\]

\[= (100)^5 + 5 (100)^4 (2) + 10 (100)^3 (2)^2 + 5 (100)^2 (2)^3 + 5 (100) (2)^4 + (2)^5\]

\[= 1000000000 + 1000000000 + 40000000 + 80000 + 8000 + 32\]

\[= 11040808032\]

8. \((101)^4\)

**Solution:**
Given \((101)^4\)

101 can be expressed as the sum or difference of two numbers and then binomial theorem can be applied.

The given question can be written as 101 = 100 + 1

\((101)^4 = (100 + 1)^4\)

\[= \binom{4}{0} (100)^4 + \binom{4}{1} (100)^3 (1) + \binom{4}{2} (100)^2 (1)^2 + \binom{4}{3} (100)(1)^3 + \binom{4}{4} (1)^4\]

\[= (100)^4 + 4 (100)^3 + 6 (100)^2 + 4 (100) + (1)^4\]

\[= 100000000 + 400000 + 60000 + 400 + 1\]

\[= 1040604001\]

9. \((99)^5\)

**Solution:**
Given \((99)^5\)

99 can be written as the sum or difference of two numbers then binomial theorem can be applied.

The given question can be written as 99 = 100 - 1

\((99)^5 = (100 - 1)^5\)

\[= \binom{5}{0} (100)^5 - \binom{5}{1} (100)^4 (1) + \binom{5}{2} (100)^3 (1)^2 - \binom{5}{3} (100)^2 (1)^3 + \binom{5}{4} (100)(1)^4 - \binom{5}{5} (1)^5\]

\[= 884736\]
\[
\begin{align*}
&= (100)^5 - 5 (100)^4 + 10 (100)^3 - 10 (100)^2 + 5 (100) - 1 \\
&= 1000000000 - 5000000000 + 10000000 - 100000 + 500 - 1 \\
&= 9509900499
\end{align*}
\]

10. Using Binomial Theorem, indicate which number is larger \((1.1)^{10000}\) or 1000.

**Solution:**

By splitting the given 1.1 and then applying binomial theorem, the first few terms of \((1.1)^{10000}\) can be obtained as

\[
(1.1)^{10000} = (1 + 0.1)^{10000} = (1 + 0.1)^{10000} C_1 (1.1) + \text{other positive terms}
\]

\[
= 1 + 10000 \times 1.1 + \text{other positive terms}
\]

\[
= 1 + 11000 + \text{other positive terms}
\]

\[
> 1000
\]

\[
(1.1)^{10000} > 1000
\]

11. Find \((a + b)^4 - (a - b)^4\). Hence, evaluate

\[
\left(\sqrt{3} + \sqrt{2}\right)^4 - \left(\sqrt{3} - \sqrt{2}\right)^4
\]

**Solution:**

Using binomial theorem, the expression \((a + b)^4\) and \((a - b)^4\), can be expanded

\[
(a + b)^4 = \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4
\]

\[
(a - b)^4 = \binom{4}{0} a^4 - \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 - \binom{4}{3} a b^3 + \binom{4}{4} b^4
\]

Now \((a + b)^4 - (a - b)^4 = \binom{4}{0} a^4 + \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 + \binom{4}{3} a b^3 + \binom{4}{4} b^4 - [\binom{4}{0} a^4 - \binom{4}{1} a^3 b + \binom{4}{2} a^2 b^2 - \binom{4}{3} a b^3 + \binom{4}{4} b^4]
\]

\[
= 2 \left(\binom{4}{1} a^3 b + \binom{4}{3} a b^3\right)
\]

\[
= 8ab (a^2 + b^2)
\]

Now by substituting \(a = \sqrt{3}\) and \(b = \sqrt{2}\) we get

\[
(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4 = 8 (\sqrt{3}) (\sqrt{2}) ((\sqrt{3})^2 + (\sqrt{2})^2)
\]

\[
= 8 (\sqrt{6}) (3 + 2)
\]

\[
= 40 \sqrt{6}
\]

12. Find \((x + 1)^6 + (x - 1)^6\). Hence or otherwise evaluate

\[
(\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6
\]
Solution:
Using binomial theorem the expressions, \((x + 1)^6\) and \((x - 1)^6\) can be expressed as
\[
\begin{align*}
(x + 1)^6 &= 6C_0 x^6 + 6C_1 x^5 + 6C_2 x^4 + 6C_3 x^3 + 6C_4 x^2 + 6C_5 x + 6C_6 \\
(x - 1)^6 &= 6C_0 x^6 - 6C_1 x^5 + 6C_2 x^4 - 6C_3 x^3 + 6C_4 x^2 - 6C_5 x + 6C_6
\end{align*}
\]
Now, \((x + 1)^6 - (x - 1)^6 = 6C_0 x^6 + 6C_1 x^5 + 6C_2 x^4 + 6C_3 x^3 + 6C_4 x^2 + 6C_5 x + 6C_6 - [6C_0 x^6 - 6C_1 x^5 + 6C_2 x^4 - 6C_3 x^3 + 6C_4 x^2 - 6C_5 x + 6C_6] = 2 [6C_0 x^6 + 6C_2 x^4 + 6C_4 x^2 + 6C_6]
\]
\[
= 2 [x^6 + 15x^4 + 15x^2 + 1]
\]
Now by substituting \(x = \sqrt{2}\) we get
\[
(\sqrt{2} + 1)^6 - (\sqrt{2} - 1)^6 = 2 [(\sqrt{2})^6 + 15(\sqrt{2})^4 + 15(\sqrt{2})^2 + 1] = 2 (8 + 60 + 30 + 1) = 2 (99) = 198
\]

13. Show that \(9^{n+1} - 8n - 9\) is divisible by 64, whenever \(n\) is a positive integer.

Solution:
In order to show that \(9^{n+1} - 8n - 9\) is divisible by 64, it has to be show that \(9^{n+1} - 8n - 9 = 64 k\), where \(k\) is some natural number
Using binomial theorem,
\[
(1 + a)^m = mC_0 a^0 + mC_1 a + mC_2 a^2 + \ldots + mC_m a^m
\]
For \(a = 8\) and \(m = n + 1\) we get
\[
(1 + 8)^{n+1} = n+1C_0 + n+1C_1 (8) + n+1C_2 (8)^2 + \ldots + n+1C_n (8)^n
\]
\[
9^{n+1} = 1 + (n + 1) 8 + 8^2 [n+1C_2 + n+1C_3 (8) + \ldots + n+1C_n (8)^{n-1}]
\]
\[
9^{n+1} = 9 + 8n + 64 [n+1C_2 + n+1C_3 (8) + \ldots + n+1C_n (8)^{n-1}]
\]
\[
9^{n+1} - 8n - 9 = 64 k
\]
Where \(k = [n+1C_2 + n+1C_3 (8) + \ldots + n+1C_n (8)^{n-1}]\) is a natural number
Thus, \(9^{n+1} - 8n - 9\) is divisible by 64, whenever \(n\) is positive integer.
Hence the proof

14. Prove that
\[
\sum_{r=0}^{n} 3^r n C_r = 4^n
\]
Solution:

By Binomial Theorem

\[ \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r = (a + b)^n \]

On the right side, we need 4\(^n\) so we will put the values as,

Putting b = 3 & a = 1 in the above equation, we get

\[ \sum_{r=0}^{n} \binom{n}{r} (1)^{n-r} (3)^r = (1 + 3)^n \]

\[ \sum_{r=0}^{n} \binom{n}{r} (1)(3)^r = (4)^n \]

\[ \sum_{r=0}^{n} \binom{n}{r} (3)^r = (4)^n \]

Hence Proved.
Find the coefficient of

1. $x^5$ in $(x + 3)^8$

Solution:
The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $x^5$ is the $T_{r+1}$ term so $a=x$, $b=3$ and $n=8$

$T_{r+1} = \binom{8}{r} x^{8-r} 3^r$............... (i)

For finding out $x^5$

We have to equate $x^5 = x^{8-r}$

$\Rightarrow r=3$

Putting value of $r$ in (i) we get

$T_4 = \binom{8}{3} x^3 3^3$

$= \frac{8!}{3! 5!} \times x^5 \times 27$

$= 1512 x^5$

Hence the coefficient of $x^5 = 1512$

2. $a^5b^7$ in $(a - 2b)^{12}$ .

Solution:
The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $a=a$, $b=-2b$ & $n=12$

Substituting the values, we get

$T_{r+1} = \binom{12}{r} a^{12-r} (-2b)^r$ ............ (i)

To find $a^5$

We equate $a^{12-r} = a^5$

$r = 7$

Putting $r = 7$ in (i)

$T_8 = \binom{12}{7} a^5 (-2b)^7$

$T_8 = \frac{12!}{7! 5!} \times a^5 \times (-2)^7 b^7$

$= -101376 a^5 b^7$

Hence the coefficient of $a^5b^7 = -101376$
Write the general term in the expansion of
3. \((x^2 - y)^6\)

Solution:
The general term \(T_{r+1}\) in the binomial expansion is given by
\[
T_{r+1} = \binom{n}{r} a^{n-r} b^r \quad \text{......... (i)}
\]
Here \(a = x^2, \ n = 6\) and \(b = -y\)
Putting values in (i)
\[
T_{r+1} = 6\binom{r}{r} x^{2(6-r)} (-y)^r = -y^r
\]

4. \((x^2 - y x)^{12}, x \neq 0\).

Solution:
The general term \(T_{r+1}\) in the binomial expansion is given by \(T_{r+1} = \binom{n}{r} a^{n-r} b^r\)
Here \(n = 12, a = x^2\) and \(b = -y x\)
Substituting the values we get
\[
T_{n+1} = 12\binom{12}{r} x^{2(12-r)} (-y x)^r
\]
\[
= -y^r
\]

5. Find the 4th term in the expansion of \((x - 2y)^{12}\).

Solution:
The general term \(T_{r+1}\) in the binomial expansion is given by \(T_{r+1} = \binom{n}{r} a^{n-r} b^r\)
Here \(a = x, \ n = 12, r = 3\) and \(b = -2y\)
By substituting the values we get
\[
T_4 = 12\binom{3}{3} x^9 (-2y)^3
\]
\[
\frac{12!}{3!9!} \times x^9 \times -8 \times y^3
= -\frac{12 \times 11 \times 10 \times 8}{3 \times 2 \times 1} \times x^9 y^3
= -1760 x^9 y^3
\]

6. Find the 13th term in the expansion of
\[
\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}, x \neq 0
\]

Solution:
The general term \(T_{r+1}\) in the binomial expansion is given by
\[
T_{r+1} = \binom{n}{r} a^{n-r} b^r
\]
Here \(a = 9x, b = -\frac{1}{3\sqrt{x}}, n = 18\) and \(r = 12\)

Putting values
\[
T_{13} = \frac{18!}{12! \times 6!} \times 9x^{18-12} \left(-\frac{1}{3\sqrt{x}}\right)^{12}
= \frac{(18 \times 17 \times 16 \times 15 \times 14 \times 13 \times 12!)}{12! \times 6 \times 5 \times 4 \times 3 \times 2 \times 1} \times 3^{12} \times x^{6} \times \frac{1}{x^{6}} \times \frac{1}{3^{12}}
= 18564
\]

Find the middle terms in the expansions of
7. \(\left(3 - \frac{x^3}{6}\right)^7\)

Solution:
Here \(n = 7\) so there would be two middle terms given by
\[
\left(\frac{n + 1}{2}\right)\text{term} = 4^{th}\ \text{and} \ \left(\frac{n + 1}{2} + 1\right)\text{th\ term} = 5^{th}
\]
We have
\[
a = 3, n = 7 \text{ and } b = -\frac{x^3}{6}
\]
For $T_4$, $r = 3$

The term will be

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_4 = \frac{7!}{3!} 3^4 \left( \frac{x^3}{6} \right)^3$$

$$= \frac{7 \times 6 \times 5 \times 4}{3 \times 2 \times 1} \times 3^4 \times \frac{x^9}{2^3 \times 3^3}$$

$$= -\frac{105}{8} x^9$$

For $T_5$ term, $r = 4$

The term $T_{r+1}$ in the binomial expansion is given by

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

$$T_5 = \frac{7!}{4!3!} 3^3 \left( \frac{x^3}{6} \right)^4$$

$$= \frac{7 \times 6 \times 5 \times 4!}{4!3!} \times \frac{3^3}{2^4 \times 3^4} \times x^3 = \frac{35 x^{12}}{48}$$

8. $\left( \frac{x}{3} + 9y \right)^{10}$

**Solution:**

Here $n$ is even so the middle term will be given by $(\frac{n+1}{2})$th term = 6th term

The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Now $a = \frac{x}{3}, b = 9y, n = 10$ and $r = 5$

Substituting the values
9. In the expansion of \((1 + a)^{m+n}\), prove that coefficients of \(a^m\) and \(a^n\) are equal.

**Solution:**

We know that the general term \(T_{r+1}\) in the binomial expansion is given by 
\[T_{r+1} = \binom{n}{r} a^{n-r} b^r\]
Here \(n = m+n\), \(a = 1\) and \(b = a\)
Substituting the values in the general form
\[T_{r+1} = \binom{m+n}{r} 1^{m+n-r} a^r\]
\[= \binom{m+n}{r} a^r\] ............... (i)

Now we have that the general term for the expression is,
\[T_{r+1} = \binom{m+n}{r} a^r\]

Now, For coefficient of \(a^m\)
\[T_{m+1} = \binom{m+n}{m} a^m\]
Hence, for coefficient of \(a^m\), value of \(r = m\)
So, the coefficient is \(\binom{m+n}{m}\)

Similarly, Coefficient of \(a^n\) is \(\binom{m+n}{n}\)
\[\frac{(m+n)!}{m!n!}\]

And also, \(\binom{m+n}{n} = \frac{(m+n)!}{m!n!}\)

The coefficient of \(a^m\) and \(a^n\) are same that is \(\frac{(m+n)!}{m!n!}\)

10. The coefficients of the \((r - 1)\text{th}\), \(r\text{th}\) and \((r + 1)\text{th}\) terms in the expansion of \((x + 1)^n\) are in the ratio 1 : 3 : 5. Find \(n\) and \(r\).

**Solution:**

The general term \(T_{r+1}\) in the binomial expansion is given by 
\[T_{r+1} = \binom{n}{r} a^{n-r} b^r\]
Here the binomial is \((1+x)^n\) with \(a = 1\) , \(b = x\) and \(n = n\)
The \((r+1)\text{th}\) term is given by 
\[T_{(r+1)} = \binom{n}{r} 1^{n-r} x^r\]
\[ T_{(r+1)} = \binom{n}{r} x^{r} \]

The coefficient of \((r+1)^{th}\) term is \(\binom{n}{r}\)

The \(r^{th}\) term is given by \((r-1)^{th}\) term

\[ T_{r+1-1} = \binom{n}{r-1} x^{r-1} \]

\[ T_r = \binom{n}{r} x^{r-1} \]

\[ \therefore \text{ the coefficient of } r^{th} \text{ term is } \binom{n}{r-1} \]

For \((r-1)^{th}\) term we will take \((r-2)^{th}\) term

\[ T_{r-1-1} = \binom{n}{r-2} x^{r-2} \]

\[ T_{r-1} = \binom{n}{r-2} x^{r-2} \]

\[ \therefore \text{ the coefficient of } (r-1)^{th} \text{ term is } \binom{n}{r-2} \]

Given that the coefficient of \((r-1)^{th}\), \(r^{th}\) and \(r+1^{th}\) term are in ratio 1:3:5

Therefore,

\[
\frac{\text{coefficient of } r-1^{th} \text{ term}}{\text{coefficient of } r^{th} \text{ term}} = \frac{1}{3}
\]

\[
\frac{\binom{n}{r-2}}{\binom{n}{r-1}} = \frac{1}{3}
\]

\[
\Rightarrow \frac{n!}{(r-2)! (n-r+2)!} \times \frac{(r-1)! (n-r-1)!}{n!} = \frac{1}{3}
\]

On rearranging we get

\[
\frac{n!}{(r-2)! (n-r+2)!} \times \frac{(r-1)! (n-r+1)!}{n!} = \frac{1}{3}
\]

By multiplying

\[
\Rightarrow \frac{(r-1)(r-2)! (n-r+1)!}{(r-2)! (n-r+2)!} = \frac{1}{3}
\]

\[
\Rightarrow \frac{(r-1)(n-r+1)!}{(n-r+2)(n-r+1)!} = \frac{1}{3}
\]

On simplifying we get
\[
\frac{(r - 1)}{(n - r + 2)} = \frac{1}{3}
\]
\[
3r - 3 = n - r + 2
\]
\[
n - 4r + 5 = 0 \quad \text{.........1}
\]

Also

\[
\frac{\text{the coefficient of } r^{\text{th}} \text{ term}}{\text{coefficient of } (r + 1)^{\text{th}} \text{ term}} = \frac{3}{5}
\]

\[
\frac{n!}{(r-1)!(n-r+1)!} = \frac{3}{5}
\]

\[
\Rightarrow \frac{r!(n-r)!}{r!(n-r)!} = \frac{3}{5}
\]

On rearranging we get

\[
\frac{n!}{(r-1)!(n-r+1)!} \times \frac{r!(n-r)!}{n!} = \frac{3}{5}
\]

By multiplying

\[
\frac{r(r-1)!(n-r)!}{(r-1)!(n-r+1)!} = \frac{3}{5}
\]

\[
\Rightarrow \frac{r(n-r)!}{(n-r+1)!} = \frac{3}{5}
\]

\[
\Rightarrow \frac{r(n-r)!}{(n-r+1)(n-r)!} = \frac{3}{5}
\]

On simplifying we get

\[
\Rightarrow \frac{r}{(n-r+1)} = \frac{3}{5}
\]

Also

\[
\frac{\text{the coefficient of } r^{\text{th}} \text{ term}}{\text{coefficient of } (r + 1)^{\text{th}} \text{ term}} = \frac{3}{5}
\]

\[
\frac{n!}{(r-1)!(n-r+1)!} = \frac{3}{5}
\]

\[
\Rightarrow \frac{r!(n-r)!}{r!(n-r)!} = \frac{3}{5}
\]

On rearranging we get

\[
5r = 3n - 3r + 3
\]
\[ 8r - 3n - 3 = 0 \] ..........2

We have 1 and 2 as
\[ n - 4r \pm 5 = 0 \] ..........1
\[ 8r - 3n - 3 = 0 \] ..........2

Multiplying equation 1 by number 2
\[ 2n - 8r + 10 = 0 \] ..........3

Adding equation 2 and 3
\[ 2n - 8r + 10 = 0 \]
\[ -3n - 8r - 3 = 0 \]
\[ \Rightarrow -n = -7 \]
\[ n = 7 \] and \[ r = 3 \]

11. Prove that the coefficient of \( x^n \) in the expansion of \( (1 + x)^{2n} \) is twice the coefficient of \( x^n \) in the expansion of \( (1 + x)^{2n-1} \).

Solution:
The general term \( T_{r+1} \) in the binomial expansion is given by \( T_{r+1} = \binom{n}{r} a^{n-r} b^r \)
The general term for binomial \( (1+x)^{2n} \) is
\[ T_{r+1} = \binom{2n}{r} x^r \] ..........1
To find the coefficient of \( x^n \)
\[ r = n \]
\[ T_{n+1} = \binom{2n}{n} x^n \]
The coefficient of \( x^n \) = \( \binom{2n}{n} \)
The general term for binomial \( (1+x)^{2n-1} \) is
\[ T_{r+1} = \binom{2n-1}{r} x^r \]
To find the coefficient of \( x^n \)
Putting \( n = r \)
\[ T_{r+1} = \binom{2n-1}{n} x^n \]
The coefficient of \( x^n \) = \( \binom{2n-1}{n} \)
We have to prove
Coefficient of \( x^n \) in \( (1+x)^{2n} \) = 2 coefficient of \( x^n \) in \( (1+x)^{2n-1} \)
Consider LHS = \( \binom{2n}{n} \)
Hence the proof.

12. Find a positive value of m for which the coefficient of $x^2$ in the expansion $(1 + x)^m$ is 6.

**Solution:**
The general term $T_{r+1}$ in the binomial expansion is given by $T_{r+1} = \binom{n}{r} a^{n-r} b^r$

Here $a = 1, b = x$ and $n = m$

Putting the value

$T_{r+1} = \binom{m}{r} 1^{m-r} x^r$

$= \binom{m}{r} x^r$

We need coefficient of $x^2$

$\therefore$ putting $r = 2$
\[ T_{2+1} = \binom{m}{2} x^2 \]

The coefficient of \( x^2 \) is \( \binom{m}{2} \).

Given that the coefficient of \( x^2 \) is \( \binom{m}{2} = 6 \),

\[
\frac{m!}{2!(m-2)!} = 6
\]

\[
\frac{m(m-1)(m-2)!}{2\times1\times(m-2)!} = 6
\]

\[
\Rightarrow m(m-1) = 12
\]

\[
\Rightarrow m^2 - m - 12 = 0
\]

\[
\Rightarrow m^2 - 4m + 3m - 12 = 0
\]

\[
\Rightarrow m(m - 4) + 3(m - 4) = 0
\]

\[
\Rightarrow (m+3)(m - 4) = 0
\]

\[
\Rightarrow m = -3, 4
\]

We need the positive value of \( m \) so \( m = 4 \).
1. Find $a$, $b$ and $n$ in the expansion of $(a + b)^n$ if the first three terms of the expansion are 729, 7290 and 30375, respectively.

Solution:
We know that $(r + 1)^{th}$ term, $(T_{r+1})$, in the binomial expansion of $(a + b)^n$ is given by

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

The first three terms of the expansion are given as 729, 7290 and 30375 respectively.

Then we have,

$$T_1 = \binom{n}{0} a^n b^0 = a^n = 729 \ldots \ 1$$
$$T_2 = \binom{n}{1} a^{n-1} b^1 = na^{n-1} b = 7290 \ldots \ 2$$
$$T_3 = \binom{n}{2} a^{n-2} b^2 = n(n-1)/2 a^{n-2} b^2 = 30375 \ldots \ 3$$

Dividing 2 by 1 we get

$$na^{n-1}b = \frac{7290}{729}$$

$$n \ b \ a = 10 \ldots \ 4$$

Dividing 3 by 2 we get

$$\frac{n(n-1)a^{n-2}b^2}{2} \cdot 2na^{n-1}b = \frac{30375}{7290}$$

$$\Rightarrow (n-1)b2a = \frac{30375}{7290}$$

$$\Rightarrow (n-1)ba = \frac{30375 \times 2}{7290} = \frac{25}{3}$$

$$\Rightarrow nba - \frac{b}{a} = \frac{25}{3}$$

$$\Rightarrow 10 - ba = \frac{25}{3}$$

$$\Rightarrow ba = 10 - \frac{25}{3} = \frac{5}{3} \ldots \ 5$$

From 4 and 5 we have

$$n \cdot \frac{5}{3} = 10$$

$$n = 6$$

Substituting $n = 6$ in 1 we get

$$a^6 = 729$$

$$a = 3$$

From 5 we have, $b/3 = 5/3$

$$b = 5$$
Thus a = 3, b = 5 and n = 76

2. Find a if the coefficients of $x^2$ and $x^3$ in the expansion of $(3 + ax)^9$ are equal.

Solution:
We know that general term of expansion $(a + b)^n$ is

$$T_{r+1} = \binom{n}{r} a^{n-r} b^r$$

For $(3+ax)^9$

Putting $a = 3$, $b = ax$ & $n = 9$

General term of $(3+ax)^9$ is

$$T_{r+1} = \binom{9}{r} 3^{n-r} (ax)^r$$

$$T_{r+1} = \binom{9}{r} 3^{n-r} a^r x^r$$

Since we need to find the coefficients of $x^2$ and $x^3$, therefore

For $r = 2$

$$T_{2+1} = \binom{9}{2} 3^{n-2} a^2 x^2$$

Thus, the coefficient of $x^2 = \binom{9}{2} 3^{n-2} a^2$

For $r = 3$

$$T_{3+1} = \binom{9}{3} 3^{n-3} a^3 x^3$$

Thus, the coefficient of $x^3 = \binom{9}{3} 3^{n-3} a^3$

Given that coefficient of $x^2 = \text{Coefficient of } x^3$

$$\Rightarrow \binom{9}{2} 3^{n-2} a^2 = \binom{9}{3} 3^{n-3} a^3$$
3. Find the coefficient of \(x^5\) in the product \((1 + 2x)^6(1 - x)^7\) using binomial theorem.

Solution:

\[
(1 + 2x)^6 = \binom{6}{0} + \binom{6}{1} (2x) + \binom{6}{2} (2x)^2 + \binom{6}{3} (2x)^3 + \binom{6}{4} (2x)^4 + \binom{6}{5} (2x)^5 + \binom{6}{6} (2x)^6
\]

\[
= 1 + 6 (2x) + 15 (2x)^2 + 20 (2x)^3 + 15 (2x)^4 + 6 (2x)^5 + (2x)^6
\]

\[
(1 - x)^7 = \binom{7}{0} - \binom{7}{1} (x) + \binom{7}{2} (x)^2 - \binom{7}{3} (x)^3 + \binom{7}{4} (x)^4 - \binom{7}{5} (x)^5 + \binom{7}{6} (x)^6 - \binom{7}{7} (x)^7
\]

\[
= 1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7
\]

\[
(1 + 2x)^6 (1 - x)^7 = \left(1 + 12 x + 60 x^2 + 160 x^3 + 240 x^4 + 192 x^5 + 64 x^6\right) \left(1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7\right)
\]

\[
192 - 21 = 171
\]

Thus, the coefficient of \(x^5\) in the expression \((1+2x)^6(1-x)^7\) is 171.

4. If \(a\) and \(b\) are distinct integers, prove that \(a - b\) is a factor of \(a^n - b^n\), whenever \(n\) is a positive integer. [Hint write \(a^n = (a - b + b)^n\) and expand]

Solution:

In order to prove that \((a - b)\) is a factor of \((a^n - b^n)\), it has to be proved that

\[a^n - b^n = k (a - b)\]

where \(k\) is some natural number.

\(a\) can be written as \(a = a - b + b\)

\[a^n = (a - b + b)^n = [(a - b) + b]^n\]

\[= \binom{n}{0} (a - b)^n + \binom{n}{1} (a - b)^{n-1} b + \ldots + \binom{n}{n} b^n\]

\[a^n - b^n = (a - b) \left[(a - b)^{n-1} + \binom{n}{1} (a - b)^{n-2} b + \ldots + \binom{n}{n} b^{n-1}\right]\]
\(a^n - b^n = (a - b) \cdot k\)

Where \(k = [(a - b)^{n-1} + ^nC_1 (a - b)^{n-1} b + \ldots + ^n C_n b^n]\) is a natural number.

This shows that \((a - b)\) is a factor of \((a^n - b^n)\), where \(n\) is a positive integer.

5. Evaluate \((\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6\)

**Solution:**

Using binomial theorem, the expression \((a + b)^6\) and \((a - b)^6\), can be expanded

\[
(a + b)^6 = ^6C_0 a^6 + ^6C_1 a^5 b + ^6C_2 a^4 b^2 + ^6C_3 a^3 b^3 + ^6C_4 a^2 b^4 + ^6C_5 a b^5 + ^6C_6 b^6
\]

\[(a - b)^6 = ^6C_0 a^6 - ^6C_1 a^5 b + ^6C_2 a^4 b^2 - ^6C_3 a^3 b^3 + ^6C_4 a^2 b^4 - ^6C_5 a b^5 + ^6C_6 b^6\]

Now \((a + b)^6 - (a - b)^6 = ^6C_0 a^6 + ^6C_1 a^5 b + ^6C_2 a^4 b^2 + ^6C_3 a^3 b^3 + ^6C_4 a^2 b^4 + ^6C_5 a b^5 + ^6C_6 b^6 - [^6C_0 a^6 - ^6C_1 a^5 b + ^6C_2 a^4 b^2 - ^6C_3 a^3 b^3 + ^6C_4 a^2 b^4 - ^6C_5 a b^5 + ^6C_6 b^6]\]

Now by substituting \(a = \sqrt{3}\) and \(b = \sqrt{2}\) we get

\[
(\sqrt{3} + \sqrt{2})^6 - (\sqrt{3} - \sqrt{2})^6 = 2 [6 (\sqrt{3})^5 (\sqrt{2}) + 20 (\sqrt{3})^3 (\sqrt{2})^3 + 6 (\sqrt{3}) (\sqrt{2})^5]
\]

\[
= 2 [54(\sqrt{6}) + 120 (\sqrt{6}) + 24 \sqrt{6}]
\]

\[
= 2 (\sqrt{6}) (198)
\]

\[
= 396 \sqrt{6}
\]

6. Find the value of \(\left(a^2 + \sqrt{a^2 - 1}\right)^4 + \left(a^2 - \sqrt{a^2 - 1}\right)^4\)

**Solution:**

Firstly the expression \((x + y)^4 + (x - y)^4\) is simplified by using binomial theorem

\[
(x + y)^4 = ^4C_0 x^4 + ^4C_1 x^3 y + ^4C_2 x^2 y^2 + ^4C_3 xy^3 + ^4C_4 y^4
\]

\[= x^4 + 4x^3 y + 6x^2 y^2 + 4xy^3 + y^4\]

\[
(x - y)^4 = ^4C_0 x^4 - ^4C_1 x^3 y + ^4C_2 x^2 y^2 - ^4C_3 xy^3 + ^4C_4 y^4
\]

\[= x^4 - 4x^3 y + 6x^2 y^2 - 4xy^3 + y^4\]

\(\therefore\) \((x + y)^4 + (x - y)^4 = 2 (x^4 + 6x^2 y^2 + y^4)\)

Putting \(x = a^2\) and \(y = \sqrt{a^2 - 1}\), we obtain
7. Find an approximation of $(0.99)^5$ using the first three terms of its expansion.

Solution:
0.99 can be written as
$0.99 = 1 - 0.01$
Now by applying binomial theorem we get
$(0.99)^5 = (1 - 0.01)^5$
$= \binom{5}{0}(1)^5 - 5(1)^4(0.01) + 10(1)^3(0.01)^2$
$= 1 - 5(0.01) + 10(0.01)^2$
$= 1 - 0.05 + 0.001$
$= 0.951$

8. Find $n$, if the ratio of the fifth term from the beginning to the fifth term from the end in the expansion of $(\sqrt[4]{2} + \frac{1}{\sqrt[4]{3}})^n$ is $\sqrt{6} : 1$

Solution:
In the expansion $(a + b)^n$, if $n$ is even then the middle term is $(n/2 + 1)^{th}$ term

$\binom{n}{4}(\sqrt[4]{2})^{n-4}\left(\frac{1}{\sqrt[4]{3}}\right)^4 = \binom{n}{4}\left(\frac{\sqrt[4]{2}}{\sqrt[4]{3}}\right)^4 \cdot \frac{1}{3} = \binom{n}{4}\left(\frac{\sqrt[4]{2}}{\sqrt[4]{3}}\right)^4 \cdot \frac{1}{3} = \frac{n!}{6!(n-4)!} \left(\frac{\sqrt[4]{2}}{\sqrt[4]{3}}\right)^4$

$\binom{n}{n-4}(\sqrt[4]{2})^4\left(\frac{1}{\sqrt[4]{3}}\right)^{n-4} = \binom{n-4}{1} \cdot 2 \cdot \left(\frac{\sqrt[4]{3}}{\sqrt[4]{2}}\right)^4 = \binom{n-4}{1} \cdot 2 \cdot \frac{3}{(\sqrt[4]{2})^4} = \frac{6n!}{(n-4)!} \cdot \frac{1}{(\sqrt[4]{3})^{n-4}}$
9. Expand using Binomial Theorem

\[
\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4, \quad x \neq 0
\]

**Solution:**

Using binomial theorem the given expression can be expanded as

\[
\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4 = \binom{4}{0} \left(1 + \frac{x}{2}\right)^4 + \binom{4}{1} \left(1 + \frac{x}{2}\right)^3 \left(-\frac{2}{x}\right) + \binom{4}{2} \left(1 + \frac{x}{2}\right) \left(-\frac{2}{x}\right)^2 + \binom{4}{3} \left(-\frac{2}{x}\right)^3 + \binom{4}{4} \left(-\frac{2}{x}\right)^4
\]

Again by using binomial theorem to expand the above terms we get

\[
\left(1 + \frac{x}{2} - \frac{2}{x}\right)^4
\]
From equation 1, 2 and 3 we get

\[
\left(1 + \frac{x}{2}\right)^3 = ^3C_0 (1)^3 + ^3C_1 (1)^2 \left(\frac{x}{2}\right) + ^3C_2 (1)\left(\frac{x}{2}\right)^2 + ^3C_3 \left(\frac{x}{2}\right)^3
\]

\[
= 1 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8}
\]

\[
= 1+ 2x + \frac{3x^2}{4} + \frac{x^3}{8} \quad \ldots (2)
\]

\[
\left(1 + \frac{x}{2}\right)^2 = ^2C_0 (1)^2 + ^2C_1 (1)\left(\frac{x}{2}\right) + ^2C_2 \left(\frac{x}{2}\right)^2
\]

\[
= 1 + \frac{3x}{2} + \frac{x^2}{4}
\]

\[
= 1+ \frac{3x}{2} + \frac{3x^2}{4} + \frac{x^3}{8} \quad \ldots (3)
\]

10. Find the expansion of \((3x^2 - 2ax + 3a^2)^3\) using binomial theorem.

Solution:
We know that \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\)

Putting \(a = 3x^2\) & \(b = -a (2x-3a)\), we get

\[
[3x^2 + (-a (2x-3a))]^3
\]

\[
= (3x^2)^3 + 3(3x^2)^2 (-a (2x-3a)) + 3(3x^2) (-a (2x-3a))^2 + (-a (2x-3a))^3
\]

\[
= 27x^6 - 27ax^4 (2x-3a) + 9a^2x^2 (2x-3a)^2 - a^3 (2x-3a)^3
\]

\[
= 27x^6 - 54ax^5 + 81a^2x^4 + 9a^2x^2 (4x^2-12ax+9a^2) - a^3 [(2x)^3 - (3a)^3 - 3(2x)(3a) + 3(2x)(3a)^2]
\]

\[
= 27x^6 - 54ax^5 + 81a^2x^4 + 36a^2x^4 - 108a^3x^3 + 81a^4x^2 - 8a^3x^3 + 27a^6 + 36a^4x^2 - 54a^5x
\]

Thus, \((3x^2 - 2ax + 3a^2)^3\)

\[
= 27x^6 - 54ax^5 + 117a^2x^4 - 116a^3x^3 + 117a^4x^2 - 54a^5x + 27a^6
\]