EXERCISE 9.4 PAGE NO: 395

For each of the differential equations in Exercises 1 to 10, find the general solution:

1. $\frac{dy}{dx} = \frac{1-\cos x}{1+\cos x}$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \frac{1 - \cos x}{1 + \cos x}$$

We know that 1 – cos x = 2 $\sin^2$ (x/2) and 1 + cos x = 2 $\cos^2$ (x/2)

Using this formula in above function we get

$$\Rightarrow \frac{dy}{dx} = \frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}$$

We have sin x/cos x = tan x using this we get

$$\Rightarrow \frac{dy}{dx} = \tan^2\frac{x}{2}$$

From the identity $\tan^2 x = \text{sex}^2\, x - 1$, the above equation can be written as

$$\Rightarrow \frac{dy}{dx} = (\sec^2\frac{x}{2} - 1)$$

Now by rearranging and taking integrals on both sides we get

$$\Rightarrow \int dy = \int \sec^2\frac{x}{2}\, dx - \int dx$$

On integrating we get

$$\Rightarrow y = 2\tan^1\frac{x}{2} - x + c$$

2. $\frac{dy}{dx} = \sqrt{4-y^2}\ (-2 < y < 2)$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \sqrt{4 - y^2}$$

On rearranging we get

$$\Rightarrow \frac{dy}{\sqrt{4-y^2}} = dx$$

Now taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{\sqrt{4-y^2}} = \int dx$$

We know that,

$$\Rightarrow \int \frac{1}{\sqrt{a^2-x^2}} dx = \sin^{-1}\frac{x}{a}$$

Then above equation becomes

$$\Rightarrow \sin^{-1}\frac{y}{2} = x + c$$

3. $\frac{dy}{dx} + y = 1 \ (y \neq 1)$

Solution:

$$\Rightarrow \frac{dy}{dx} + y = 1$$

On rearranging we get

$\Rightarrow$ dy = (1 - y) dx

Separating variables by variable separable method we get

$$\Rightarrow \frac{dy}{1-y} = dx$$

Now by taking integrals on both sides we get

$$\Rightarrow \int \frac{dy}{1-y} = \int dx$$

On integrating

$\Rightarrow$ - log (1 - y) = x + log c

$\Rightarrow$ - log (1 - y) – log c = x

$\Rightarrow$ log (1 - y) c = -x

$$\Rightarrow (1-y)c = e^{-x}$$

Above equation can be written as

$\Rightarrow (1-y) = \frac{1}{c}e^{-x}$

$y = 1 + \frac{1}{c}e^{-x}$

$Y = 1 + A e^{-x}$

4. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$

Solution:

Given

$\Rightarrow \sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy$

Dividing both sides by (tan x) (tan y) we get

$\therefore \frac{\sec^2 x \tan y\, dx}{\tan x \tan y} + \frac{\sec^2 y \tan x\, dy}{\tan x \tan y} = 0$

On simplification we get

$\Rightarrow \frac{\sec^2 x\, dx}{\tan x} + \frac{\sec^2 y\, dy}{\tan y} = 0$

Integrating both sides,

$\Rightarrow \int \frac{\sec^2 x\, dx}{\tan x} = \int \frac{\sec^2 y\, dy}{\tan y}$

⇒ let tan x = t &tan y = u

Then

$\sec^2 x\, dx = dt \& \sec^2 y\, dy = du$

By substituting these in above equation we get

$\therefore \int \frac{dt}{t} = -\int \frac{du}{u}$

On integrating

⇒ log t = -log u + log c

Or,

⇒ log (tan x) = -log (tan y) + log c

$\Rightarrow \log \tan x = \log \frac{c}{\tan y}$

⇒ (tan x) (tan y) = c

5. $(e^x + e^{-x})\,dy - (e^x - e^{-x})\,dx = 0$

Solution:

Given

$\Rightarrow (e^x + e^{-x})dy - (e^x - e^{-x})dx = 0$

On rearranging the above equation we get

$$\Rightarrow dy = \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$$

Taking Integrals both sides,

$$\Rightarrow \int dy = \int \frac{(e^x - e^{-x})dx}{e^x + e^{-x}}$$

Now let $(e^x + e^{-x}) = t$

Then, $(e^x - e^{-x})dx = dt$

$$\therefore y = \int \frac{dt}{t}$$

On integrating

$$\because \int \frac{dx}{x} = \log x$$

So,

$\Rightarrow y = \log t$

Now by substituting the value of t we get

$\Rightarrow y = \log(e^x + e^{-x}) + C$

6. $\frac{dy}{dx} = (1 + x^2)(1 + y^2)$

Solution:

$$\Rightarrow \frac{dy}{dx} = (1 + x^2)(1 + y^2)$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dy}{1 + y^2} = dx(1 + x^2)$$

Now taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{1 + y^2} = \int dx + \int x^2 dx$$

On integrating we get

$$\Rightarrow \tan^{-1} y = x + \frac{x^3}{3} + c$$

7. $y \log y\, dx - x\, dy = 0$

Solution:

Given

y log y dx – x dy = 0

On rearranging we get

⇒ (y log y) dx = x dy

Separating variables by using variable separable method we get

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y\log y}$$

Now integrals on both sides,

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y\log y}$$

⇒ let logy = t

Then

$$\Rightarrow \frac{1}{y} dy = dt$$

$$\Rightarrow \log x = \int \frac{dt}{t}$$

⇒ Log x + log c = log t

Now by substituting the value of t

⇒ Log x + log c = log (log y)

Now by using logarithmic formulae we get

⇒ Log c x = log y

⇒ Log y = cx

$$\Rightarrow y = e^{cx}$$

8. $x^5 \frac{dy}{dx} = -y^5$

Solution:

Given

$$\Rightarrow x^5 \frac{dy}{dx} = -y^5$$

Separating variables by using variable separable method we get

$$\Rightarrow \frac{dy}{y^5} = \frac{-dx}{x^5}$$

On rearranging

$$\Rightarrow \frac{dy}{y^5} + \frac{dx}{x^5} = 0$$

Integrating both sides,

$$\Rightarrow \int \frac{dy}{y^5} + \int \frac{dx}{x^5} = a$$

Let a be a constant,

$$\Rightarrow \int y^{-5} dy + \int x^{-5} dx = a$$

On integrating we get

$$\Rightarrow -4y^{-4} - 4x^{-4} + c = a$$

On simplification we get

$$\Rightarrow -x^{-4} - y^{-4} = c$$

The above equation can be written as

$$\Rightarrow \frac{1}{x^4} + \frac{1}{y^4} = c$$

9. $\frac{dy}{dx} = \sin^{-1} x$

Solution:

Given

$$\Rightarrow \frac{dy}{dx} = \sin^{-1} x$$

Separating variables by using variable separable method we get

$\Rightarrow \ dy = \sin^{-1} x \, dx$

Taking integrals on both sides,

$\Rightarrow \int dy = \int \sin^{-1} x \ dx$

Now to integrate $\sin^{-1}x$ we have to multiply it by 1

to use product rule

$\int u.v \, dx = u \int v \ dx - \int \left(\frac{d}{dx} u\right) (\int vdx) dx\}$

Then we get

$\Rightarrow y = \int 1. \sin^{-1} x \, dx$

According to product rule and ILATE rule, the above equation can be written as

$\therefore y = \{\sin^{-1} x \int 1. dx - \int (\frac{d}{dx} \sin^{-1} x)(\int 1. dx) dx\}$

On integrating we get

$\Rightarrow y = x \sin^{-1} x - \int \frac{x}{\sqrt{1 - x^2}} dx$

Now

$\Rightarrow$ let $1 - x^2 = t$

Then

$\Rightarrow -2x \, dx = dt$

$\Rightarrow xdx = -\frac{dt}{2}$

Substituting these in above equation we get

$\Rightarrow y = x \sin^{-1} x + \int \frac{1}{2\sqrt{t}} dt$

On simplification above equation can be written as

$\Rightarrow y = x\sin^{-1} x + \frac{1}{2} \int t^{-\frac{1}{2}} dt$

$\Rightarrow y = x\sin^{-1} x + \frac{1}{2} \sqrt{t} + c$

Substituting the value of t, we get

$\Rightarrow y = x\sin^{-1} x + \sqrt{1 - x^2} + c$

10. $e^x \tan y\, dx + (1 - e^x) \sec^2 y\, dy = 0$

Solution:

Given

$\Rightarrow\ e^x \tan y\, dx + 1(1 - e^x) \sec^2 y\, dy = 0$

On rearranging above equation can be written as

$\Rightarrow\ (1 - e^x) \sec^2 y\, dy = -e^x \tan y\, dy = 0$

Separating the variables by using variable separable method,

$$\Rightarrow \frac{\sec^2 y}{\tan y} dy = -\frac{e^x}{1 - e^x} dx$$

Now by taking integrals on both sides, we get

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy = \int \frac{e^{-x}}{1 - e^x} dx$$

Let tan y = t and 1 – e^x = u

Then on differentiating

$(\sec^2 y\, dy = dt) \& (e^x dx = du)$

Substituting these in above equation we get

$$\therefore \int \frac{dt}{t} = \int \frac{du}{u}$$

On integrating we get

⇒ Log t = log u + log c

Substituting the values of t and u on above equation.

$\Rightarrow\ \log(\tan y) = \log(1 - e^x) + \log c$

$\Rightarrow\ \log \tan y = \log c(1 - e^x)$

By using logarithmic formula above equation can be written as

$\Rightarrow\ \tan y = c(1 - e^x)$

For each of the differential equations in Exercises 11 to 14, find a particular solution Satisfying the given condition:

11. $(x^3 + x^2 + x + 1)\dfrac{dy}{dx} = 2x^2 + x;\ y = 1$ when $x = 0$

Solution:

Given

$\Rightarrow (x^3 + x^2 + x + 1)\frac{dy}{dx} = 2x^2 + x$

Separating variables by using variable separable method,

$$\Rightarrow dy = \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx$$

Taking integrals on both sides, we get

$$\Rightarrow \int dy = \int \frac{2x^2 + x}{(x + 1)(x^2 + 1)}dx \quad ......1$$

Integrating it partially using partial fraction method,

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{A}{x + 1} + \frac{Bx + C}{x^2 + 1}$$

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{Ax^2 + A(Bx + C)(x + 1)}{(x + 1)(x^2 + 1)}$$

$$\Rightarrow 2x^2 + x = Ax^2 + A + Bx + Cx + C$$

$$\Rightarrow 2x^2 + x = (A + B)x^2 + (B + C)x + A + C$$

Now comparing the coefficients of x^2 and x

$\Rightarrow A + B = 2$

$\Rightarrow B + C = 1$

$\Rightarrow A + C = 0$

Solving them we will get the values of A, B, C

$$A = \frac{1}{2}, B = \frac{3}{2}, C = -\frac{1}{2}$$

Putting the values of A, B, C in 1 we get

$$\Rightarrow \frac{2x^2 + x}{(x + 1)(x^2 + 1)} = \frac{1}{2}\frac{1}{(x + 1)} + \frac{1}{2}\frac{3x - 1}{x^2 + 1}$$

Now taking integrals on both sides

$$\Rightarrow \int dy = \frac{1}{2}\int \frac{1}{x + 1}dx + \frac{1}{2}\int \frac{3x - 1}{x^2 + 1}dx$$

On integrating

$$\Rightarrow y = \frac{1}{2}\log(x + 1) + \frac{3}{2}\int \frac{x}{x^2 + 1}dx - \frac{1}{2}\int \frac{dx}{x^2 + 1}$$

$$\Rightarrow y = \frac{1}{2}\log(x + 1) + \frac{3}{4}\int \frac{2x}{x^2 + 1}dx - \frac{1}{2}\tan^{-1}x \quad ....2$$

For second term

let $x^2 + 1 = t$

Then, 2x dx = dt

$$\therefore \frac{3}{4}\int \frac{2x}{x^2+1}dx = \frac{3}{4}\int \frac{dt}{t}$$

$$\text{so}, I = \frac{3}{4}\log t$$

Substituting the value of t we get

$$I = \frac{3}{4}\log(x^2+1)$$

Then 2 becomes

$$\Rightarrow y = \frac{1}{2}\log(x+1) + \frac{3}{4}\log(x^2+1) - \frac{1}{2}\tan^{-1}x + c$$

Taking 4 common

$$\Rightarrow y = \frac{1}{4}[2\log(x+1) + 3\log(x^2+1)] - \frac{1}{2}\tan^{-1}x + c$$

$$\Rightarrow y = \frac{1}{4}[\log(x+1)^2 + \log(x^2+1)^3] - \frac{1}{2}\tan^{-1}x + c$$

$$\Rightarrow y = \frac{1}{4}[\log\{(x+1)^2(x^2+1)^3\}] - \frac{1}{2}\tan^{-1}x + c \quad ....3$$

Now, we are given that y = 1 when x = 0

$$\therefore 1 = \frac{1}{4}[\log\{(0+1)^2(0^2+1)\}] - \frac{1}{2}\tan^{-1}0 + c$$

$$1 = \frac{1}{4}\times 0 - \frac{1}{2}\times 0 + c$$

Therefore,

C = 1

Putting the value of c in 3 we get

$$y = \frac{1}{4}[\log\{(x+1)^2(x^2+1)^3\}] - \frac{1}{2}\tan^{-1}x + 1$$

12. $x(x^2-1)\frac{dy}{dx} = 1$; $y = 0$ when $x = 2$

Solution:

Given

$$x(x^2 + 1)\frac{dy}{dx} = 1$$

Separating variables by variable separable method,

$$\Rightarrow dy = \frac{dx}{x(x^2 + 1)}$$

$X^2 + 1$ can be written as $(x + 1)(x - 1)$ we get

$$\Rightarrow dy = \frac{dx}{x(x + 1)(x - 1)}$$

Taking integrals on both sides,

$$\Rightarrow \int dy = \int \frac{dx}{x(x + 1)(x - 1)} \quad .....1$$

Now by using partial fraction method,

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{A}{x} + \frac{B}{x + 1} + \frac{c}{x - 1} \quad .....2$$

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{A(x - 1)(x + 1) + B(x)(x - 1) + C(x)(x + 1)}{x(x + 1)(x - 1)}$$

Or

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = \frac{(A + B + C)x^2 + (B - C)x - A}{x(x + 1)(x - 1)}$$

Now comparing the values of A, B, C

A + B + C = 0

B-C = 0

A = -1

Solving these we will get that B = ½ and C = ½

Now putting the values of A, B, C in 2

$$\Rightarrow \frac{1}{x(x + 1)(x - 1)} = -\frac{1}{x} + \frac{1}{2}\left(\frac{1}{x + 1}\right) + \frac{1}{2}\left(\frac{1}{x - 1}\right)$$

Now taking integrals we get

$$\Rightarrow \int dy = -\int \frac{1}{x}dx + \frac{1}{2}\int \left(\frac{1}{x + 1}\right)dx + \frac{1}{2}\int \left(\frac{1}{x - 1}\right)dx$$

On integrating

$$\Rightarrow y = -\log x + \frac{1}{2}\log(x + 1) + \frac{1}{2}\log(x - 1) + \log c$$

$$\Rightarrow y = \frac{1}{2}\log\left[\frac{c^2(x-1)(x+1)}{x^2}\right\} \quad .....3$$

Now we are given that y = 0 when x = 2

$$0 = \frac{1}{2}\log\left[\frac{c^2(2-1)(2+1)}{4}\right\}$$

$$\Rightarrow \log\frac{3c^2}{4} = 0$$

We know $e^0 = 1$ by substituting we get

$$\Rightarrow \frac{3c^2}{4} = 1$$

$$\Rightarrow 3c^2 = 4$$

$$\Rightarrow c^2 = 4/3$$

Now putting the value of c^2 in 3

Then,

$$y = \frac{1}{2}\log\left[\frac{4(x-1)(x+1)}{3x^2}\right]$$

$$y = \frac{1}{2}\log\left[\frac{4(x^2-1)}{3x^2}\right]$$

13. $\cos\left(\frac{dy}{dx}\right) = a \quad (a \in \mathbf{R});\ y = 1 \text{ when } x = 0$

Solution:

Given

$$\cos\left(\frac{dy}{dx}\right) = a$$

On rearranging we get

$$\Rightarrow \frac{dy}{dx} = \cos^{-1} a$$

$dy = \cos^{-1} a\, dx$

Integrating both sides, we get

$$\int dy = \cos^{-1} a \int dx$$

$y = x \cos^{-1} a + C$ 1

Now y = 1 when x = 0

Then

$1 = 0 \cos^{-1} a + C$

Hence C = 1

Substituting C = 1 in equation (1), we get:

$y = x \cos^{-1} a + 1$

$(y - 1)/x = \cos^{-1} a$

$$\Rightarrow \cos\left(\frac{y-1}{x}\right) = a$$

14. $\frac{dy}{dx} = y \tan x$; $y = 1$ when $x = 0$

Solution:

Given

$$\frac{dy}{dx} = y \tan x$$

Separating variables by variable separable method,

$$\Rightarrow \frac{dy}{y} = \tan x \, dx$$

Taking Integrals both sides, we get

$$\Rightarrow \int \frac{dy}{y} = \int \tan x \, dx$$

On integrating

⇒ Log y = -log (cos x) + log c

Using standard trigonometric identity we get

⇒ Log y = log (sec x) + log c

Using logarithmic formula in above equation we get

⇒ Log y = log c (sec x)

⇒ y = c (sec x) 1

Now we are given that y = 1 when x = 0

⇒ 1 = c (sec 0)

⇒ 1 = c × 1

⇒ c = 1

Putting the value of c in 1

⇒ y = sec x

15. Find the equation of a curve passing through the point (0, 0) and whose differential equation is y′ = e^x sin x

Solution:

To find the equation of a curve that passes through point (0, 0) and has differential equation y′ = e^x sin x

So, we need to find the general solution of the given differential equation and the put the given point in to find the value of constant.

So, $\Rightarrow \frac{dy}{dx} = e^x \sin x$

Separating variables by variable separable method, we get

$\Rightarrow dy = e^x \sin x \ dx$

Integrating both sides,

$\Rightarrow \int dy = \int e^x \sin x \, dx$ 1

Now by using product rule we get

$\int u.v \, dx = u \int v \, dx - \int \{\frac{d}{dx} u \int v \, dx\} dx$

Now let

$I = \int e^x \sin x \, dx$

$\Rightarrow I = \sin x \int e^x dx - \int (\frac{d}{dx} \sin x \int e^x dx) dx$

$\Rightarrow I = e^x \sin x - \int \cos x \, e^x dx$

Now by integrating we get

$\Rightarrow I = e^x \sin x - [\cos x \int e^x dx + \int \sin x \, e^x dx]$

From 1 we have

$\Rightarrow I = e^x \sin x - e^x \cos x - I$

Now on simplifying

$\Rightarrow\ 2I = e^x \sin x - e^x \cos x$

$\Rightarrow\ 2I = e^x(\sin x - \cos x)$

$\Rightarrow\ I = e^x \dfrac{(\sin x - \cos x)}{2}$

Substituting I in 1 we get

$\Rightarrow\ y = e^x \dfrac{(\sin x - \cos x)}{2} + c$ 2

Now we are given that the curve passes through point (0, 0)

$\therefore 0 = e^0 \dfrac{(\sin 0 - \cos 0)}{2} + c$

$\Rightarrow\ 0 = \dfrac{1(0-1)}{2} + c$

$\Rightarrow\ c = \dfrac{1}{2}$

Substituting the value of C in 2

$\Rightarrow\ y = e^x \dfrac{(\sin x - \cos x)}{2} + \dfrac{1}{2}$

On rearranging

$\Rightarrow\ 2y = e^x(\sin x - \cos x) + 1$

Hence

$\Rightarrow\ 2y - 1 = e^x(\sin x - \cos x)$

16. For the differential equation $xy\dfrac{dy}{dx} = (x+2)(y+2)$

Find the solution curve passing through the point (1, –1).

Solution:

For this question, we need to find the particular solution at point (1,-1) for the given differential equation.

Given differential equation is

$\Rightarrow\ xy\dfrac{dy}{dx} = (x + 2)(y + 2)$

Separating variables by variable separable method, we get

$\Rightarrow\ \dfrac{y}{y + 2}dy = \dfrac{(x + 2)dx}{x}$

Taking Integrals both sides, we get

$$\Rightarrow \int\left(1-\frac{2}{y+2}\right)dy = \int\left(1+\frac{2}{x}\right)dx$$

Splitting the integrals

$$\Rightarrow \int dy - 2\int\frac{1}{y+2}dy = \int dx + 2\int\frac{1}{x}dx$$

$$\Rightarrow y-2\log(y+2) = x+2\log x + c \quad ...1$$

Now separating like terms on each side,

$$\Rightarrow y-x-c = 2\log x + 2\log(y+2)$$

$$\Rightarrow y-x-c = \log x^2 + \log(y+2)^2$$

Using logarithmic formula we get

$$\Rightarrow y-x-c = \log\{x^2(y+2)^2\} - i)$$

Now we are given that, the curve passes through (1, -1)

Substituting the values of x and y, to find the value of c

$$\Rightarrow -1-1-c = \log\{1(-1+2)^2\}$$

⇒ -2-c = log (1)

We know that log 1 0

⇒ c = -2 + 0

So c = -2

Substituting the value of c in 1

$$y-x-c = \log\{x^2(y+2)^2\}$$

$$y-x+2 = \log\{x^2(y+2)^2\}$$

17. Find the equation of a curve passing through the point (0, –2) given that at any point (*x*, *y*) on the curve, the product of the slope of its tangent and *y* coordinate of the point is equal to the *x* coordinate of the point.

Solution:

We know that slope of a tangent is = $\frac{dy}{dx}$.

So we are given that the product of the slope of its tangent and y coordinate of the point is equal to the x coordinate of the point.

$$y\frac{dy}{dx} = x$$

Now separating variables by variable separable method,

$\Rightarrow$ y dy = x dx

Taking integrals both sides,

$$\Rightarrow \int ydy = \int xdx$$

On integrating we get

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + c$$

$$\Rightarrow y^2 - x^2 = 2c \quad ...1$$

Now the curve passes through (0, -2).

∴ 4-0 = 2c

$\Rightarrow$ c = 2

Putting the value of c in 1 we get

$$\Rightarrow y^2 - x^2 = 4$$

18. At any point (*x*, *y*) of a curve, the slope of the tangent is twice the slope of the line segment joining the point of contact to the point (– 4, –3). Find the equation of the curve given that it passes through (–2, 1).

Solution:

We know that (x, y) is the point of contact of curve and its tangent.

Slope (m1) for line joining (x, y) and (-4, -3) is $\frac{y+3}{x+4}$......1

Also we know that slope of tangent of a curve is $\frac{dy}{dx}$.

$$\therefore \text{slope (m2) of tangent} = \frac{dy}{dx} \quad .....2$$

Now, according to the question, we can write as

(m2) = 2(m1)

$$\Rightarrow \frac{dy}{dx} = \frac{2(y+3)}{x+4}$$

Separating variables by variable separable method, we get

$$\Rightarrow \frac{dy}{y+3} = \frac{2dx}{x+4}$$

Taking integrals on both sides,

$$\Rightarrow \int \frac{dy}{y+3} = 2\int \frac{dx}{x+4}$$

On integrating we get

$\Rightarrow \log(y + 3) = 2\log(x + 4) + \log c$

Using logarithmic formula above equation can be written as

$$\Rightarrow \log(y+3) = \log c(x+4)^2$$

$$\Rightarrow y+3 = c(x+4)^2 \quad ....3$$

Now, this equation passes through the point (-2, 1).

$$\Rightarrow 1+3 = c(-2+4)^2$$

$\Rightarrow 4 = 4c$

$\Rightarrow c = 1$

Substitute the value of c in 3

$$\Rightarrow y+3 = (x+4)^2$$

19. The volume of spherical balloon being inflated changes at a constant rate. If initially its radius is 3 units and after 3 seconds it is 6 units. Find the radius of balloon after *t* seconds.

Solution:

Let the rate of change of the volume of the balloon be k where k is a constant

$$\therefore \frac{dy}{dt} = k$$

$$\frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \ \{\text{volume of sphere} = \frac{4}{3}\pi r^3\}$$

On differentiating with respect to r we get

$$\Rightarrow \frac{4}{3}\pi 3r^2 \frac{dr}{dt} = k$$

On rearranging

$$\Rightarrow 4\pi r^2 dr = kdt$$

Taking integrals on both sides,

$$\Rightarrow 4\pi \int r^2 dr = k\int dt$$

On integrating we get

$$\Rightarrow \frac{4\pi r^3}{3} = kt + c \quad ....1$$

Now, from the question we have

At t = 0, r = 3:

$\Rightarrow 4\pi \times 33 = 3(k \times 0 + c)$

$\Rightarrow 108\,\pi = 3c$

$\Rightarrow c = 36\pi$

At t = 3, r = 6:

$\Rightarrow\ 4\pi \times 6^3\ =\ 3(k \times 3\ +\ c)$

$\Rightarrow k = 84\,\pi$

Substituting the values of k and c in 1

$\Rightarrow\ 4\pi r^3\ =\ 3(84\pi t\ +\ 36\pi)$

$\Rightarrow\ 4\pi r^3\ =\ 4\pi(63t\ +\ 27)$

$\Rightarrow\ r^3\ =\ 63t\ +\ 27$

$\Rightarrow\ r\ =\ \sqrt[3]{63t\ +\ 27}$

So the radius of balloon after t seconds is $\sqrt[3]{63t\ +\ 27}$

20. In a bank, principal increases continuously at the rate of *r*% per year. Find the value of *r* if Rs 100 double itself in 10 years ($\log_e 2 = 0.6931$).

Solution:

Let t be time, p be principal and r be rate of interest

According the information principal increases at the rate of r% per year.

$\therefore \frac{dp}{dt}\ =\ \left(\frac{r}{100}\right)p$

Separating variables by variable separable method, we get

$\Rightarrow\ \frac{dp}{p}\ =\ \left(\frac{r}{100}\right)dt$

Taking integrals on both sides,

$\Rightarrow\ \int\frac{dp}{p}\ =\ \frac{r}{100}\int\ dt$

On integrating we get

$\Rightarrow\ \log p\ =\ \frac{rt}{100}\ +\ k$

$\Rightarrow\ p\ =\ e^{\frac{rt}{100}+k}$ 1

Given that t = 0, p = 100.

$\Rightarrow 100 = e^{k}$ 2

Now, if t = 10, then p = 2 × 100 = 200

So,

$\Rightarrow 200 = e^{\frac{rt}{10}+k}$

$\Rightarrow 200 = e^{\frac{rt}{10}}.e^{k}$

From 2

$\Rightarrow 200 = e^{\frac{rt}{10}} \times 100$

$\Rightarrow e^{\frac{r}{10}} = 2$

$\Rightarrow \frac{r}{10} = \log 2$

$\Rightarrow r = 6.93$

So r is 6.93%.

21. In a bank, principal increases continuously at the rate of 5% per year. An amount of Rs 1000 is deposited with this bank, how much will it worth after 10 years ($e^{0.5}$ = 1.648).

Solution:

Let p and t be principal and time respectively.

Given that principal increases continuously at rate of 5% per year.

$\therefore \frac{dp}{dt} = \left(\frac{5}{100}\right)p$

Separating variables by variable separable method,

$\Rightarrow \frac{dp}{p} = \frac{p}{25}$

Taking integrals on both sides,

$\Rightarrow \int \frac{dp}{p} = \frac{1}{20}\int dt$

$\Rightarrow \log p = e^{\frac{t}{20}+c}$ ...1

When t = 0, p = 1000

$\Rightarrow 1000 = e^{c}$

At t = 10

$\Rightarrow\ p = e^{\frac{1}{2}+c}$

The above equation can be written as

$\Rightarrow\ p = e^{0.5} \times e^{c}$

$\Rightarrow\ p = 1.648 \times 1000\ (e^{0.5} = 1.648)$

$\Rightarrow\ p = 1648$

So after 10 years the total amount would be Rs.1648

22. In a culture, the bacteria count is 1, 00,000. The number is increased by 10% in 2 hours. In how many hours will the count reach 2, 00,000, if the rate of growth of bacteria is proportional to the number present?

Solution:

Let y be the number of bacteria at any instant t.

Given that the rate of growth of bacteria is proportional to the number present

$\therefore \frac{dy}{dt} \propto y$

$\Rightarrow \frac{dy}{dt} = ky$ (k is a constant)

Separating variables by variable separable method we get,

$\Rightarrow \frac{dy}{dt} = kdt$

Taking integrals on both sides,

$\Rightarrow \int \frac{dy}{y} = k\int dt$

On integrating we get

$\Rightarrow$ Log y = k t + c…1

Let y' be the number of bacteria at t = 0.

$\Rightarrow$ Log y' = c

Substituting the value of c in 1

$\Rightarrow$ log y = k t + log y'

$\Rightarrow$ Log y- log y' = k t

Using logarithmic formula we get

$\Rightarrow \log\frac{y}{y'} = kt$ 2

Also, given that number of bacteria increases by 10% in 2 hours.

Therefore,

$\Rightarrow y = \frac{110}{100}y'$

$\Rightarrow \frac{y}{y'} = \frac{11}{10}$ 3

Substituting this value in 2, we get

$\Rightarrow k \times 2 = \log\frac{11}{10}$

$\Rightarrow k = \frac{1}{2}\log\frac{11}{10}$

So, 2 becomes

$\Rightarrow \frac{1}{2}\log\frac{11}{10} \times t = \log\frac{y}{y'}$

$\Rightarrow t = \frac{2\log\frac{y}{y'}}{\log\frac{11}{10}}$ ...4

Now, let the time when number of bacteria increase from 100000 to 200000 be t'.

$\Rightarrow y = 2y'$ at $t = t'$

So from 4, we have

$\Rightarrow t' = \frac{2\log\frac{y}{y'}}{\log\frac{11}{10}} = \frac{2\log 2}{\log\frac{11}{10}}$

So bacteria increases from 100000 to 200000 in $\frac{2\log 2}{\log\frac{11}{10}}$ hours.

23. The general solution of the differential equation $\frac{dy}{dx} = e^{x+y}$ is

(A) $e^x + e^{-y} = C$ (B) $e^x + e^y = C$

(C) $e^{-x} + e^y = C$ (D) $e^{-x} + e^{-y} = C$

Solution:

(A) $e^x + e^{-y} = C$

Explanation:

We have

$$\Rightarrow \frac{dy}{dx} = e^{x+y}$$

Using laws of exponents we get

$$\Rightarrow \frac{dy}{dx} = e^x \times e^y$$

Separating variables by variable separable method we get

$$\Rightarrow e^{-y}dy = e^x dx$$

Now taking integrals on both sides

$$\Rightarrow \int e^{-y}dy = \int e^x dx$$

On integrating

$$\Rightarrow -e^{-y} = e^x + c$$

$$\Rightarrow e^x + e^{-y} = -c$$

Or,

$$e^x + e^{-y} = c$$

So the correct option is A.