Exercise 9.4

Find the sum to n terms of each of the series in Exercises 1 to 7.

1. 1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + …

Solution:

Given series is 1 × 2 + 2 × 3 + 3 × 4 + 4 × 5 + …

It's seen that,

n^{th} term, $a_n = n\,(n + 1)$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k(k+1)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left(\frac{2n+1}{3}+1\right)$$

$$= \frac{n(n+1)}{2}\left(\frac{2n+4}{3}\right)$$

$$= \frac{n(n+1)(n+2)}{3}$$

2. 1 × 2 × 3 + 2 × 3 × 4 + 3 × 4 × 5 + …

Solution:

Given series is 1 × 2 × 3 + 2 × 3 × 4 + 3 × 4 × 5 + …

It's seen that,

n^{th} term, $a_n = n\,(n + 1)\,(n + 2)$

$= (n^2 + n)\,(n + 2)$

$= n^3 + 3n^2 + 2n$

Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} k^3 + 3\sum_{k=1}^{n} k^2 + 2\sum_{k=1}^{n} k$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2}$$

$$= \left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{2} + n(n+1)$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + 2n + 1 + 2\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{n^2 + n + 4n + 6}{2}\right]$$

$$= \frac{n(n+1)}{4}\left(n^2 + 5n + 6\right)$$

$$= \frac{n(n+1)}{4}\left(n^2 + 2n + 3n + 6\right)$$

$$= \frac{n(n+1)\left[n(n+2) + 3(n+2)\right]}{4}$$

$$= \frac{n(n+1)(n+2)(n+3)}{4}$$

3. $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$
Solution:

Given series is $3 \times 1^2 + 5 \times 2^2 + 7 \times 3^2 + \ldots$
It's seen that,
n^{th} term, $a_n = (2n + 1) n^2 = 2n^3 + n^2$
Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k$$

$$= \sum_{k=1}^{n} = \left(2k^3 + k^2\right) = 2\sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} k^2$$

$$= 2\left[\frac{n(n+1)}{2}\right]^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n^2(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left[n(n+1) + \frac{2n+1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 3n + 2n + 1}{3}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2 + 5n + 1}{3}\right]$$

$$= \frac{n(n+1)\left(3n^2 + 5n + 1\right)}{6}$$

4. Find the sum to *n* terms of the series $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots$
Solution:

Given series is, $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots$
It's seen that,
n^{th} term, $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ (By partial fractions)

$$a_1 = \frac{1}{1} - \frac{1}{2}$$
$$a_2 = \frac{1}{2} - \frac{1}{3}$$
$$a_3 = \frac{1}{3} - \frac{1}{4} \ldots$$
$$a_n = \frac{1}{n} - \frac{1}{n+1}$$

On adding the above terms column wise, we get

$$a_1 + a_2 + \ldots + a_n = \left[\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{n}\right] - \left[\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots\frac{1}{n+1}\right]$$
$$\therefore S_n = 1 - \frac{1}{n+1} = \frac{n+1-1}{n+1} = \frac{n}{n+1}$$

5. Find the sum to *n* terms of the series $5^2 + 6^2 + 7^2 + \ldots + 20^2$
Solution:

Given series is $5^2 + 6^2 + 7^2 + \ldots + 20^2$
It's seen that,
n^{th} term, $a_n = (n + 4)^2 = n^2 + 8n + 16$
Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}\left(k^2 + 8k + 16\right)$$
$$= \sum_{k=1}^{n} k^2 + 8\sum_{k=1}^{n} k + \sum_{k=1}^{n} 16$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{8n(n+1)}{2} + 16n$$

Now, its found that
16^{th} term is $(16 + 4)^2 = 20^2$
Thus,

$$S_{16} = \frac{16(16+1)(2\times 16+1)}{6} + \frac{8\times 16\times(16+1)}{2} + 16\times 16$$

$$= \frac{(16)(17)(33)}{6} + \frac{(8)\times 16\times(16+1)}{2} + 16\times 16$$
$$= \frac{(16)(17)(33)}{6} + \frac{(8)(16)(17)}{2} + 256$$
$$= 1496 + 1088 + 256$$
$$= 2840$$

Hence, $5^2 + 6^2 + 7^2 + \ldots\ldots + 20^2 = 2840$

6. Find the sum to *n* terms of the series 3 × 8 + 6 × 11 + 9 × 14 +…
Solution:

Given series is 3 × 8 + 6 × 11 + 9 × 14 + …
It's found out that,
a_n = (n^{th} term of 3, 6, 9 …) × (n^{th} term of 8, 11, 14, …)
= $(3n)(3n + 5)$
= $9n^2 + 15n$
Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n}(9k^2 + 15k)$$
$$= 9\sum_{k=1}^{n} k^2 + 15\sum_{k=1}^{n} k$$
$$= 9\times\frac{n(n+1)(2n+1)}{6} + 15\times\frac{n(n+1)}{2}$$
$$= \frac{3n(n+1)(2n+1)}{2} + \frac{15n(n+1)}{2}$$
$$= \frac{3n(n+1)}{2}(2n+1+5)$$
$$= \frac{3n(n+1)}{2}(2n+6)$$
$$= 3n(n+1)(n+3)$$

7. Find the sum to *n* terms of the series $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$
Solution:

Given series is $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \ldots$
Finding the n^{th} term, we have
$a_n = (1^2 + 2^2 + 3^2 + \ldots\ldots + n^2)$

$$= \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n(2n^2+3n+1)}{6}$$
$$= \frac{2n^3+3n^2+n}{6}$$
$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n$$

Now, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k$$
$$= \sum_{k=1}^{n}\left(\frac{1}{3}k^3 + \frac{1}{2}k^2 + \frac{1}{6}k\right)$$
$$= \frac{1}{3}\sum_{k=1}^{n}k^3 + \frac{1}{2}\sum_{k=1}^{n}k^2 + \frac{1}{6}\sum_{k=1}^{n}k$$
$$= \frac{1}{3}\frac{n^2(n+1)^2}{(2)^2} + \frac{1}{2}\times\frac{n(n+1)(2n+1)}{6} + \frac{1}{6}\times\frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)}{2} + \frac{(2n+1)}{2} + \frac{1}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+1+1}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{n^2+n+2n+2}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{n(n+1)+2(n+1)}{2}\right]$$
$$= \frac{n(n+1)}{6}\left[\frac{(n+1)(n+2)}{2}\right]$$
$$= \frac{n(n+1)^2(n+2)}{12}$$

8. Find the sum to *n* terms of the series whose n^{th} term is given by *n* (*n* + 1) (*n* + 4).
Solution:

Given,
$a_n = n(n+1)(n+4) = n(n^2 + 5n + 4) = n^3 + 5n^2 + 4n$
Now, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$$

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+20n+10+24}{6}\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+23n+34}{6}\right]$$

$$= \frac{n(n+1)(3n^2+23n+34)}{12}$$

9. Find the sum to *n* terms of the series whose n^{th} terms is given by $n^2 + 2^n$
Solution:

Given,
n^{th} term of the series as:
$a_n = n^2 + 2^n$
Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} k^2 + 2^k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2^k \qquad (1)$$

Consider $\sum_{k=1}^{n} 2^k = 2^1 + 2^2 + 2^3 + \ldots$

The above series 2, 2^2, 2^3, … is a G.P. with both the first term and common ratio equal to 2.

$$\therefore \sum_{k=1}^{n} 2^k = \frac{(2)\left[(2)^n - 1\right]}{2-1} = 2(2^n - 1) \qquad (2)$$

Therefore, from (1) and (2), we obtain

$$S_n = \sum_{k=1}^{n} k^2 + 2(2^n - 1) = \frac{n(n+1)(2n+1)}{6} + 2(2^n - 1)$$

10. Find the sum to *n* terms of the series whose n^{th} terms is given by $(2n – 1)^2$
Solution:

Given,
n^{th} term of the series as:
$a_n = (2n - 1)^2 = 4n^2 - 4n + 1$
Then, the sum of n terms of the series can be expressed as

$$S_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} \left(4k^2 - 4k + 1\right)$$

$$= 4\sum_{k=1}^{n} k^2 - 4\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$$

$$= \frac{2n(n+1)(2n+1)}{3} - 2n(n+1) + n$$

$$= n\left[\frac{2\left(2n^2+3n+1\right)}{3} - 2(n+1) + 1\right]$$

$$= n\left[\frac{4n^2 + 6n + 2 - 6n - 6 + 3}{3}\right]$$

$$= n\left[\frac{4n^2 - 1}{3}\right]$$

$$= \frac{n(2n+1)(2n-1)}{3}$$