Miscellaneous Exercise

1. Show that the sum of $(m + n)^{th}$ and $(m – n)^{th}$ terms of an A.P. is equal to twice the m^{th} term.
Solution:

Let's take a and d to be the first term and the common difference of the A.P. respectively.
We know that, the k^{th} term of an A. P. is given by
$a_k = a + (k – 1) d$
So, $a_{m+n} = a + (m + n – 1) d$
And, $a_{m-n} = a + (m – n – 1) d$
$a_m = a + (m – 1) d$
Thus,
$a_{m+n} + a_{m-n} = a + (m + n – 1) d + a + (m – n – 1) d$
$= 2a + (m + n – 1 + m – n – 1) d$
$= 2a + (2m – 2) d$
$= 2a + 2 (m – 1) d$
$= 2 [a + (m – 1) d]$
$= 2a_m$
Therefore, the sum of $(m + n)^{th}$ and $(m – n)^{th}$ terms of an A.P. is equal to twice the m^{th} term

2. If the sum of three numbers in A.P., is 24 and their product is 440, find the numbers.
Solution:

Let's consider the three numbers in A.P. as $a – d$, a, and $a + d$.
Then, from the question we have
$(a – d) + (a) + (a + d) = 24$ … (i)
$3a = 24$
$\therefore a = 8$
And,
$(a – d) a (a + d) = 440$ … (ii)
$(8 – d) (8) (8 + d) = 440$
$(8 – d) (8 + d) = 55$
$64 – d^2 = 55$
$d^2 = 64 – 55 = 9$
$\therefore d = \pm 3$
Thus,
When $d = 3$, the numbers are 5, 8, and 11 and
When $d = –3$, the numbers are 11, 8, and 5.
Therefore, the three numbers are 5, 8, and 11.

3. Let the sum of n, $2n$, $3n$ terms of an A.P. be S_1, S_2 and S_3, respectively, show that $S_3 = 3 (S_2 – S_1)$
Solution:

Let's take a and d to be the first term and the common difference of the A.P. respectively.
So, we have

$$S_1 = \frac{n}{2}[2a+(n-1)d] \quad ...(1)$$

$$S_2 = \frac{2n}{2}\left[2a+(2n-1)d\right] = n\left[2a+(2n-1)d\right] \quad ...(2)$$

$$S_3 = \frac{3n}{2}\left[2a+(3n-1)d\right] \quad ...(3)$$

From (1) and (2), we get

$$S_2 - S_1 = n\left[2a+(2n-1)d\right] - \frac{n}{2}\left[2a+(n-1)d\right]$$

$$= n\left\{\frac{4a+4nd-2d-2a-nd+d}{2}\right\}$$

$$= n\left[\frac{2a+3nd-d}{2}\right]$$

$$= \frac{n}{2}[2a+(3n-1)d]$$

Now,

$$3(S_2 - S_1) = \frac{3n}{2}[2a+(3n-1)d] = S_3 \quad \text{[From (3)]}$$

Hence proved.

4. Find the sum of all numbers between 200 and 400 which are divisible by 7.
Solution:

First let's find the numbers between 200 and 400 which are divisible by 7.
The numbers are:
203, 210, 217, … 399
Here, the first term, $a = 203$
Last term, $l = 399$ and
Common difference, $d = 7$
Let's consider the number of terms of the A.P. to be n.
Hence, $a_n = 399 = a + (n-1)d$
$399 = 203 + (n-1) 7$
$7(n-1) = 196$
$n-1 = 28$
$n = 29$
Then, the sum of 29 terms of the A.P is given by:

$$\therefore S_{29} = \frac{29}{2}(203+399)$$

$$= \frac{29}{2}(602)$$

$$= (29)(301)$$

$$= 8729$$

Therefore, the required sum is 8729.

5. Find the sum of integers from 1 to 100 that are divisible by 2 or 5.
Solution:

First let's find the integers from 1 to 100, which are divisible by 2.
And, they are 2, 4, 6… 100.
Clearly, this forms an A.P. with the first term and common difference both equal to 2.
So, we have
100 = 2 + (n –1) 2
n = 50
Hence, the sum is

$$2+4+6+...+100=\frac{50}{2}\left[2(2)+(50-1)(2)\right]$$
$$=\frac{50}{2}[4+98]$$
$$=(25)(102)$$
$$=2550$$

Now, the integers from 1 to 100, which are divisible by 5, are 5, 10… 100.
This also forms an A.P. with the first term and common difference both equal to 5.
So, we have
100 = 5 + (n –1) 5
5n = 100
n = 20
Hence, the sum is

$$5+10+...+100=\frac{20}{2}\left[2(5)+(20-1)5\right]$$
$$=10\left[10+(19)5\right]$$
$$=10\left[10+95\right]=10\times105$$
$$=1050$$

Lastly, the integers which are divisible by both 2 and 5, are 10, 20, … 100.
And this also forms an A.P. with the first term and common difference both equal to 10.
So, we have
100 = 10 + (n –1) (10)
100 = 10n
n = 10

$$10+20+...+100=\frac{10}{2}\left[2(10)+(10-1)(10)\right]$$
$$=5\left[20+90\right]=5(110)=550$$

Thus, the required sum = 2550 + 1050 – 550 = 3050
Therefore, the sum of the integers from 1 to 100, which are divisible by 2 or 5, is 3050.

6. Find the sum of all two digit numbers which when divided by 4, yields 1 as remainder.
Solution:

We have to first find the two-digit numbers, which when divided by 4, yield 1 as remainder.
They are: 13, 17, … 97.
As it's seen that this series forms an A.P. with first term (a) 13 and common difference (d) 4.
Let n be the number of terms of the A.P.
We know that, the n^{th} term of an A.P. is given by,
$a_n = a + (n - 1) d$
So, $97 = 13 + (n - 1)(4)$
$4(n - 1) = 84$
$n - 1 = 21$
$n = 22$
Now, the sum of n terms of an A.P. is given by,

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right]$$

$$\therefore S_{22} = \frac{22}{2}\left[22(13) + (22-1)(4)\right]$$

$$= 11\left[26 + 84\right]$$

$$= 1210$$

Therefore, the required sum is 1210.

7. If *f* is a function satisfying *f(x + y) = f(x) f(y)* for all x, y ∈ N such that $f(1) = 3 \text{ and } \sum_{x=1}^{n} f(x) = 120$ **, find the value of *n*.**
Solution:

Given that,
$f(x + y) = f(x) \times f(y)$ for all $x, y \in N$ … (1)
$f(1) = 3$
Taking $x = y = 1$ in (1), we have
$f(1 + 1) = f(2) = f(1) f(1) = 3 \times 3 = 9$
Similarly,
$f(1 + 1 + 1) = f(3) = f(1 + 2) = f(1) f(2) = 3 \times 9 = 27$
And, $f(4) = f(1 + 3) = f(1) f(3) = 3 \times 27 = 81$
Thus, $f(1), f(2), f(3), …$, that is 3, 9, 27, …, forms a G.P. with the first term and common ratio both equal to 3.
We know that sum of terms in G.P is given by,

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

And it's given that,

$$\sum_{x=1}^{n} f(x) = 120$$

Hence, the sum of terms of the function is 120.

$$120 = \frac{3(3^n - 1)}{3 - 1}$$
$$120 = \frac{3}{2}(3^n - 1)$$
$$3^n - 1 = 80$$
$$3^n = 81 = 3^4$$
$$\therefore n = 4$$

Therefore, the value of n is 4.

8. The sum of some terms of G.P. is 315 whose first term and the common ratio are 5 and 2, respectively. Find the last term and the number of terms.
Solution:

Given that the sum of some terms in a G.P is 315.
Let the number of terms be n.
We know that, sum of terms is

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Given that the first term a is 5 and common ratio r is 2.

$$315 = \frac{5(2^n - 1)}{2 - 1}$$
$$2^n - 1 = 63$$
$$2^n = 64 = (2)^6$$
$$n = 6$$

Hence, the last term of the G.P = 6^{th} term = ar^{6-1} = $(5)(2)^5$ = (5)(32) = 160
Therefore, the last term of the G.P. is 160.

9. The first term of a G.P. is 1. The sum of the third term and fifth term is 90. Find the common ratio of G.P.
Solution:

Let's consider a and r to be the first term and the common ratio of the G.P. respectively.
Given, $a = 1$
$a_3 = ar^2 = r^2$
$a_5 = ar^4 = r^4$
Then, from the question we have
$r^2 + r^4 = 90$
$r^4 + r^2 - 90 = 0$

$$r^2 = \frac{-1 + \sqrt{1 + 360}}{2} = \frac{-1 \pm \sqrt{361}}{2} = \frac{-1 \pm 19}{2} = -10 \text{ or } 9$$
$$\therefore r = \pm 3 \qquad \text{(Taking real roots)}$$

Therefore, the common ratio of the G.P. is ±3.

10. The sum of three numbers in G.P. is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an arithmetic progression. Find the numbers.
Solution:

Let's consider the three numbers in G.P. to be as a, ar, and ar^2.
Then from the question, we have
$a + ar + ar^2 = 56$
$a(1 + r + r^2) = 56$
$\Rightarrow a = \frac{56}{1+r+r^2}$ … (1)
Also, given
$a - 1$, $ar - 7$, $ar^2 - 21$ forms an A.P.
So, $(ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$
$ar - a - 6 = ar^2 - ar - 14$
$ar^2 - 2ar + a = 8$
$ar^2 - ar - ar + a = 8$
$a(r^2 + 1 - 2r) = 8$
$a(r - 1)^2 = 8$ … (2)
$\Rightarrow \frac{56}{1+r+r^2}(r-1)^2 = 8$ [Using (1)]
$7(r^2 - 2r + 1) = 1 + r + r^2$
$7r^2 - 14r + 7 - 1 - r - r^2 = 0$
$6r^2 - 15r + 6 = 0$
$6r^2 - 12r - 3r + 6 = 0$
$6r(r - 2) - 3(r - 2) = 0$
$(6r - 3)(r - 2) = 0$
r = 2, 1/2
When $r = 2$, $a = 8$
When r = ½, a = 32
Thus,
When $r = 2$, the three numbers in G.P. are 8, 16, and 32.
When r = 1/2, the three numbers in G.P. are 32, 16, and 8.
Therefore in either case, the required three numbers are 8, 16, and 32.

11. A G.P. consists of an even number of terms. If the sum of all the terms is 5 times the sum of terms occupying odd places, then find its common ratio.
Solution:

Let's consider the terms in the G.P.to be $T_1, T_2, T_3, T_4, \dots T_{2n}$.
The number of terms = $2n$
Then, from the question we have
$T_1 + T_2 + T_3 + \dots + T_{2n} = 5[T_1 + T_3 + \dots + T_{2n-1}]$
$T_1 + T_2 + T_3 + \dots + T_{2n} - 5[T_1 + T_3 + \dots + T_{2n-1}] = 0$

$T_2 + T_4 + \ldots + T_{2n} = 4 [T_1 + T_3 + \ldots + T_{2n-1}]$ …… (1)
Now, let the terms in G.P. be $a, ar, ar^2, ar^3, \ldots$
Then (1) becomes,

$$\frac{ar(r^n - 1)}{r-1} = \frac{4 \times a(r^n - 1)}{r-1}$$ [Using sum of terms in G.P.]

ar = 4a
r = 4
Thus, the common ratio of the G.P. is 4.

12. The sum of the first four terms of an A.P. is 56. The sum of the last four terms is 112. If its first term is 11, then find the number of terms.
Solution:

Let's consider the terms in A.P. to be $a, a + d, a + 2d, a + 3d, \ldots a + (n - 2) d, a + (n - 1)d$.
From the question, we have
Sum of first four terms = $a + (a + d) + (a + 2d) + (a + 3d) = 4a + 6d$
Sum of last four terms = $[a + (n - 4) d] + [a + (n - 3) d] + [a + (n - 2) d] + [a + n - 1) d]$
$= 4a + (4n - 10) d$
Then according to the given condition,
$4a + 6d = 56$
$4(11) + 6d = 56$ [Since $a = 11$ (given)]
$6d = 12$
$d = 2$
Hence, $4a + (4n - 10) d = 112$
$4(11) + (4n - 10)2 = 112$
$(4n - 10)2 = 68$
$4n - 10 = 34$
$4n = 44$
$n = 11$
Therefore, the number of terms of the A.P. is 11.

13. If $\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx} = \frac{c+dx}{c-dx} (x \neq 0)$, then show that *a, b, c* and *d* are in G.P.
Solution:

Given,

$$\frac{a+bx}{a-bx} = \frac{b+cx}{b-cx}$$

On cross multiplying, we have

$(a+bx)(b-cx)=(b+cx)(a-bx)$

$ab-acx+b^2x-bcx^2=ab-b^2x+acx-bcx^2$

$2b^2x=2acx$

$b^2=ac$

$\frac{b}{a}=\frac{c}{b}$ ...(1)

Also, given $\frac{b+cx}{b-cx}=\frac{c+dx}{c-dx}$

On cross multiplying, we have

$(b+cx)(c-dx)=(b-cx)(c+dx)$

$bc-bdx+c^2x-cdx^2=bc+bdx-c^2x-cdx^2$

$2c^2x=2bdx$

$c^2=bd$

$\frac{c}{d}=\frac{d}{c}$ ...(2)

From (1) and (2), we get

b/a = c/b = d/c

Therefore, a, b, c and d are in G.P.

14. Let S be the sum, P the product and R the sum of reciprocals of n terms in a G.P. Prove that $P^2R^n = S^n$

Solution:

Let the terms in G.P. be $a, ar, ar^2, ar^3, \dots ar^{n-1}\dots$

Form the question, we have

$$S=\frac{a(r^n-1)}{r-1}$$

$$P=a^n\times r^{1+2+\dots+n-1}$$

$$=a^n\, r^{\frac{n(n-1)}{2}} \quad \left[\because \text{ Sum of first } n \text{ natural numbers is } n\frac{(n+1)}{2}\right]$$

$$R=\frac{1}{a}+\frac{1}{ar}+\dots+\frac{1}{ar^{n-1}}$$

$$=\frac{r^{n-1}+r^{n-2}+\dots r+1}{ar^{n-1}}$$

$$=\frac{1(r^n-1)}{(r-1)}\times\frac{1}{ar^{n-1}} \quad \left[\because 1, r, \dots r^{n-1} \text{ forms a G.P}\right]$$

$$= \frac{r^n - 1}{ar^{n-1}(r-1)}$$

$$\therefore P^2R^n = a^{2n}r^{n(n-1)}\frac{(r^n-1)^n}{a^n r^{n(n-1)}(r-1)^n}$$

$$= \frac{a^n(r^n-1)^n}{(r-1)^n}$$

$$= \left[\frac{a(r^n-1)}{(r-1)}\right]^n$$

$$= S^n$$

Hence, $P^2 R^n = S^n$

15. The p^{th}, q^{th} and r^{th} terms of an A.P. are *a, b, c* respectively.
Show that (q - r) a + (r - p) b + (p -q) c = 0
Solution:

Let's assume *t* and *d* to be the first term and the common difference of the A.P. respectively.
Then the n^{th} term of the A.P. is given by, $a_n = t + (n - 1)d$
Thus,
$a_p = t + (p - 1)d = a$ … (1)
$a_q = t + (q - 1)d = b$ … (2)
$a_r = t + (r - 1)d = c$ … (3)
On subtracting equation (2) from (1), we get
$(p - 1 - q + 1)d = a - b$
$(p - q)d = a - b$

$$d = \frac{a-b}{p-q} \quad ...(4)$$

On subtracting equation (3) from (2), we get
$(q - 1 - r + 1)d = b - c$
$(q - r)d = b - c$

$$d = \frac{b-c}{q-r} \quad ...(5)$$

Equating both the values of *d* obtained in (4) and (5), we get

$$\frac{a-b}{p-q} = \frac{b-c}{q-r}$$

$$(a-b)(q-r) = (b-c)(p-q)$$

$$aq - bq - ar + br = bp - bq - cp + cq$$

$$bp - cp + cq - aq + ar - br = 0$$

$$(-aq + ar) + (bp - br) + (-cp + cq) = 0 \qquad \text{(By rearranging terms)}$$

-a (q - r) – b (r - p) - c (p - q) = 0
a (q - r) + b (r - p) + c (p - q) = 0
Therefore, the given result is proved.

16. If $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ **are in A.P., prove that *a, b, c* are in A.P.**
Solution:

Given, $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$ are in A.P.

$$b\left(\frac{1}{c}+\frac{1}{a}\right)-a\left(\frac{1}{b}+\frac{1}{c}\right)=c\left(\frac{1}{a}+\frac{1}{b}\right)-b\left(\frac{1}{c}+\frac{1}{a}\right)$$

$$\frac{b(a+c)}{ac}-\frac{a(b+c)}{bc}=\frac{c(a+b)}{ab}-\frac{b(a+c)}{ac}$$

$$\frac{b^2a+b^2c-a^2b-a^2c}{abc}=\frac{c^2a+c^2b-b^2a-b^2c}{abc}$$

$$b^2a-a^2b+b^2c-a^2c=c^2a-b^2a+c^2b-b^2c$$

$$ab(b-a)+c(b^2-a^2)=a(c^2-b^2)+bc(c-b)$$

$$ab(b-a)+c(b-a)(b+a)=a(c-b)(c+b)+bc(c-b)$$

$$(b-a)(ab+cb+ca)=(c-b)(ac+ab+bc)$$

$$b-a=c-b$$

Therefore, a, b and c are in A.P.

17. If *a, b, c, d* are in G.P, prove that $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P.
Solution:

Given, *a*, *b*, *c*,and *d* are in G.P.
So, we have
$\therefore b^2 = ac$ … (i)
$c^2 = bd$ … (ii)
$ad = bc$ … (iii)
Required to prove $(a^n + b^n), (b^n + c^n), (c^n + d^n)$ are in G.P. i.e.,
$(b^n + c^n)^2 = (a^n + b^n)(c^n + d^n)$
Taking L.H.S.
$(b^n + c^n)^2 = b^{2n} + 2b^nc^n + c^{2n}$
$= (b^2)^n + 2b^nc^n + (c^2)^n$
$= (ac)^n + 2b^nc^n + (bd)^n$ [Using (i) and (ii)]
$= a^n c^n + b^nc^n + b^n c^n + b^n d^n$
$= a^n c^n + b^nc^n + a^n d^n + b^n d^n$ [Using (iii)]
$= c^n (a^n + b^n) + d^n (a^n + b^n)$
$= (a^n + b^n)(c^n + d^n)$

= R.H.S.
Therefore, $(a^n + b^n)$, $(b^n + c^n)$, and $(c^n + d^n)$ are in G.P
- Hence proved.

18. If a and b are the roots of $x^2 – 3x + p = 0$ and c, dare roots of $x^2 – 12x + q = 0$, where $a, b, c, d,$ form a G.P. Prove that $(q + p): (q – p) = 17:15$.
Solution:

Given, a and b are the roots of $x^2 – 3x + p = 0$
So, we have $a + b = 3$ and $ab = p$ … (i)
Also, c and d are the roots of $x^2 – 12x + q = 0$
So, $c + d = 12$ and $cd = q$ … (ii)
And given a, b, c, d are in G.P.
Let's take $a = x$, $b = xr$, $c = xr^2$, $d = xr^3$
From (i) and (ii), we get
$x + xr = 3$
$x (1 + r) = 3$
And,
$xr^2 + xr^3 =12$
$xr^2 (1 + r) = 12$
On dividing, we get

$$\frac{xr^2(1+r)}{x(1+r)} = \frac{12}{3}$$
$$r^2 = 4$$
$$r = \pm 2$$

When r = 2, x = 3/(1 + 2) = 3/3 = 1
When r = -2, x = 3/(1 - 2) = 3/-1 = -3
Case I:
When $r = 2$ and $x =1$,
$ab = x^2r = 2$
$cd = x^2r^5 = 32$

$$\frac{q+p}{q-p} = \frac{32+2}{32-2} = \frac{34}{30} = \frac{17}{15}$$
$$(q+p):(q-p) = 17:15$$

Case II:
When $r = –2$, $x = –3$,
$ab = x^2r = –18$
$cd = x^2r^5 = – 288$

$$\frac{q+p}{q-p} = \frac{-288-18}{-288+18} = \frac{-306}{-270} = \frac{17}{15}$$
$$(q+p):(q-p) = 17:15$$

Therefore, in both the cases, we get $(q + p): (q – p) = 17:15$\

19. The ratio of the A.M and G.M. of two positive numbers *a* and *b*, is *m*: *n*. Show that $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$**.**

Solution:

Let the two numbers be *a* and *b*.

A.M = (a + b)/ 2 and G.M. = √ab

From the question, we have

$$\frac{a+b}{2\sqrt{ab}}=\frac{m}{n}$$

$$\frac{(a+b)^2}{4(ab)}=\frac{m^2}{n^2}$$

$$(a+b)^2=\frac{4ab\,m^2}{n^2}$$

$$(a+b)=\frac{2\sqrt{ab}\,m}{n} \qquad ...(1)$$

By using this in identity $(a - b)^2 = (a + b)^2 - 4ab$, we get

$$(a-b)^2=\frac{4ab\,m^2}{n^2}-4ab=\frac{4ab\left(m^2-n^2\right)}{n^2}$$

$$(a-b)=\frac{2\sqrt{ab}\sqrt{m^2-n^2}}{n} \qquad ...(2)$$

Adding (1) and (2), we get

$$2a=\frac{2\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$a=\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

Substituting the value of a in (1), we get

$$b=\frac{2\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)$$

$$=\frac{\sqrt{ab}}{n}m-\frac{\sqrt{ab}}{n}\sqrt{m^2-n^2}$$

$$=\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)$$

$$\therefore a:b=\frac{a}{b}=\frac{\frac{\sqrt{ab}}{n}\left(m+\sqrt{m^2-n^2}\right)}{\frac{\sqrt{ab}}{n}\left(m-\sqrt{m^2-n^2}\right)}=\frac{\left(m+\sqrt{m^2-n^2}\right)}{\left(m-\sqrt{m^2-n^2}\right)}$$

Therefore, $a:b=\left(m+\sqrt{m^2-n^2}\right):\left(m-\sqrt{m^2-n^2}\right)$

20. If *a, b, c* are in A.P,; *b, c, d* are in G.P and 1/c, 1/d, 1/e are in A.P. prove that *a, c, e* are in G.P.
Solution:

Given a, b, c are in A.P.
Hence, $b - a = c - b$ … (1)
And, given that b, c, d are in G.P.
So, $c^2 = bd$ … (2)
Also, 1/c, 1/d, 1/e are in A.P.
So,

$$\frac{1}{d}-\frac{1}{c}=\frac{1}{e}-\frac{1}{d}$$

$$\frac{2}{d}=\frac{1}{c}+\frac{1}{e} \qquad ...(3)$$

Now, required to prove that a, c, e are in G.P. i.e., $c^2 = ae$
From (1), we have
$2b = a + c$
$b = (a + c)/ 2$
And from (2), we have
$d = c^2/ b$
On substituting these values in (3), we get

$$\frac{2b}{c^2}=\frac{1}{c}+\frac{1}{e}$$

$$\frac{2(a+c)}{2c^2}=\frac{1}{c}+\frac{1}{e}$$

$$\frac{a+c}{c^2}=\frac{e+c}{ce}$$

$$\frac{a+c}{c}=\frac{e+c}{e}$$

$$(a+c)e=(e+c)c$$

$$ae+ce=ec+c^2$$

$$c^2=ae$$

Therefore, a, c, and e are in G.P.

21. Find the sum of the following series up to *n* terms:
(i) 5 + 55 + 555 + … **(ii) .6 + .66 + . 666 + …**
Solution:

(i) Given, 5 + 55 + 555 + …
Let S_n = 5 + 55 + 555 + ….. up to n terms

$$=\frac{5}{9}[9+99+999+\ldots\text{to n terms}]$$

$$=\frac{5}{9}\left[(10-1)+(10^2-1)+(10^3-1)+\ldots\text{to n terms}\right]$$

$$=\frac{5}{9}\left[(10+10^2+10^3+\ldots\text{n terms})-(1+1+\ldots\text{ n terms})\right]$$

$$=\frac{5}{9}\left[\frac{10(10^n-1)}{10-1}-n\right]$$

$$=\frac{5}{9}\left[\frac{10(10^n-1)}{9}-n\right]$$

$$=\frac{50}{81}(10^n-1)-\frac{5n}{9}$$

(ii) Given, .6 + .66 + . 666 + …
Let S_n = 06. + 0.66 + 0.666 + … up to n terms

$$=6[0.1+0.11+0.111+\ldots\text{to n terms}]$$

$$=\frac{6}{9}[0.9+0.99+0.999+\ldots\text{to n terms}]$$

$$=\frac{6}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{10^2}\right)+\left(1-\frac{1}{10^3}\right)+\ldots\text{to n terms}\right]$$

$$=\frac{2}{3}\left[(1+1+\ldots\text{n terms})-\frac{1}{10}\left(1+\frac{1}{10}+\frac{1}{10^2}+\ldots\text{n terms}\right)\right]$$

$$=\frac{2}{3}\left[n-\frac{1}{10}\left(\frac{1-\left(\frac{1}{10}\right)^n}{1-\frac{1}{10}}\right)\right]$$

$$=\frac{2}{3}n-\frac{2}{30}\times\frac{10}{9}\left(1-10^{-n}\right)$$

$$=\frac{2}{3}n-\frac{2}{27}\left(1-10^{-n}\right)$$

22. Find the 20th term of the series 2 × 4 + 4 × 6 + 6 × 8 + … + n terms.
Solution:

Given series is 2 × 4 + 4 × 6 + 6 × 8 + … n terms
∴ n^{th} term = $a_n = 2n \times (2n + 2) = 4n^2 + 4n$

The 20th term,
$a_{20} = 4(20)^2 + 4(20) = 4(400) + 80 = 1600 + 80 = 1680$
Therefore, the 20th term of the series is 1680.

23. Find the sum of the first *n* terms of the series: 3 + 7 + 13 + 21 + 31 + …
Solution:

The given series is 3 + 7 + 13 + 21 + 31 + …
$S = 3 + 7 + 13 + 21 + 31 + \ldots + a_{n-1} + a_n$
$S = 3 + 7 + 13 + 21 + \ldots + a_{n-2} + a_{n-1} + a_n$
On subtracting both the equations, we get
$S - S = [3 + (7 + 13 + 21 + 31 + \ldots + a_{n-1} + a_n)] - [(3 + 7 + 13 + 21 + 31 + \ldots + a_{n-1}) + a_n]$
$S - S = 3 + [(7 - 3) + (13 - 7) + (21 - 13) + \ldots + (a_n - a_{n-1})] - a_n$
$0 = 3 + [4 + 6 + 8 + \ldots (n-1) \text{ terms}] - a_n$
$a_n = 3 + [4 + 6 + 8 + \ldots (n-1) \text{ terms}]$

$$\Rightarrow a_n = 3 + \left(\frac{n-1}{2}\right)\left[2 \times 4 + (n-1-1)2\right]$$
$$= 3 + \left(\frac{n-1}{2}\right)\left[8 + (n-2)2\right]$$
$$= 3 + \frac{(n-1)}{2}(2n+4)$$
$$= 3 + (n-1)(n+2)$$
$$= 3 + (n^2 + n - 2)$$
$$= n^2 + n + 1$$
$$\therefore \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$
$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$
$$= n\left[\frac{(n+1)(2n+1) + 3(n+1) + 6}{6}\right]$$
$$= n\left[\frac{2n^2 + 3n + 1 + 3n + 3 + 6}{6}\right]$$
$$= n\left[\frac{2n^2 + 6n + 10}{6}\right]$$
$$= \frac{n}{3}(n^2 + 3n + 5)$$

24. If S_1, S_2, S_3 are the sum of first *n* natural numbers, their squares and their cubes, respectively, show that $9S_2^2 = S_3(1 + 8S_1)$.

Solution:

From the question, we have

$$S_1 = \frac{n(n+1)}{2}$$

$$S_3 = \frac{n^2(n+1)^2}{4}$$

$$\text{Here, } S_3(1+8S_1) = \frac{n^2(n+1)^2}{4}\left[1+\frac{8n(n+1)}{2}\right]$$

$$= \frac{n^2(n+1)^2}{4}\left[1+4n^2+4n\right]$$

$$= \frac{n^2(n+1)^2}{4}(2n+1)^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \quad \ldots(1)$$

$$\text{Also, } 9S_2^2 = 9\frac{\left[n(n+1)(2n+1)\right]^2}{(6)^2}$$

$$= \frac{9}{36}\left[n(n+1)(2n+1)\right]^2$$

$$= \frac{\left[n(n+1)(2n+1)\right]^2}{4} \quad \ldots(2)$$

Therefore, from (1) and (2), we have $9S_2^2 = S_3(1 + 8S_1)$.

25. Find the sum of the following series up to *n* terms: $\frac{1^3}{1}+\frac{1^3+2^3}{1+3}+\frac{1^3+2^3+3^3}{1+3+5}+\ldots$

Solution:

The n^{th} term of the given series is $\frac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots+(2n-1)} = \frac{\left[\frac{n(n+1)}{2}\right]^2}{1+3+5+\ldots+(2n-1)}$

Here, $1, 3, 5, \ldots (2n-1)$ is an A.P. with first term a, last term $(2n-1)$ and number of terms as n

So,

$$1+3+5+\ldots\ldots+(2n-1) = \frac{n}{2}\left[2\times1+(n-1)2\right] = n^2$$

And,

$$a_n = \frac{n^2(n+1)^2}{4n^2} = \frac{(n+1)^2}{4} = \frac{1}{4}n^2+\frac{1}{2}n+\frac{1}{4}$$

Thus,

$$S_n = \sum_{K=1}^{n} a_K = \sum_{K=1}^{n}\left(\frac{1}{4}K^2 + \frac{1}{2}K + \frac{1}{4}\right)$$

$$= \frac{1}{4}\frac{n(n+1)(2n+1)}{6} + \frac{1}{2}\frac{n(n+1)}{2} + \frac{1}{4}n$$

$$= \frac{n\left[(n+1)(2n+1)+6(n+1)+6\right]}{24}$$

$$= \frac{n\left[2n^2+3n+1+6n+6+6\right]}{24}$$

$$= \frac{n\left(2n^2+9n+13\right)}{24}$$

26. Show that $\dfrac{1\times 2^2 + 2\times 3^2 + \ldots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots + n^2\times(n+1)} = \dfrac{3n+5}{3n+1}$

Solution:

n^{th} term of the numerator = $n(n + 1)^2 = n^3 + 2n^2 + n$
n^{th} term of the denominator = $n^2(n + 1) = n^3 + n^2$

$$\frac{1\times 2^2 + 2\times 3^2 + \ldots + n\times(n+1)^2}{1^2\times 2 + 2^2\times 3 + \ldots + n^2\times(n+1)} = \frac{\sum_{K=1}^{n} a_K}{\sum_{K=1}^{n} a_K} = \frac{\sum_{K=1}^{n}\left(K^3+2K^2+K\right)}{\sum_{K=1}^{n}\left(K^3+K^2\right)} \quad \ldots(1)$$

Here, $\sum_{K=1}^{n}\left(K^3+2K^2+K\right)$

$$= \frac{n^2(n+1)^2}{4} + \frac{2\,n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left[\frac{n(n+1)}{2} + \frac{2}{3}(2n+1) + 1\right]$$

$$= \frac{n(n+1)}{2}\left[\frac{3n^2+3n+8n+4+6}{6}\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2+11n+10\right]$$

$$= \frac{n(n+1)}{12}\left[3n^2+6n+5n+10\right]$$

$$= \frac{n(n+1)}{12}\left[3n(n+2)+5(n+2)\right]$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12} \quad \ldots(2)$$

$$\text{Also, } \sum_{K=1}^{n}\left(K^3+K^2\right)=\frac{n^2(n+1)^2}{4}+\frac{n(n+1)(2n+1)}{6}$$

$$=\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+\frac{2n+1}{3}\right]$$

$$=\frac{n(n+1)}{2}\left[\frac{3n^2+3n+4n+2}{6}\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+7n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n^2+6n+n+2\right]$$

$$=\frac{n(n+1)}{12}\left[3n(n+2)+1(n+2)\right]$$

$$=\frac{n(n+1)(n+2)(3n+1)}{12} \quad \ldots(3)$$

From (1), (2) and (3), we obtain

$$\frac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)}=\frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}}$$

$$=\frac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}=\frac{3n+5}{3n+1}$$

Hence proved.

27. A farmer buys a used tractor for Rs 12000. He pays Rs 6000 cash and agrees to pay the balance in annual installments of Rs 500 plus 12% interest on the unpaid amount. How much will be the tractor cost him?
Solution:

Given, the farmer pays Rs 6000 in cash.
So, the unpaid amount = Rs 12000 – Rs 6000 = Rs 6000
From the question, the interest paid annually will be
12% of 6000, 12% of 5500, 12% of 5000, …, 12% of 500
Hence, the total interest to be paid = 12% of 6000 + 12% of 5500 + 12% of 5000 + … + 12% of 500
= 12% of (6000 + 5500 + 5000 + … + 500)
= 12% of (500 + 1000 + 1500 + … + 6000)
It’s seen that, the series 500, 1000, 1500 … 6000 is an A.P. with the first term and common difference both equal to 500.
Let’s take the number of terms of the A.P. to be *n*.
So, 6000 = 500 + (*n* – 1) 500

$1 + (n - 1) = 12$
$n = 12$
Now,
The sum of the A.P = 12/2 [2(500) + (12 - 1)(500)] = 6 [1000 + 5500] = 6(6500) = 39000
Hence, the total interest to be paid = 12% of (500 + 1000 + 1500 + … + 6000)
= 12% of 39000 = Rs 4680
Therefore, the tractor will cost the farmer = (Rs 12000 + Rs 4680) = Rs 16680

28. Shamshad Ali buys a scooter for Rs 22000. He pays Rs 4000 cash and agrees to pay the balance in annual installment of Rs 1000 plus 10% interest on the unpaid amount. How much will the scooter cost him?
Solution:

Given, Shamshad Ali buys a scooter for Rs 22000 and pays Rs 4000 in cash.
So, the unpaid amount = Rs 22000 – Rs 4000 = Rs 18000
Form the question, it's understood that the interest paid annually is
10% of 18000, 10% of 17000, 10% of 16000 … 10% of 1000
Hence, the total interest to be paid = 10% of 18000 + 10% of 17000 + 10% of 16000 + … + 10% of 1000
= 10% of (18000 + 17000 + 16000 + … + 1000)
= 10% of (1000 + 2000 + 3000 + … + 18000)
It's seen that, 1000, 2000, 3000 … 18000 forms an A.P. with first term and common difference both equal to 1000.
Let's take the number of terms to be n.
So, $18000 = 1000 + (n - 1)(1000)$
$n = 18$
Now, the sum of the A.P is given by:

$$\therefore 1000 + 2000 + \ldots + 18000 = \frac{18}{2}\left[2(1000) + (18-1)(1000)\right]$$
$$= 9\left[2000 + 17000\right]$$
$$= 171000$$

Thus,
Total interest paid = 10% of (18000 + 17000 + 16000 + … + 1000)
= 10% of Rs 171000 = Rs 17100
Therefore, the cost of scooter = Rs 22000 + Rs 17100 = Rs 39100

29. A person writes a letter to four of his friends. He asks each one of them to copy the letter and mail to four different persons with instruction that they move the chain similarly. Assuming that the chain is not broken and that it costs 50 paise to mail one letter. Find the amount spent on the postage when 8^{th} set of letter is mailed.
Solution:

It's seen that,
The numbers of letters mailed forms a G.P.: $4, 4^2, \ldots 4^8$
Here, first term = 4 and common ratio = 4

And the number of terms = 8
The sum of n terms of a G.P. is given by:

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$\therefore S_8 = \frac{4(4^8 - 1)}{4 - 1} = \frac{4(65536 - 1)}{3} = \frac{4(65535)}{3} = 4(21845) = 87380$$

Also, given that the cost to mail one letter is 50 paisa.
Hence, Cost of mailing 87380 letters = Rs 87380 x (50/100) = Rs 43690 = Rs 43690
Therefore, the amount spent when 8^{th} set of letter is mailed will be Rs 43690.

30. A man deposited Rs 10000 in a bank at the rate of 5% simple interest annually. Find the amount in 15^{th} year since he deposited the amount and also calculate the total amount after 20 years.
Solution:

Given, the man deposited Rs 10000 in a bank at the rate of 5% simple interest annually.
Hence, the interest in first year = (5/100) x Rs 10000 = Rs 500
So, Amount in 15^{th} year = Rs $10000 + \underbrace{500 + 500 + ... + 500}_{14 \text{times}}$

= Rs 10000 + 14 × Rs 500
= Rs 10000 + Rs 7000
= Rs 17000

And, the amount after 20 years = $\text{Rs } 10000 + \underbrace{500 + 500 + ... + 500}_{20 \text{times}}$

= Rs 10000 + 20 × Rs 500
= Rs 10000 + Rs 10000
= Rs 20000

Therefore, the amount in the 15^{th} year is Rs 17000 and the total amount after 20 years will be Rs 20000.

31. A manufacturer reckons that the value of a machine, which costs him Rs 15625, will depreciate each year by 20%. Find the estimated value at the end of 5 years.
Solution:

Given, the cost of machine = Rs 15625
Also, given that the machine depreciates by 20% every year.
Hence, its value after every year is 80% of the original cost i.e., $4/5^{th}$ of the original cost.
Therefore, the value at the end of 5 years = $15625 \times \underbrace{\frac{4}{5} \times \frac{4}{5} \times .... \times \frac{4}{5}}_{5 \text{times}}$

= 5 × 1024 = 5120

Thus, the value of the machine at the end of 5 years will be Rs 5120.

32. 150 workers were engaged to finish a job in a certain number of days. 4 workers dropped out on second day, 4 more workers dropped out on third day and so on. It took 8 more days to finish the work. Find the number of days in which the work was completed.
Solution:

Let's assume x to be the number of days in which 150 workers finish the work.
Then from the question, we have
$150x = 150 + 146 + 142 + \ldots. (x + 8)$ terms
The series $150 + 146 + 142 + \ldots. (x + 8)$ terms is an A.P.
With first term (a) = 150, common difference (d) = –4 and number of terms (n) = $(x + 8)$
Now, finding the sum of terms:

$$150x = \frac{(x+8)}{2}\left[2(150)+(x+8-1)(-4)\right]$$
$$150x = (x+8)\left[150+(x+7)(-2)\right]$$
$$150x = (x+8)(150-2x-14)$$
$$150x = (x+8)(136-2x)$$
$$75x = (x+8)(68-x)$$
$$75x = 68x - x^2 + 544 - 8x$$
$$x^2 + 75x - 60x - 544 = 0$$
$$x^2 + 15x - 544 = 0$$
$$x^2 + 32x - 17x - 544 = 0$$
$$x(x+32) - 17(x+32) = 0$$
$$(x-17)(x+32) = 0$$
$$x = 17 \text{ or } x = -32$$

As x cannot be negative. [Number of days is always a positive quantity]
$x = 17$
Hence, the number of days in which the work should have been completed is 17.
But, due to the dropping out of workers the number of days in which the work is completed
= (17 + 8) = 25