MISCELLANEOUS EXERCISE PAGE NO: 233

1. Find the values of k for which the line (k – 3) x – (4 – k^2) y + k^2 – 7k + 6 = 0 is
(a) Parallel to the x-axis,
(b) Parallel to the y-axis,
(c) Passing through the origin.
Solution:

It is given that
$(k – 3) x – (4 – k^2) y + k^2 – 7k + 6 = 0$ … (1)

(a) Here if the line is parallel to the x-axis
Slope of the line = Slope of the x-axis
It can be written as
$(4 – k^2) y = (k – 3) x + k^2 – 7k + 6 = 0$
We get

$$y=\frac{(k-3)}{(4-k^2)}x+\frac{k^2-7k+6}{(4-k^2)}$$

Which is of the form y = mx + c

Here the slope of the given line

$$=\frac{(k-3)}{(4-k^2)}$$

Consider the slope of x-axis = 0

$$\frac{(k-3)}{(4-k^2)}=0$$

By further calculation
k – 3 = 0
k = 3

Hence, if the given line is parallel to the x-axis, then the value of k is 3.

(b) Here if the line is parallel to the y-axis, it is vertical and the slope will be undefined.
So the slope of the given line

$= \frac{(k-3)}{(4-k^2)}$

Here,

$\frac{(k-3)}{(4-k^2)}$ is undefined at $k^2 = 4$

$k^2 = 4$

$k = \pm 2$

Hence, if the given line is parallel to the y-axis, then the value of k is ± 2.

(c) Here if the line is passing through (0, 0) which is the origin satisfies the given equation of line.

$(k - 3)(0) - (4 - k^2)(0) + k^2 - 7k + 6 = 0$

By further calculation

$k^2 - 7k + 6 = 0$

Separating the terms

$k^2 - 6k - k + 6 = 0$

We get

$(k - 6)(k - 1) = 0$

$k = 1$ or 6

Hence, if the given line is passing through the origin, then the value of k is either 1 or 6.

2. Find the values of θ and p, if the equation x cos θ + y sin θ = p is the normal form of the line √3x + y + 2 = 0.

Solution:

It is given that

$\sqrt{3}x + y + 2 = 0$

It can be reduced as

$\sqrt{3}x + y + 2 = 0$

$-\sqrt{3}x - y = 2$

By dividing both sides by $\sqrt{(-\sqrt{3})^2 + (-1)^2} = 2$, we get

$$-\frac{\sqrt{3}}{2}x - \frac{1}{2}y = \frac{2}{2}$$

It can be written as

$$\left(-\frac{\sqrt{3}}{2}\right)x + \left(-\frac{1}{2}\right)y = 1 \qquad ...(1)$$

By comparing equation (1) to x cos θ + y sin θ = p, we get

$$\cos\theta = -\frac{\sqrt{3}}{2}, \ \sin\theta = -\frac{1}{2}, \text{ and } p = 1$$

Here the values of sin θ and cos θ are negative

$$\theta = \pi + \frac{\pi}{6} = \frac{7\pi}{6}$$

Hence, the respective values of θ and p are 7π/6 and 1.

3. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and –6, respectively.

Solution:

Consider the intercepts cut by the given lines on a and b axes.

a + b = 1 …… (1)

ab = - 6 …….. (2)

By solving both the equations we get

a = 3 and b = -2 or a = - 2 and b = 3

We know that the equation of the line whose intercepts on a and b axes is

$$\frac{x}{a} + \frac{y}{b} = 1 \text{ or } bx + ay - ab = 0$$

Case I – a = 3 and b = - 2

So the equation of the line is – 2x + 3y + 6 = 0, i.e. 2x – 3y = 6.

Case II – a = -2 and b = 3
So the equation of the line is 3x – 2y + 6 = 0, i.e. -3x + 2y = 6

Hence, the required equation of the lines are 2x – 3y = 6 and -3x + 2y = 6.

4. What are the points on the *y*-axis whose distance from the line x/3 + y/4 = 1 is 4 units.
Solution:

Consider (0, b) as the point on the y-axis whose distance from line x/3 + y/4 = 1 is 4 units.
It can be written as 4x + 3y – 12 = 0 ……. (1)
By comparing equation (1) to the general equation of line Ax + By + C = 0, we get
A = 4, B = 3 and C = - 12
We know that the perpendicular distance (d) of a line Ax + By + C = 0 from (x_1, y_1) is written as

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

If (0, b) is the point on the y-axis whose distance from line x/3 + y/4 = 1 is 4 units, then

$$4 = \frac{|4(0) + 3(b) - 12|}{\sqrt{4^2 + 3^2}}$$

By further calculation

$$4 = \frac{|3b - 12|}{5}$$

By cross multiplication
20 = |3b – 12|
We get
20 = ± (3b – 12)
Here 20 = (3b – 12) or 20 = - (3b – 12)
It can be written as
3b = 20 + 12 or 3b = -20 + 12
So we get
b = 32/3 or b = -8/3

Hence, the required points are (0, 32/3) and (0, -8/3).

5. Find the perpendicular distance from the origin to the line joining the

points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

Solution:

Here the equation of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$ is written as

$$y - \sin\theta = \frac{\sin\phi - \sin\theta}{\cos\phi - \cos\theta}(x - \cos\theta)$$

By cross multiplication

$$y(\cos\phi - \cos\theta) - \sin\theta(\cos\phi - \cos\theta) = x(\sin\phi - \sin\theta) - \cos\theta(\sin\phi - \sin\theta)$$

By multiplying the terms we get

$$x(\sin\theta - \sin\phi) + y(\cos\phi - \cos\theta) + \cos\theta\sin\phi - \cos\theta\sin\theta - \sin\theta\cos\phi + \sin\theta\cos\theta = 0$$

On further simplification

$$x(\sin\theta - \sin\phi) + y(\cos\phi - \cos\theta) + \sin(\phi - \theta) = 0$$

So we get

$Ax + By + C = 0$, where $A = \sin\theta - \sin\phi$, $B = \cos\phi - \cos\theta$, and $C = \sin(\phi - \theta)$

We know that the perpendicular distance (d) of a line $Ax + By + C = 0$ from (x_1, y_1) is written as

$$d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

So the perpendicular distance (d) of the given line from $(x_1, y_1) = (0, 0)$ is

$$d = \frac{|(\sin\theta - \sin\phi)(0) + (\cos\phi - \cos\theta)(0) + \sin(\phi - \theta)|}{\sqrt{(\sin\theta - \sin\phi)^2 + (\cos\phi - \cos\theta)^2}}$$

By expanding using formula

$$= \frac{|\sin(\phi-\theta)|}{\sqrt{\sin^2\theta+\sin^2\phi-2\sin\theta\sin\phi+\cos^2\phi+\cos^2\theta-2\cos\phi\cos\theta}}$$

Grouping of terms

$$= \frac{|\sin(\phi-\theta)|}{\sqrt{(\sin^2\theta+\cos^2\theta)+(\sin^2\phi+\cos^2\phi)-2(\sin\theta\sin\phi+\cos\theta\cos\phi)}}$$

By further simplification

$$= \frac{|\sin(\phi-\theta)|}{\sqrt{1+1-2(\cos(\phi-\theta))}}$$

Taking out 2 as common

$$= \frac{|\sin(\phi-\theta)|}{\sqrt{2(1-\cos(\phi-\theta))}}$$

Using the formula

$$= \frac{|\sin(\phi-\theta)|}{\sqrt{2\left(2\sin^2\left(\frac{\phi-\theta}{2}\right)\right)}}$$

We get

$$= \frac{|\sin(\phi-\theta)|}{\left|2\sin\left(\frac{\phi-\theta}{2}\right)\right|}$$

6. Find the equation of the line parallel to *y*-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.
Solution:

Here the equation of any line parallel to the y-axis is of the form
x = a ……. (1)
Two given lines are
x – 7y + 5 = 0 …… (2)
3x + y = 0 …… (3)
By solving equations (2) and (3) we get
x = -5/22 and y = 15/22
(-5/ 22, 15/22) is the point of intersection of lines (2) and (3)
If the line x = a passes through point (-5/22, 15/22) we get a = -5/22

Hence, the required equation of the line is x = -5/22.

7. Find the equation of a line drawn perpendicular to the line x/4 + y/6 = 1 through the point, where it meets the y-axis.
Solution:

It is given that
x/4 + y/6 = 1
We can write it as
3x + 2y – 12 = 0
So we get
y = -3/2 x + 6, which is of the form y = mx + c
Here the slope of the given line = -3/2
So the slope of line perpendicular to the given line = -1/ (-3/2) = 2/3

Consider the given line intersect the y-axis at (0, y)
By substituting x as zero in the equation of the given line
y/6 = 1
y = 6
Hence, the given line intersects the y-axis at (0, 6)

We know that the equation of the line that has a slope of 2/3 and passes through point (0, 6) is
(y – 6) = 2/3 (x – 0)
By further calculation
3y – 18 = 2x
So we get
2x – 3y + 18 = 0

Hence, the required equation of the line is 2x – 3y + 18 = 0.

8. Find the area of the triangle formed by the lines *y – x* = 0, *x* + *y* = 0 and *x – k* = 0.
Solution:

It is given that
y – x = 0 …… (1)
x + y = 0 …… (2)
x – k = 0 ……. (3)

Here the point of intersection of
Lines (1) and (2) is
x = 0 and y = 0
Lines (2) and (3) is
x = k and y = - k
Lines (3) and (1) is
x = k and y = k
So the vertices of the triangle formed by the three given lines are (0, 0), (k, -k) and (k, k)

Here the area of triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is
$\frac{1}{2} |x_1 (y_2 – y_3) + x_2 (y_3 – y_1) + x_3 (y_1 – y_2)|$
So the area of triangle formed by the three given lines
= ½ |0 (-k – k) + k (k – 0) + k (0 + k)| square units
By further calculation
= ½ $|k^2 + k^2|$ square units
So we get
= ½ $|2k^2|$
= k^2 square units

9. Find the value of *p* so that the three lines 3*x* + *y* – 2 = 0, *px* + 2*y* – 3 = 0 and 2*x* – *y* – 3 = 0 may intersect at one point.
Solution:

It is given that
3x + y – 2 = 0 …… (1)
px + 2y – 3 = 0 ….. (2)
2x – y – 3 = 0 …… (3)
By solving equations (1) and (3) we get
x = 1 and y = -1
Here the three lines intersect at one point and the point of intersection of lines (1) and (3) will also satisfy line (2)
p (1) + 2 (-1) – 3 = 0
By further calculation
p – 2 – 3 = 0
So we get
p = 5

Hence, the required value of p is 5.

10. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1 (c_2 – c_3) + m_2 (c_3 – c_1) + m_3 (c_1 – c_2) = 0$.

Solution:

It is given that

$y = m_1x + c_1$ ….. (1)

$y = m_2x + c_2$ ….. (2)

$y = m_3x + c_3$ ….. (3)

By subtracting equation (1) from (2) we get

$0 = (m_2 – m_1) x + (c_2 – c_1)$

$(m_1 – m_2) x = c_2 – c_1$

So we get

$$x = \frac{c_2 - c_1}{m_1 - m_2}$$

By substituting this value in equation (1) we get

$$y = m_1\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_1$$

By multiplying the terms

$$y = \frac{m_1c_2 - m_1c_1}{m_1 - m_2} + c_1$$

Taking LCM

$$y = \frac{m_1c_2 - m_1c_1 + m_1c_1 - m_2c_1}{m_1 - m_2}$$

On further simplification

$$y = \frac{m_1c_2 - m_2c_1}{m_1 - m_2}$$

Here

$\left(\frac{c_2 - c_1}{m_1 - m_2}, \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)$ is the point of intersection of lines (1) and (2)

Lines (1), (2) and (3) are concurrent. So the point of intersection of lines (1) and (2) will satisfy equation (3)

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = m_3\left(\frac{c_2 - c_1}{m_1 - m_2}\right) + c_3$$

By multiplying the terms and taking LCM

$$\frac{m_1c_2 - m_2c_1}{m_1 - m_2} = \frac{m_3c_2 - m_3c_1 + c_3m_1 - c_3m_2}{m_1 - m_2}$$

By cross multiplication

$m_1 c_2 - m_2c_1 - m_3c_2 + m_3c_1 - c_3m_1 + c_3m_2 = 0$
Taking out the common terms
$m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0$

Therefore, $m_1 (c_2 - c_3) + m_2 (c_3 - c_1) + m_3 (c_1 - c_2) = 0$.

11. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line *x* –2*y* = 3.
Solution:

Consider m_1 as the slope of the required line
It can be written as
y = 1/2 x – 3/2 which is of the form y = mx + c
So the slope of the given line $m_2 = 1/2$
We know that the angle between the required line and line x – 2y = 3 is 45°
If θ is the acute angle between lines l_1 and l_2 with slopes m_1 and m_2

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|$$

We get

$$\tan 45^\circ = \frac{|m_1 - m_2|}{1 + m_1 m_2}$$

Substituting the values

$$1 = \left|\frac{\frac{1}{2} - m_1}{1 + \frac{m_1}{2}}\right|$$

By taking LCM

$$1 = \left|\frac{\left(\frac{1 - 2m_1}{2}\right)}{\frac{2 + m_1}{2}}\right|$$

On further calculation

$$1 = \left|\frac{1 - 2m_1}{2 + m_1}\right|$$

We get

$$1 = \pm\left(\frac{1 - 2m_1}{2 + m_1}\right)$$

Here

$$1 = \frac{1 - 2m_1}{2 + m_1} \text{ or } 1 = -\left(\frac{1 - 2m_1}{2 + m_1}\right)$$

It can be written as

$2 + m_1 = 1 – 2m_1$ or $2 + m_1 = - 1 + 2m_1$

$m_1 = - 1/3$ or $m_1 = 3$

Case I – $m_1 = 3$

Here the equation of the line passing through (3, 2) and having a slope 3 is

$y – 2 = 3 (x – 3)$

By further calculation

y – 2 = 3x – 9
So we get
3x – y = 7

Case II – m_1 = -1/3
Here the equation of the line passing through (3, 2) and having a slope -1/3 is
y – 2 = - 1/3 (x – 3)
By further calculation
3y – 6 = - x + 3
So we get
x + 3y = 9

Hence, the equations of the lines are 3x – y = 7 and x + 3y = 9.

12. Find the equation of the line passing through the point of intersection of the lines 4*x* + 7*y* – 3 = 0 and 2*x* – 3*y* + 1 = 0 that has equal intercepts on the axes.
Solution:

Consider the equation of the line having equal intercepts on the axes as
x/a + y/a = 1
It can be written as
x + y = a ….. (1)
By solving equations 4x + 7y – 3 = 0 and 2x – 3y + 1 = 0 we get
x = 1/13 and y = 5/13
(1/13, 5/13) is the point of intersection of two given lines
We know that equation (1) passes through point (1/13, 5/13)
1/13 + 5/13 = a
a = 6/13
So the equation (1) passes through (1/13, 5/13)
1/13 + 5/13 = a
We get
a = 6/13
Here the equation (1) becomes
x + y = 6/13
13x + 13y = 6

Hence, the required equation of the line is 13x + 13y = 6.

13. Show that the equation of the line passing through the origin and making an

angle θ with the line y = mx + c is $\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$.

Solution:

Consider y = m_1x as the equation of the line passing through the origin

It is given that the line makes an angle θ with line $y = mx + c$, then angle θ is written as

$$\tan\theta = \left|\frac{m_1 - m}{1 + m_1 m}\right|$$

By substituting the values

$$\tan\theta = \left|\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right|$$

We get

$$\tan\theta = \pm\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

Here

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m} \quad \text{or} \quad \tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}\right)$$

Case I –

$$\tan\theta = \frac{\frac{y}{x} - m}{1 + \frac{y}{x}m}$$

We can write it as

$$\tan\theta + \frac{y}{x} m \tan\theta = \frac{y}{x} - m$$

By further simplification

$$m + \tan\theta = \frac{y}{x}(1 - m\tan\theta)$$

So we get

$$\frac{y}{x} = \frac{m + \tan\theta}{1 - m\tan\theta}$$

Case II –

$$\tan\theta = -\left(\frac{\frac{y}{x} - m}{1 + \frac{y}{x} m}\right)$$

We can write it as

$$\tan\theta + \frac{y}{x} m \tan\theta = -\frac{y}{x} + m$$

By further simplification

$$\frac{y}{x}(1 + m\tan\theta) = m - \tan\theta$$

So we get

$$\frac{y}{x} = \frac{m - \tan\theta}{1 + m\tan\theta}$$

Hence, the required line is given by

$$\frac{y}{x} = \frac{m \pm \tan\theta}{1 \mp m\tan\theta}$$

14. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line $x + y = 4$?
Solution:

We know that the equation of the line joining the points (-1, 1) and (5, 7) is given by

$$y-1=\frac{7-1}{5+1}(x+1)$$

By further calculation

$$y-1=\frac{6}{6}(x+1)$$

So we get

$x - y + 2 = 0$ ……. (1)

So the equation of the given line is

$x + y - 4 = 0$ ….. (2)

Here the point of intersection of lines (1) and (2) is given by

$x = 1$ and $y = 3$

Consider (1, 3) divide the line segment joining (-1, 1) and (5, 7) in the ratio 1: k.

Using the section formula

$$(1,3)=\left(\frac{k(-1)+1(5)}{1+k},\frac{k(1)+1(7)}{1+k}\right)$$

By further calculation

$$(1,3)=\left(\frac{-k+5}{1+k},\frac{k+7}{1+k}\right)$$

So we get

$$\frac{-k+5}{1+k}=1,\frac{k+7}{1+k}=3$$

We can write it as

$$\frac{-k+5}{1+k}=1$$

By cross multiplication

$- k + 5 = 1 + k$

We get

$2k = 4$

$k = 2$

Hence, the line joining the points (-1, 1) and (5, 7) is divided by the line x + y = 4 in the ratio 1: 2.

15. Find the distance of the line 4*x* + 7*y* + 5 = 0 from the point (1, 2) along the line 2*x* – *y* = 0.

Solution:

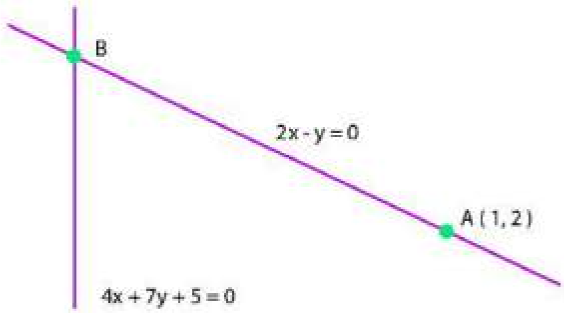
It is given that

2x – y = 0 ….. (1)

4x + 7y + 5 = 0 …… (2)

Here A (1, 2) is a point on the line (1)

Consider B as the point of intersection of lines (1) and (2)



By solving equations (1) and (2) we get x = -5/18 and y = - 5/9

So the coordinates of point B are (-5/18, -5/9)

From distance formula the distance between A and B

$$AB = \sqrt{\left(1+\frac{5}{18}\right)^2 + \left(2+\frac{5}{9}\right)^2} \text{ units}$$

By taking LCM

$$= \sqrt{\left(\frac{23}{18}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

It can be written as

$$= \sqrt{\left(\frac{23}{2\times 9}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

So we get

$$= \sqrt{\left(\frac{23}{9}\right)^2\left(\frac{1}{2}\right)^2 + \left(\frac{23}{9}\right)^2} \text{ units}$$

By taking the common terms out

$$= \sqrt{\left(\frac{23}{9}\right)^2\left(\frac{1}{4}+1\right)} \text{ units}$$

We get

$$= \frac{23}{9}\sqrt{\frac{5}{4}} \text{ units}$$

$$= \frac{23}{9} \times \frac{\sqrt{5}}{2} \text{ units}$$

So we get

$$= \frac{23\sqrt{5}}{18} \text{ units}$$

Hence, the required distance is $\frac{23\sqrt{5}}{18}$ units.

16. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.
Solution:

Consider y = mx + c as the line passing through the point (-1, 2)
So we get

2 = m (-1) + c
By further calculation
2 = -m + c
c = m + 2
Substituting the value of c
y = mx + m + 2 …… (1)
So the given line is
x + y = 4 ……. (2)
By solving both the equations we get

$$x = \frac{2-m}{m+1} \text{ and } y = \frac{5m+2}{m+1}$$

$\left(\frac{2-m}{m+1}, \frac{5m+2}{m+1}\right)$ is the point of intersection of lines (1) and (2)

Here the point is at a distance of 3 units from (-1, 2)

From distance formula

$$\sqrt{\left(\frac{2-m}{m+1}+1\right)^2 + \left(\frac{5m+2}{m+1}-2\right)^2} = 3$$

Squaring on both sides

$$\left(\frac{2-m+m+1}{m+1}\right)^2 + \left(\frac{5m+2-2m-2}{m+1}\right)^2 = 3^2$$

By further calculation

$$\frac{9}{(m+1)^2} + \frac{9m^2}{(m+1)^2} = 9$$

Dividing the equation by 9

$$\frac{1+m^2}{(m+1)^2} = 1$$

By cross multiplication
$1 + m^2 = m^2 + 1 + 2m$
So we get
2m = 0
m = 0

Hence, the slope of the required line must be zero i.e. the line must be parallel to the x-axis.

17. The hypotenuse of a right angled triangle has its ends at the points (1, 3) and (−4, 1). Find the equation of the legs (perpendicular sides) of the triangle.
Solution:

Consider ABC as the right angles triangle where $\angle C = 90^o$
Here infinity such lines are present.
m is the slope of AC
So the slope of BC = -1/m

Equation of AC -
y – 3 = m (x – 1)
By cross multiplication
x – 1 = 1/m (y – 3)

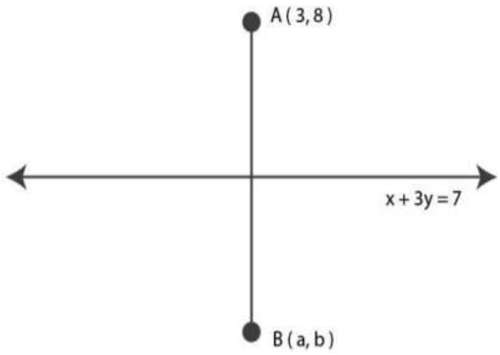
Equation of BC –
y – 1 = - 1/m (x + 4)
By cross multiplication
x + 4 = - m (y – 1)

By considering values of m we get
If m = 0,
So we get
y – 3 = 0, x + 4 = 0

If m = ∞,
So we get
x – 1 = 0, y – 1 = 0 we get x = 1, y = 1

18. Find the image of the point (3, 8) with respect to the line *x* + 3*y* = 7 assuming the line to be a plane mirror.
Solution:

It is given that
x + 3y = 7 ….. (1)
Consider B (a, b) as the image of point A (3, 8)
So line (1) is perpendicular bisector of AB.



Here

$$\text{Slope of AB} = \frac{b-8}{a-3}$$

$$\text{slope of line } (1) = -\frac{1}{3}$$

Line (1) is perpendicular to AB

$$\left(\frac{b-8}{a-3}\right) \times \left(-\frac{1}{3}\right) = -1$$

By further calculation

$$\frac{b-8}{3a-9} = 1$$

By cross multiplication

b – 8 = 3a – 9

3a – b = 1 …… (2)

We know that

$$\text{Mid-point of AB} = \left(\frac{a+3}{2}, \frac{b+8}{2}\right)$$

So the mid-point of line segment AB will satisfy line (1)

From equation (1)

$$\left(\frac{a+3}{2}\right) + 3\left(\frac{b+8}{2}\right) = 7$$

By further calculation

a + 3 + 3b + 24 = 14

On further simplification

a + 3b = - 13 ….. (3)
By solving equations (2) and (3) we get
a = - 1 and b = - 4

Hence, the image of the given point with respect to the given line is (-1, -4).

19. If the lines *y* = 3*x* + 1 and 2*y* = *x* + 3 are equally inclined to the line *y* = *mx* + 4, find the value of *m*.
Solution:

It is given that
y = 3x + 1 …… (1)
2y = x + 3 …… (2)
y = mx + 4 …… (3)
Here the slopes of
Line (1), $m_1 = 3$
Line (2), $m_2 = ½$
Line (3), $m_3 = m$
We know that the lines (1) and (2) are equally inclined to line (3) which means that the angle between lines (1) and (3) equals the angle between lines (2) and (3).

$$\left|\frac{m_1 - m_3}{1 + m_1 m_3}\right| = \left|\frac{m_2 - m_3}{1 + m_2 m_3}\right|$$

Substituting the values we get

$$\left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{\frac{1}{2} - m}{1 + \frac{1}{2}m}\right|$$

By taking LCM

$$\left|\frac{3 - m}{1 + 3m}\right| = \left|\frac{1 - 2m}{m + 2}\right|$$

It can be written as

$$\frac{3 - m}{1 + 3m} = \pm\left(\frac{1 - 2m}{m + 2}\right)$$

Here

$$\frac{3 - m}{1 + 3m} = \frac{1 - 2m}{m + 2} \text{ or } \frac{3 - m}{1 + 3m} = -\left(\frac{1 - 2m}{m + 2}\right)$$

If

$$\frac{3 - m}{1 + 3m} = \frac{1 - 2m}{m + 2}$$

By cross multiplication

(3 – m) (m + 2) = (1 – 2m) (1 + 3m)

On further calculation

$- m^2 + m + 6 = 1 + m – 6m^2$

So we get

$5m^2 + 5 = 0$

Dividing the equation by 5

$m^2 + 1 = 0$

m = √-1, which is not real.

Therefore, this case is not possible.

If

$$\frac{3-m}{1+3m} = -\left(\frac{1-2m}{m+2}\right)$$

By cross multiplication

$(3 – m) (m + 2) = - (1 – 2m) (1 + 3m)$

On further calculation

$- m^2 + m + 6 = - (1 + m – 6m^2)$

So we get

$7m^2 – 2m – 7 = 0$

Here we get

$$m = \frac{2 \pm \sqrt{4-4(7)(-7)}}{2(7)}$$

By further simplification

$$m = \frac{2 \pm 2\sqrt{1+49}}{14}$$

We can write it as

$$m = \frac{1 \pm 5\sqrt{2}}{7}$$

Hence, the required value of m is

$$\frac{1 \pm 5\sqrt{2}}{7}$$

20. If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y – 5 = 0$ and $3x – 2y + 7 = 0$ is always 10. Show that P must move on a line.

Solution:

It is given that

x + y – 5 = 0 …. (1)

3x – 2y + 7 = 0 ….. (2)

Here the perpendicular distances of P (x, y) from lines (1) and (2) are written as

$$d_1 = \frac{|x+y-5|}{\sqrt{(1)^2+(1)^2}} \text{ and } d_2 = \frac{|3x-2y+7|}{\sqrt{(3)^2+(-2)^2}}$$

So we get

$$d_1 = \frac{|x+y-5|}{\sqrt{2}} \text{ and } d_2 = \frac{|3x-2y+7|}{\sqrt{13}}$$

We know that $d_1 + d_2 = 10$

Substituting the values

$$\frac{|x+y-5|}{\sqrt{2}} + \frac{|3x-2y+7|}{\sqrt{13}} = 10$$

By further calculation

$$\sqrt{13}|x+y-5| + \sqrt{2}|3x-2y+7| - 10\sqrt{26} = 0$$

It can be written as

$$\sqrt{13}(x+y-5) + \sqrt{2}(3x-2y+7) - 10\sqrt{26} = 0$$

Now by assuming (x + y – 5) and (3x – 2y + 7) are positive

$$\sqrt{13}x + \sqrt{13}y - 5\sqrt{13} + 3\sqrt{2}x - 2\sqrt{2}y + 7\sqrt{2} - 10\sqrt{26} = 0$$

Taking out the common terms

$x(\sqrt{13}+3\sqrt{2}) + y(\sqrt{13}-2\sqrt{2}) + (7\sqrt{2}-5\sqrt{13}-10\sqrt{26}) = 0$, which is the equation of a line.

In the same way we can find the equation of line for any signs of (x + y – 5) and (3x – 2y + 7)

Hence, point P must move on a line.

21. Find equation of the line which is equidistant from parallel lines 9*x* + 6*y* – 7 = 0 and 3*x* + 2*y* + 6 = 0.

Solution:

It is given that

9x + 6y – 7 = 0 …… (1)

3x + 2y + 6 = 0 …… (2)

Consider P (h, k) be the arbitrary point that is equidistant from lines (1) and (2)

Here the perpendicular distance of P (h, k) from line (1) is written as

$$d_1 = \frac{|9h+6k-7|}{(9)^2+(6)^2} = \frac{|9h+6k-7|}{\sqrt{117}} = \frac{|9h+6k-7|}{3\sqrt{13}}$$

Similarly the perpendicular distance of P (h, k) from line (2) is written as

$$d_2 = \frac{|3h+2k+6|}{\sqrt{(3)^2+(2)^2}} = \frac{|3h+2k+6|}{\sqrt{13}}$$

We know that P (h, k) is equidistant from lines (1) and (2) $d_1 = d_2$

Substituting the values

$$\frac{|9h+6k-7|}{3\sqrt{13}} = \frac{|3h+2k+6|}{\sqrt{13}}$$

By further calculation

$$|9h+6k-7| = 3|3h+2k+6|$$

It can be written as

$$|9h+6k-7| = \pm 3(3h+2k+6)$$

Here

9h + 6k – 7 = 3 (3h + 2k + 6) or 9h + 6k – 7 = - 3 (3h + 2k + 6)

9h + 6k – 7 = 3 (3h + 2k + 6) is not possible as

9h + 6k – 7 = 3 (3h + 2k + 6)

By further calculation

- 7 = 18

We know that

9h + 6k – 7 = -3 (3h + 2k + 6)

By multiplication

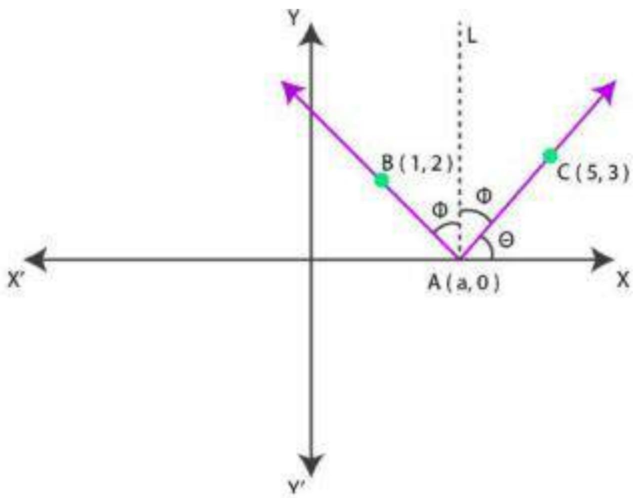
9h + 6k – 7 = -9h – 6k – 18

We get

18h + 12k + 11 = 0

Hence, the required equation of the line is 18x + 12y + 11 = 0.

22. A ray of light passing through the point (1, 2) reflects on the *x*-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.
Solution:



Consider the coordinates of point A as (a, 0)
Construct a line (AL) which is perpendicular to the x-axis
Here the angle of incidence is equal to angle of reflection
$\angle BAL = \angle CAL = \Phi$
$\angle CAX = \theta$
It can be written as
$\angle OAB = 180° – (\theta + 2\Phi) = 180° – [\theta + 2(90° – \theta)]$
On further calculation
$= 180° – \theta – 180° + 2\theta$
$= \theta$
So we get
$\angle BAX = 180° – \theta$

$$\text{slope of line AC} = \frac{3-0}{5-a}$$

$$\tan\theta = \frac{3}{5-a} \quad ...(1)$$

$$\text{Slope of line AB} = \frac{2-0}{1-a}$$

We get

$$\tan(180° - \theta) = \frac{2}{1-a}$$

By further calculation

$$-\tan\theta = \frac{2}{1-a}$$

$$\tan\theta = \frac{2}{a-1} \quad ...(2)$$

From equations (1) and (2) we get

$$\frac{3}{5-a} = \frac{2}{a-1}$$

By cross multiplication

3a – 3 = 10 – 2a

We get

a = 13/5

Hence, the coordinates of point A are (13/5, 0).

23. Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2-b^2}, 0\right)$ and $\left(-\sqrt{a^2-b^2}, 0\right)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

Solution:

It is given that

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

We can write it as

bx cos θ + ay sin θ – ab = 0 ….. (1)

Here the length of the perpendicular from point $\left(\sqrt{a^2-b^2},0\right)$ to line (1)

$$p_1=\frac{\left|b\cos\theta\left(\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\qquad ...(2)$$

Similarly the length of the perpendicular from point $\left(-\sqrt{a^2-b^2},0\right)$ to line (2)

$$p_2=\frac{\left|b\cos\theta\left(-\sqrt{a^2-b^2}\right)+a\sin\theta(0)-ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}=\frac{\left|b\cos\theta\sqrt{a^2-b^2}+ab\right|}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\qquad ...(3)$$

By multiplying equations (2) and (3) we get

$$p_1p_2=\frac{\left|b\cos\theta\sqrt{a^2-b^2}-ab\right|\left|\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}\right)^2}$$

We get

$$=\frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}-ab\right)\left(b\cos\theta\sqrt{a^2-b^2}+ab\right)\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

From the formula

$$=\frac{\left|\left(b\cos\theta\sqrt{a^2-b^2}\right)^2-(ab)^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

By squaring the numerator we get

$$=\frac{\left|b^2\cos^2\theta\left(a^2-b^2\right)-a^2b^2\right|}{\left(b^2\cos^2\theta+a^2\sin^2\theta\right)}$$

By expanding using formula

$$=\frac{\left|a^2b^2\cos^2\theta-b^4\cos^2\theta-a^2b^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

Taking out the common terms

$$=\frac{b^2\left|a^2\cos^2\theta-b^2\cos^2\theta-a^2\right|}{b^2\cos^2\theta+a^2\sin^2\theta}$$

We get

$$= \frac{b^2 \left| a^2 \cos^2\theta - b^2 \cos^2\theta - a^2 \sin^2\theta - a^2 \cos^2\theta \right|}{b^2 \cos^2\theta + a^2 \sin^2\theta}$$

Here $\sin^2 \theta + \cos^2 \theta = 1$

$$= \frac{b^2 \left| -\left(b^2 \cos^2\theta + a^2 \sin^2\theta\right) \right|}{b^2 \cos^2\theta + a^2 \sin^2\theta}$$

So we get

$$= \frac{b^2 \left(b^2 \cos^2\theta + a^2 \sin^2\theta\right)}{\left(b^2 \cos^2\theta + a^2 \sin^2\theta\right)}$$

$= b^2$

Therefore, it is proved.

24. A person standing at the junction (crossing) of two straight paths represented by the equations $2x – 3y + 4 = 0$ and $3x + 4y – 5 = 0$ wants to reach the path whose equation is $6x – 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Solution:

It is given that

$2x – 3y + 4 = 0$ …… (1)

$3x + 4y – 5 = 0$ ……. (2)

$6x – 7y + 8 = 0$ …… (3)

Here the person is standing at the junction of the paths represented by lines (1) and (2).

By solving equations (1) and (2) we get

$x = - 1/17$ and $y = 22/17$

Hence, the person is standing at point (-1/17, 22/17).

We know that the person can reach path (3) in the least time if he walks along the perpendicular line to (3) from point (-1/17, 22/17)

Here the slope of the line (3) = 6/7

We get the slope of the line perpendicular to line (3) = -1/ (6/7) = - 7/6

So the equation of line passing through (-1/17, 22/17) and having a slope of -7/6 is written as

$$\left(y - \frac{22}{17}\right) = -\frac{7}{6}\left(x + \frac{1}{17}\right)$$

By further calculation

6 (17y – 22) = - 7 (17x + 1)

By multiplication

102y – 132 = - 119x – 7

We get

1119x + 102y = 125

Therefore, the path that the person should follow is 119x + 102y = 125.