Miscellaneous Exercise

1. The relation *f* is defined by $f(x)=\begin{cases}x^2, & 0\le x\le 3\\ 3x, & 3\le x\le 10\end{cases}$

The relation *g* is defined by $g(x)=\begin{cases}x^2, & 0\le x\le 2\\ 3x, & 2\le x\le 10\end{cases}$

Show that *f* is a function and *g* is not a function.
Solution:

The given relation *f* is defined as:

$$f(x)=\begin{cases}x^2, & 0\le x\le 3\\ 3x, & 3\le x\le 10\end{cases}$$

It is seen that, for $0 \le x < 3$,
$f(x) = x^2$ and for $3 < x \le 10$,
$f(x) = 3x$
Also, at $x = 3$
$f(x) = 3^2 = 9$ or $f(x) = 3 \times 3 = 9$
i.e., at $x = 3$, $f(x) = 9$ [Single image]
Hence, for $0 \le x \le 10$, the images of $f(x)$ are unique.
Therefore, the given relation is a function.
Now,
In the given relation *g* is defined as

$$g(x)=\begin{cases}x^2, & 0\le x\le 2\\ 3x, & 2\le x\le 10\end{cases}$$

It is seen that, for $x = 2$
$g(x) = 2^2 = 4$ and $g(x) = 3 \times 2 = 6$
Thus, element 2 of the domain of the relation *g* corresponds to two different images i.e., 4 and 6.
Therefore, this relation is not a function.

2. If $f(x) = x^2$, find $\frac{f(1.1)-f(1)}{(1.1-1)}$
Solution:

Given,
$f(x) = x^2$
Hence,

$$\frac{f(1.1)-f(1)}{(1.1-1)}=\frac{(1.1)^2-(1)^2}{(1.1-1)}=\frac{1.21-1}{0.1}=\frac{0.21}{0.1}=2.1$$

3. Find the domain of the function $f(x)=\frac{x^2+2x+1}{x^2-8x+12}$
Solution:

Given function,

$$f(x)=\frac{x^2+2x+1}{x^2-8x+12}.$$

$$f(x)=\frac{x^2+2x+1}{x^2-8x+12}=\frac{x^2+2x+1}{(x-6)(x-2)}$$

It clearly seen that, the function *f* is defined for all real numbers except at $x = 6$ and $x = 2$ as the denominator becomes zero otherwise.
Therefore, the domain of *f* is R – {2, 6}.

4. Find the domain and the range of the real function *f* defined by *f*(x) = √(x - 1).
Solution:

Given real function,
$f(x) = \sqrt{(x - 1)}$
Clearly, $\sqrt{(x - 1)}$ is defined for $(x – 1) \geq 0$.
So, the function $f(x) = \sqrt{(x - 1)}$ is defined for $x \geq 1$.
Thus, the domain of *f* is the set of all real numbers greater than or equal to 1.
Domain of $f = [1, \infty)$.
Now,
As $x \geq 1 \Rightarrow (x – 1) \geq 0 \Rightarrow \sqrt{(x - 1)} \geq 0$
Thus, the range of *f* is the set of all real numbers greater than or equal to 0.
Range of $f = [0, \infty)$.

5. Find the domain and the range of the real function *f* defined by *f* (*x*) = |*x* – 1|.
Solution:

Given real function,
$f(x) = |x – 1|$
Clearly, the function $|x – 1|$ is defined for all real numbers.
Hence,
Domain of *f* = R
Also, for $x \in$ R, $|x – 1|$ assumes all real numbers.
Therefore, the range of *f* is the set of all non-negative real numbers.

6. Let $f=\left\{\left(x,\ \frac{x^2}{1+x^2}\right): x\in \mathbf{R}\right\}$ **be a function from R into R. Determine the range of *f*.**

Solution:

Given function,

$$f=\left\{\left(x,\ \frac{x^2}{1+x^2}\right): x\in \mathbf{R}\right\}$$

Substituting values and determining the images, we have

$$= \left\{(0,\ 0),\ \left(\pm 0.5,\ \frac{1}{5}\right),\ \left(\pm 1,\ \frac{1}{2}\right),\ \left(\pm 1.5,\ \frac{9}{13}\right),\ \left(\pm 2,\ \frac{4}{5}\right),\ \left(3,\ \frac{9}{10}\right),\ \left(4,\ \frac{16}{17}\right),\ \ldots\right\}$$

The range of *f* is the set of all second elements. It can be observed that all these elements are greater than or equal to 0 but less than 1.

[As the denominator is greater than the numerator.]

Or,
We know that, for x ∈ R,
$x^2 \geq 0$
Then,
$x^2 + 1 \geq x^2$
$1 \geq x^2 / (x^2 + 1)$

Therefore, the range of *f* = [0, 1)

7. Let *f*, *g*: R → R be defined, respectively by *f*(*x*) = *x* + 1, *g*(*x*) = 2*x* – 3. Find *f* + *g*, *f* – *g* and *f*/*g*.
Solution:

Given, the functions *f*, *g*: R → R is defined as
f(*x*) = *x* + 1, *g*(*x*) = 2*x* – 3
Now,
(*f* + *g*) (*x*) = *f*(*x*) + *g*(*x*) = (*x* + 1) + (2*x* – 3) = 3*x* – 2
Thus, (*f* + *g*) (*x*) = 3*x* – 2
(*f* – *g*) (*x*) = *f*(*x*) – *g*(*x*) = (*x* + 1) – (2*x* – 3) = *x* + 1 – 2*x* + 3 = – *x* + 4
Thus, (*f* – *g*) (*x*) = –*x* + 4
f/*g*(x) = *f*(x)/*g*(x), g(x) ≠ 0, x ∈ R
f/*g*(x) = *x* + 1/ 2*x* – 3, 2*x* – 3 ≠ 0
Thus, *f*/*g*(x) = *x* + 1/ 2*x* – 3, *x* ≠ 3/2

8. Let *f* = {(1, 1), (2, 3), (0, –1), (–1, –3)} be a function from Z to Z defined by *f*(*x*) = *ax* + *b*, for some integers *a*, *b*. Determine *a*, *b*.
Solution:

Given, *f* = {(1, 1), (2, 3), (0, –1), (–1, –3)}
And the function defined as, *f*(*x*) = *ax* + *b*
For (1, 1) ∈ *f*
We have, *f*(1) = 1
So, *a* × 1 + *b* = 1
a + *b* = 1 …. (i)
And for (0, –1) ∈ *f*
We have *f*(0) = –1
a × 0 + *b* = –1
b = –1
On substituting *b* = –1 in (i), we get
a + (–1) = 1 ⇒ *a* = 1 + 1 = 2.

Therefore, the values of a and b are 2 and –1 respectively.

9. Let R be a relation from N to N defined by R = {(a, b): a, $b \in$ N and $a = b^2$}. Are the following true?
(i) (a, a) ∈ R, for all $a \in$ N
(ii) (a, b) ∈ R, implies (b, a) ∈ R
(iii) (a, b) ∈ R, (b, c) ∈ R implies (a, c) ∈ R.
Justify your answer in each case.
Solution:

Given relation R = {(a, b): a, $b \in$ N and $a = b^2$}
(i) It can be seen that 2 ∈ N; however, $2 \neq 2^2 = 4$.
Thus, the statement "(a, a) ∈ R, for all $a \in$ N" is not true.

(ii) Its clearly seen that (9, 3) ∈ N because 9, 3 ∈ N and $9 = 3^2$.
Now, $3 \neq 9^2 = 81$; therefore, (3, 9) ∉ N
Thus, the statement "(a, b) ∈ R, implies (b, a) ∈ R" is not true.

(iii) Its clearly seen that (16, 4) ∈ R, (4, 2) ∈ R because 16, 4, 2 ∈ N and $16 = 4^2$ and $4 = 2^2$.
Now, $16 \neq 2^2 = 4$; therefore, (16, 2) ∉ N
Thus, the statement "(a, b) ∈ R, (b, c) ∈ R implies (a, c) ∈ R" is not true.

10. Let A = {1, 2, 3, 4}, B = {1, 5, 9, 11, 15, 16} and f = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)}. Are the following true?
(i) f is a relation from A to B (ii) f is a function from A to B.
Justify your answer in each case.
Solution:

Given,
A = {1, 2, 3, 4} and B = {1, 5, 9, 11, 15, 16}
So,
A × B = {(1, 1), (1, 5), (1, 9), (1, 11), (1, 15), (1, 16), (2, 1), (2, 5), (2, 9), (2, 11), (2, 15), (2, 16), (3, 1), (3, 5), (3, 9), (3, 11), (3, 15), (3, 16), (4, 1), (4, 5), (4, 9), (4, 11), (4, 15), (4, 16)}
Also given that,
f = {(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)}
(i) A relation from a non-empty set A to a non-empty set B is a subset of the Cartesian product A × B.
It's clearly seen that f is a subset of A × B.
Therefore, f is a relation from A to B.

(ii) As the same first element i.e., 2 corresponds to two different images (9 and 11), relation f is not a function.

11. Let f be the subset of Z × Z defined by f = {(ab, $a + b$): a, $b \in$ Z}. Is f a function from Z to Z: justify your answer.
Solution:

Given relation f is defined as
$f = \{(ab, a + b): a, b \in Z\}$
We know that a relation f from a set A to a set B is said to be a function if every element of set A has unique images in set B.
As 2, 6, –2, –6 ∈ Z, $(2 \times 6, 2 + 6), (-2 \times -6, -2 + (-6)) \in f$
i.e., $(12, 8), (12, -8) \in f$
It's clearly seen that, the same first element, 12 corresponds to two different images (8 and –8).
Therefore, the relation f is not a function.

12. Let A = {9, 10, 11, 12, 13} and let *f*: A → N be defined by *f*(*n*) = the highest prime factor of *n*. Find the range of *f*.
Solution:

Given,
A = {9, 10, 11, 12, 13}
Now, f: A → **N** is defined as
$f(n)$ = The highest prime factor of n
So,
Prime factor of 9 = 3
Prime factors of 10 = 2, 5
Prime factor of 11 = 11
Prime factors of 12 = 2, 3
Prime factor of 13 = 13
Thus, it can be expressed as
$f(9)$ = The highest prime factor of 9 = 3
$f(10)$ = The highest prime factor of 10 = 5
$f(11)$ = The highest prime factor of 11 = 11
$f(12)$ = The highest prime factor of 12 = 3
$f(13)$ = The highest prime factor of 13 = 13
The range of f is the set of all $f(n)$, where $n \in A$.
Therefore,
Range of f = {3, 5, 11, 13}