Miscellaneous Exercise

1. Evaluate: $\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$

Solution:

$$\left[i^{18}+\left(\frac{1}{i}\right)^{25}\right]^3$$

$$=\left[i^{4\times4+2}+\frac{1}{i^{4\times6+1}}\right]^3$$

$$=\left[\left(i^4\right)^4\cdot i^2+\frac{1}{\left(i^4\right)^6\cdot i}\right]^3$$

$$=\left[i^2+\frac{1}{i}\right]^3 \qquad \left[i^4=1\right]$$

$$=\left[-1+\frac{1}{i}\times\frac{i}{i}\right]^3 \qquad \left[i^2=-1\right]$$

$$=\left[-1+\frac{i}{i^2}\right]^3$$

$$=\left[-1-i\right]^3$$

$$=(-1)^3\left[1+i\right]^3$$

$$=-\left[1^3+i^3+3\cdot1\cdot i(1+i)\right]$$

$$=-\left[1+i^3+3i+3i^2\right]$$

$$=-\left[1-i+3i-3\right]$$

$$=-\left[-2+2i\right]$$

$$=2-2i$$

2. For any two complex numbers z_1 and z_2, prove that
Re (z_1z_2) = Re z_1 Re z_2 – Im z_1 Im z_2
Solution:

Lets's assume $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as two complex numbers

Product of these complex numbers, z_1z_2

$$z_1z_2 = (x_1 + iy_1)(x_2 + iy_2)$$
$$= x_1(x_2 + iy_2) + iy_1(x_2 + iy_2)$$
$$= x_1x_2 + ix_1y_2 + iy_1x_2 + i^2y_1y_2$$
$$= x_1x_2 + ix_1y_2 + iy_1x_2 - y_1y_2 \qquad [i^2 = -1]$$
$$= (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)$$

Now,

$$\text{Re}(z_1z_2) = x_1x_2 - y_1y_2$$
$$\Rightarrow \text{Re}(z_1z_2) = \text{Re}\, z_1 \,\text{Re}\, z_2 - \text{Im}\, z_1 \,\text{Im}\, z_2$$

Hence, proved.

3. Reduce $\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right)$ to the standard form

Solution:

$$\left(\frac{1}{1-4i} - \frac{2}{1+i}\right)\left(\frac{3-4i}{5+i}\right) = \left[\frac{(1+i)-2(1-4i)}{(1-4i)(1+i)}\right]\left[\frac{3-4i}{5+i}\right]$$
$$= \left[\frac{1+i-2+8i}{1+i-4i-4i^2}\right]\left[\frac{3-4i}{5+i}\right] = \left[\frac{-1+9i}{5-3i}\right]\left[\frac{3-4i}{5+i}\right]$$
$$= \left[\frac{-3+4i+27i-36i^2}{25+5i-15i-3i^2}\right] = \frac{33+31i}{28-10i} = \frac{33+31i}{2(14-5i)}$$

$$= \frac{(33+31i)}{2(14-5i)} \times \frac{(14+5i)}{(14+5i)} \qquad [\text{On multiplying numerator and denominator by } (14 + 5i)]$$
$$= \frac{462+165i+434i+155i^2}{2\left[(14)^2-(5i)^2\right]} = \frac{307+599i}{2(196-25i^2)}$$
$$= \frac{307+599i}{2(221)} = \frac{307+599i}{442} = \frac{307}{442} + \frac{599i}{442}$$

Hence, this is the required standard form.

4. If $x - iy = \sqrt{\frac{a-ib}{c-id}}$ prove that $(x^2+y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$.

Solution:

Given,

$$x - iy = \sqrt{\frac{a - ib}{c - id}}$$

$$= \sqrt{\frac{a - ib}{c - id} \times \frac{c + id}{c + id}} \left[\text{On multiplying numerator and denominator by } (c + id)\right]$$

$$= \sqrt{\frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}}$$

So,

$$(x - iy)^2 = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

$$x^2 - y^2 - 2ixy = \frac{(ac + bd) + i(ad - bc)}{c^2 + d^2}$$

On comparing real and imaginary parts, we get

$$x^2 - y^2 = \frac{ac + bd}{c^2 + d^2}, \quad -2xy = \frac{ad - bc}{c^2 + d^2} \qquad (1)$$

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2$$

$$= \left(\frac{ac + bd}{c^2 + d^2}\right)^2 + \left(\frac{ad - bc}{c^2 + d^2}\right)^2 \qquad [\text{Using } (1)]$$

$$= \frac{a^2c^2 + b^2d^2 + 2acbd + a^2d^2 + b^2c^2 - 2adbc}{(c^2 + d^2)^2}$$

$$= \frac{a^2c^2 + b^2d^2 + a^2d^2 + b^2c^2}{(c^2 + d^2)^2}$$

$$= \frac{a^2(c^2 + d^2) + b^2(c^2 + d^2)}{(c^2 + d^2)^2}$$

$$= \frac{(c^2 + d^2)(a^2 + b^2)}{(c^2 + d^2)^2}$$

$$= \frac{a^2 + b^2}{c^2 + d^2}$$

- Hence Proved

5. Convert the following in the polar form:

(i) $\frac{1+7i}{(2-i)^2}$, **(ii)** $\frac{1+3i}{1-2i}$

Solution:

(i) Here, $z=\frac{1+7i}{(2-i)^2}$

$$=\frac{1+7i}{(2-i)^2}=\frac{1+7i}{4+i^2-4i}=\frac{1+7i}{4-1-4i}$$

$$=\frac{1+7i}{3-4i}\times\frac{3+4i}{3+4i}=\frac{3+4i+21i+28i^2}{3^2+4^2}$$ [Multiplying by its conjugate in the numerator and denominator]

$$=\frac{3+4i+21i-28}{3^2+4^2}=\frac{-25+25i}{25}$$

$$=-1+i$$

Let $r\cos\theta=-1$ and $r\sin\theta=1$

On squaring and adding, we get

$r^2(\cos^2\theta+\sin^2\theta)=1+1$

$r^2(\cos^2\theta+\sin^2\theta)=2$

$r^2=2$ $[\cos^2\theta+\sin^2\theta=1]$

$r=\sqrt{2}$ [Conventionally, $r>0$]

So,

$\sqrt{2}\cos\theta=-1$ and $\sqrt{2}\sin\theta=1$

$\Rightarrow\cos\theta=\frac{-1}{\sqrt{2}}$ and $\sin\theta=\frac{1}{\sqrt{2}}$

$\therefore\theta=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$ [As θ lies in II quadrant]

Expressing as, $z=r\cos\theta+i\,r\sin\theta$

$$=\sqrt{2}\cos\frac{3\pi}{4}+i\sqrt{2}\sin\frac{3\pi}{4}=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

Therefore, this is the required polar form.

(ii) Let, $z = \frac{1+3i}{1-2i}$

$$= \frac{1+3i}{1-2i} \times \frac{1+2i}{1+2i}$$

$$= \frac{1+2i+3i-6}{1+4}$$

$$= \frac{-5+5i}{5} = -1+i$$

Now,

Let $r \cos \theta = -1$ and $r \sin \theta = 1$

On squaring and adding, we get

$r^2 (\cos^2 \theta + \sin^2 \theta) = 1 + 1$

$r^2 (\cos^2 \theta + \sin^2 \theta) = 2$

$r^2 = 2$ $\quad [\cos^2 \theta + \sin^2 \theta = 1]$

$\Rightarrow r = \sqrt{2}$ $\quad$ [Conventionally, $r > 0$]

$\therefore \sqrt{2} \cos\theta = -1$ and $\sqrt{2} \sin\theta = 1$

$$\cos\theta = \frac{-1}{\sqrt{2}} \text{ and } \sin\theta = \frac{1}{\sqrt{2}}$$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ $\quad$ [As θ lies in II quadrant]

Expressing as, $z = r \cos \theta + i\, r \sin \theta$

$$z = \sqrt{2} \cos\frac{3\pi}{4} + i\sqrt{2} \sin\frac{3\pi}{4} = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

Therefore, this is the required polar form.

Solve each of the equation in Exercises 6 to 9.

6. $3x^2 - 4x + 20/3 = 0$

Solution:

Given quadratic equation, $\mathbf{3x^2 - 4x + 20/3 = 0}$

It can be re-written as: $9x^2 - 12x + 20 = 0$

On comparing it with $ax^2 + bx + c = 0$, we get

$a = 9$, $b = -12$, and $c = 20$

So, the discriminant of the given equation will be

$D = b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$

Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-12) \pm \sqrt{-576}}{2 \times 9} = \frac{12 \pm \sqrt{576}\, i}{18}$$

$$= \frac{12 \pm 24i}{18} = \frac{6(2 \pm 4i)}{18} = \frac{2 \pm 4i}{3} = \frac{2}{3} \pm \frac{4}{3} i$$

7. $x^2 – 2x + 3/2 = 0$
Solution:

Given quadratic equation, $x^2 – 2x + 3/2 = 0$
It can be re-written as $2x^2 – 4x + 3 = 0$
On comparing it with $ax^2 + bx + c = 0$, we get
$a = 2$, $b = –4$, and $c = 3$
So, the discriminant of the given equation will be
$D = b^2 – 4ac = (–4)^2 – 4 \times 2 \times 3 = 16 – 24 = –8$
Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-4) \pm \sqrt{-8}}{2 \times 2} = \frac{4 \pm 2\sqrt{2}\, i}{4} \qquad \left[\sqrt{-1} = i\right]$$

$$= \frac{2 \pm \sqrt{2}\, i}{2} = 1 \pm \frac{\sqrt{2}}{2} i$$

8. $27x^2 – 10x + 1 = 0$
Solution:

Given quadratic equation, $27x^2 – 10x + 1 = 0$
On comparing it with $ax^2 + bx + c = 0$, we get
$a = 27$, $b = –10$, and $c = 1$
So, the discriminant of the given equation will be
$D = b^2 – 4ac = (–10)^2 – 4 \times 27 \times 1 = 100 – 108 = –8$
Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}\, i}{54}$$

$$= \frac{5 \pm \sqrt{2}\, i}{27} = \frac{5}{27} \pm \frac{\sqrt{2}}{27} i$$

9. $21x^2 – 28x + 10 = 0$
Solution:

Given quadratic equation, $21x^2 – 28x + 10 = 0$
On comparing it with $ax^2 + bx + c = 0$, we have
$a = 21$, $b = –28$, and $c = 10$
So, the discriminant of the given equation will be
$D = b^2 – 4ac = (–28)^2 – 4 \times 21 \times 10 = 784 – 840 = –56$
Hence, the required solutions are

$$x = \frac{-b \pm \sqrt{D}}{2a} = \frac{-(-28) \pm \sqrt{-56}}{2 \times 21} = \frac{28 \pm \sqrt{56}\, i}{42}$$

$$= \frac{28 \pm 2\sqrt{14}\, i}{42} = \frac{28}{42} \pm \frac{2\sqrt{14}}{42} i = \frac{2}{3} \pm \frac{\sqrt{14}}{21} i$$

10. If $z_1 = 2 - i$, $z_2 = 1 + i$, find $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$

Solution:

Given, $z_1 = 2 - i$, $z_2 = 1 + i$

So,

$$\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right| = \left|\frac{(2-i)+(1+i)+1}{(2-i)-(1+i)+1}\right|$$

$$= \left|\frac{4}{2-2i}\right| = \left|\frac{4}{2(1-i)}\right|$$

$$= \left|\frac{2}{1-i} \times \frac{1+i}{1+i}\right| = \left|\frac{2(1+i)}{1^2 - i^2}\right|$$

$$= \left|\frac{2(1+i)}{1+1}\right| \qquad [i^2 = -1]$$

$$= \left|\frac{2(1+i)}{2}\right|$$

$$= |1+i| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Hence, the value of $\left|\frac{z_1 + z_2 + 1}{z_1 - z_2 + 1}\right|$ is $\sqrt{2}$.

11. If $a + ib = \frac{(x+i)^2}{2x^2+1}$, prove that $a^2 + b^2 = \frac{(x^2+1)^2}{(2x^2+1)^2}$.

Solution:

Given,

$$a + ib = \frac{(x+i)^2}{2x^2+1}$$

$$= \frac{x^2 + i^2 + 2xi}{2x^2+1}$$

$$= \frac{x^2 - 1 + i2x}{2x^2+1}$$

$$= \frac{x^2-1}{2x^2+1} + i\left(\frac{2x}{2x^2+1}\right)$$

Comparing the real and imaginary parts, we have

$$a = \frac{x^2-1}{2x^2+1} \text{ and } b = \frac{2x}{2x^2+1}$$

$$\therefore a^2+b^2=\left(\frac{x^2-1}{2x^2+1}\right)^2+\left(\frac{2x}{2x^2+1}\right)^2$$

$$=\frac{x^4+1-2x^2+4x^2}{(2x+1)^2}$$

$$=\frac{x^4+1+2x^2}{(2x^2+1)^2}$$

$$=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

Hence proved,

$$a^2+b^2=\frac{(x^2+1)^2}{(2x^2+1)^2}$$

12. Let $z_1 = 2 - i$, $z_2 = -2 + i$. Find

(i) $\text{Re}\left(\frac{z_1 z_2}{\overline{z}_1}\right)$, **(ii)** $\text{Im}\left(\frac{1}{z_1\overline{z}_1}\right)$

Solution:

Given,

$z_1 = 2-i,\ z_2 = -2+i$

(i) $z_1z_2=(2-i)(-2+i)=-4+2i+2i-i^2=-4+4i-(-1)=-3+4i$

$\overline{z}_1 = 2+i$

$$\therefore \frac{z_1z_2}{\overline{z}_1}=\frac{-3+4i}{2+i}$$

On multiplying numerator and denominator by (2 - *i*), we get

$$\frac{z_1z_2}{\overline{z}_1}=\frac{(-3+4i)(2-i)}{(2+i)(2-i)}=\frac{-6+3i+8i-4i^2}{2^2+1^2}=\frac{-6+11i-4(-1)}{2^2+1^2}$$

$$=\frac{-2+11i}{5}=\frac{-2}{5}+\frac{11}{5}i$$

Comparing the real parts, we have

$$\text{Re}\left(\frac{z_1z_2}{\overline{z}_1}\right)=\frac{-2}{5}$$

(ii) $\frac{1}{z_1\bar{z}_1} = \frac{1}{(2-i)(2+i)} = \frac{1}{(2)^2+(1)^2} = \frac{1}{5}$

On comparing the imaginary part, we get

$\text{Im}\left(\frac{1}{z_1\bar{z}_1}\right) = 0$

13. Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$
Solution:

Let $z = \frac{1+2i}{1-3i}$, then

$z = \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2+3^2} = \frac{1+5i+6(-1)}{1+9}$

$= \frac{-5+5i}{10} = \frac{-5}{10} + \frac{5i}{10} = \frac{-1}{2} + \frac{1}{2}i$

Let $z = r\cos\theta + ir\sin\theta$

So,

$r\cos\theta = \frac{-1}{2}$ and $r\sin\theta = \frac{1}{2}$

On squaring and adding, we get

$r^2(\cos^2\theta + \sin^2\theta) = \left(\frac{-1}{2}\right)^2 + \left(\frac{1}{2}\right)^2$

$r^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$ [Conventionally, $r > 0$]

$r = \frac{1}{\sqrt{2}}$

Now,

$\frac{1}{\sqrt{2}}\cos\theta = \frac{-1}{2}$ and $\frac{1}{\sqrt{2}}\sin\theta = \frac{1}{2}$

$\Rightarrow \cos\theta = \frac{-1}{\sqrt{2}}$ and $\sin\theta = \frac{1}{\sqrt{2}}$

$\therefore \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$ [As θ lies in the II quadrant]

14. Find the real numbers *x* and *y* if $(x - iy)(3 + 5i)$ is the conjugate of $-6 - 24i$.
Solution:

Let's assume $z = (x - iy)(3 + 5i)$

$z = 3x + 5xi - 3yi - 5yi^2 = 3x + 5xi - 3yi + 5y = (3x + 5y) + i(5x - 3y)$

$\therefore \overline{z} = (3x + 5y) - i(5x - 3y)$

Also given, $\overline{z} = -6 - 24i$

And,
(3x + 5y) – *i*(5x – 3y) = -6 -24*i*
On equating real and imaginary parts, we have
3x + 5y = -6 …… (i)
5x – 3y = 24 …… (ii)
Performing (i) x 3 + (ii) x 5, we get
(9x + 15y) + (25x – 15y) = -18 + 120
34x = 102
x = 102/34 = 3
Putting the value of *x* in equation (i), we get
3(3) + 5y = -6
5y = -6 – 9 = -15
y = -3
Therefore, the values of *x* and *y* are 3 and –3 respectively.

15. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+i}$
Solution:

$$\frac{1+i}{1-i} - \frac{1-i}{1+i} = \frac{(1+i)^2 - (1-i)^2}{(1-i)(1+i)}$$
$$= \frac{1+i^2+2i-1-i^2+2i}{1^2+1^2}$$
$$= \frac{4i}{2} = 2i$$
$$\therefore \left|\frac{1+i}{1-i} - \frac{1-i}{1+i}\right| = |2i| = \sqrt{2^2} = 2$$

16. If $(x + iy)^3 = u + iv$, then show that $\frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$
Solution:

Given,
$(x+iy)^3 = u + iv$
$x^3 + (iy)^3 + 3 \cdot x \cdot iy(x+iy) = u + iv$
$x^3 + i^3y^3 + 3x^2yi + 3xy^2i^2 = u + iv$
$x^3 - iy^3 + 3x^2yi - 3xy^2 = u + iv$
$(x^3 - 3xy^2) + i(3x^2y - y^3) = u + iv$

On equating real and imaginary parts, we get

$u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

$$\frac{u}{x} + \frac{v}{y} = \frac{x^3 - 3xy^2}{x} + \frac{3x^2y - y^3}{y}$$

$$= \frac{x(x^2 - 3y^2)}{x} + \frac{y(3x^2 - y^2)}{y}$$

$$= x^2 - 3y^2 + 3x^2 - y^2$$

$$= 4x^2 - 4y^2$$

$$= 4(x^2 - y^2)$$

$$\therefore \frac{u}{x} + \frac{v}{y} = 4(x^2 - y^2)$$

Hence proved.

17. If α and β are different complex numbers with |β| = 1, then find $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$
Solution:

Let α = *a* + *ib* and β = *x* + *iy*

Given, $|\beta| = 1$

So, $\sqrt{x^2 + y^2} = 1$

$\Rightarrow x^2 + y^2 = 1 \quad \ldots \text{(i)}$

$$\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{(x + iy) - (a + ib)}{1 - (a - ib)(x + iy)}\right|$$

$$= \left|\frac{(x - a) + i(y - b)}{1 - (ax + aiy - ibx + by)}\right|$$

$$= \left|\frac{(x - a) + i(y - b)}{(1 - ax - by) + i(bx - ay)}\right|$$

$$= \frac{|(x - a) + i(y - b)|}{|(1 - ax - by) + i(bx - ay)|} \quad \left[\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$$

$$= \frac{\sqrt{(x - a)^2 + (y - b)^2}}{\sqrt{(1 - ax - by)^2 + (bx - ay)^2}}$$

$$= \frac{\sqrt{x^2 + a^2 - 2ax + y^2 + b^2 - 2by}}{\sqrt{1 + a^2x^2 + b^2y^2 - 2ax + 2abxy - 2by + b^2x^2 + a^2y^2 - 2abxy}}$$