Polynomial

Basics Revisited

Algebraic Expressions

An algebraic expression is an expression made up of **variables** and **constants** along with mathematical operators.

An algebraic expression is a sum of terms, which are considered to be building blocks for expressions.

A **term** is a product of **variables** and **constants**. A term can be an algebraic expression in itself.

Examples of a term - 3 which is just a constant.

- $2x$, which is the product of constant '2' and the variable 'x'
- $4xy$, which is the product of the constant '4' and the variables 'x' and 'y'.
- $5x^2y$, which is the product of 5, x, x and y.

The constant in each term is referred to as the **coefficient** .

Example of an algebraic expression - $3x^2y + 4xy + 5x + 6$ - which is the sum of four terms - $3x^2y,\ 4xy,\ 5x$ and 6

An algebraic expression can have **any number of terms**. The **coefficient** in each term can be **any real number**. There can be **any number of variables** in an algebraic expression. The **exponent** on the variables, however, must be **rational numbers.**

Polynomial

An algebraic expression can have exponents that are **rational numbers.** However, a **polynomial** is an algebraic expression in which the exponent on any variable is a **whole number.**

$5x^3 + 3x + 1$ is an example of a polynomial. It is an algebraic expression as well

$2x + 3\sqrt{x}$ is an algebraic expression, but not a polynomial. - since the exponent on x is $\frac{1}{2}$ which is not a whole number.

Degree of a Polynomial

For a polynomial in one variable - the **highest exponent** on the **variable** in a polynomial is the **degree** of the polynomial.

Example: The degree of the polynomial $x^2 + 2x + 3$ is 2, as the highest power of x in the given expression is x^2.

TYPES OF POLYNOMIALS

Polynomials can be classified based on
a) Number of terms
b) Degree of the polynomial.

Types of polynomials based on the number of terms

a) **Monomial** - A polynomial with just one term. Example - $2x$, $6x^2$, $9xy$

b) **Binomial** - A polynomial with two terms. Example - $4x^2 + x$, $5x + 4$

a) **Trinomial** - A polynomial with three terms. Example - $x^2 + 3x + 4$

Types of polynomials based on degree:

Linear Polynomial

A polynomial whose degree is one is called a ***linear polynomial***.
For example, $2x + 1$ is a linear polynomial.

Quadratic Polynomial

A polynomial of degree two is called a ***quadratic polynomial***.
For example, $3x^2 + 8x + 5$ is a quadratic polynomial.

Cubic Polynomial

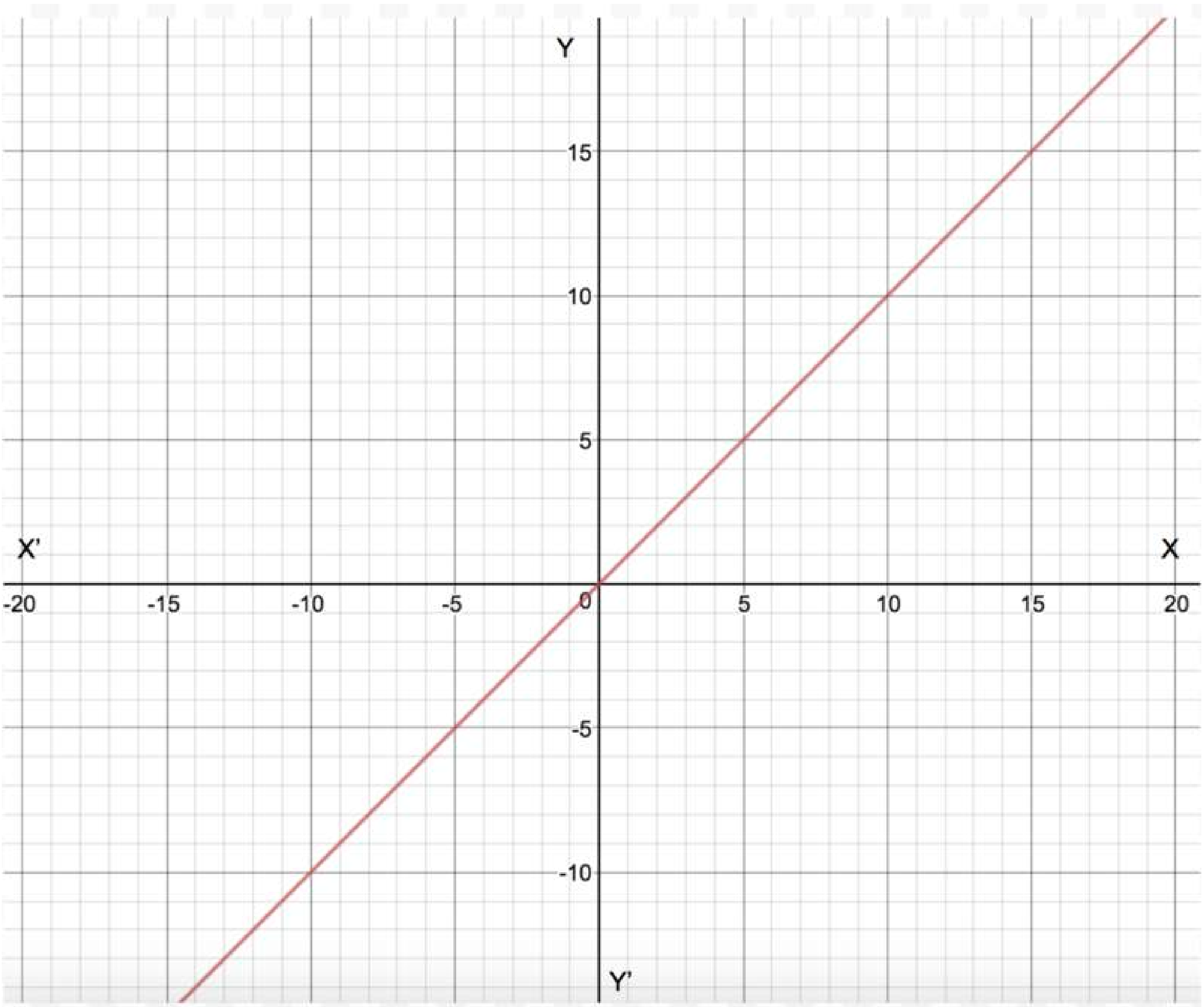
A polynomial of degree three is called a ***cubic polynomial***.
For example, $2x^3 + 5x^2 + 9x + 15$ is a cubic polynomial.

Graphical Representations

Representing Equations on a Graph

Any equation can be represented as a graph on the Cartesian plane, where each point on the graph represents the x and y coordinates of the point that satisfies the equation. An equation can be seen as a constraint placed on the x and y coordinates of a point, and any point that satisfies that constraint will lie on the curve

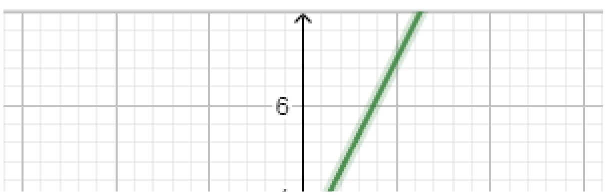
For example, the equation y = x, on a graph, will be a straight line that joins all the points which have their x coordinate equal to their y coordinate. Example - (1,1), (2,2) and so on.

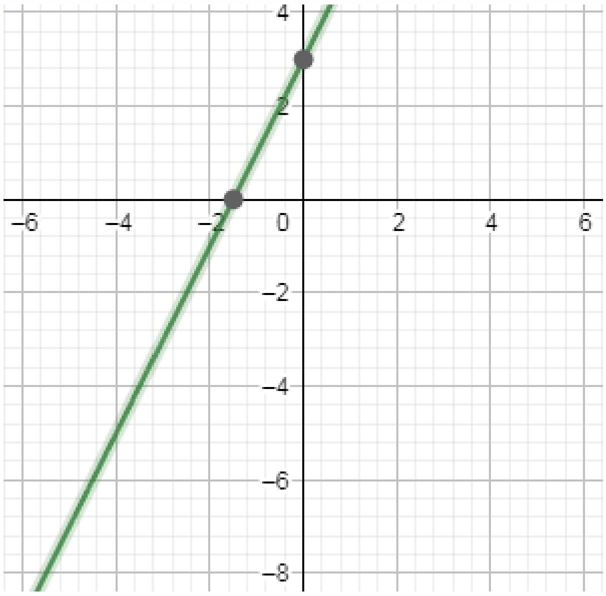


Visualization of a Polynomial

Geometrical Representation of a Linear Polynomial

The graph of a linear polynomial is a straight line. It cuts the X-axis at exactly one point.

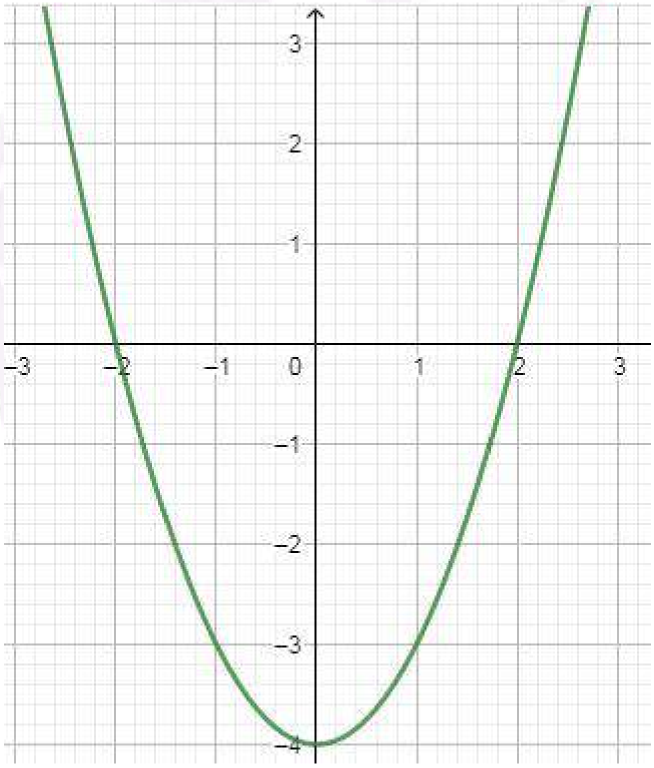




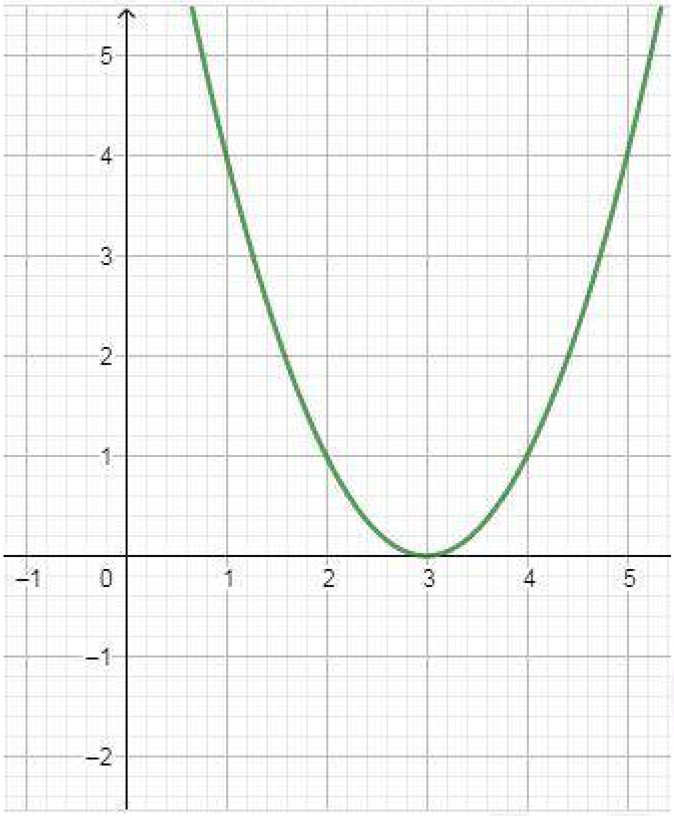
Linear graph

Geometrical Representation of a Quadratic Polynomial

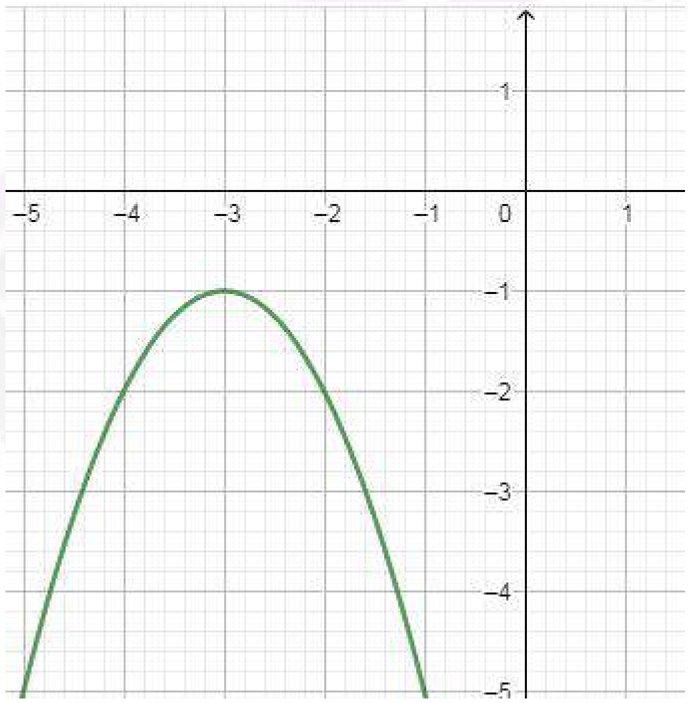
- The graph of a quadratic polynomial is a parabola.
- It looks like a U which either opens upwards or opens downwards depending on the value of a in $ax^2 + bx + c$.
- If a is positive then parabola opens upwards and if a is negative then it opens downwards.
- It can cut the x-axis at 0, 1 or two points.



Graph of a polynomial which cuts the x-axis in two distinct points (a>0)



Graph of a Quadratic polynomial which touches the x-axis at one point (a>0)

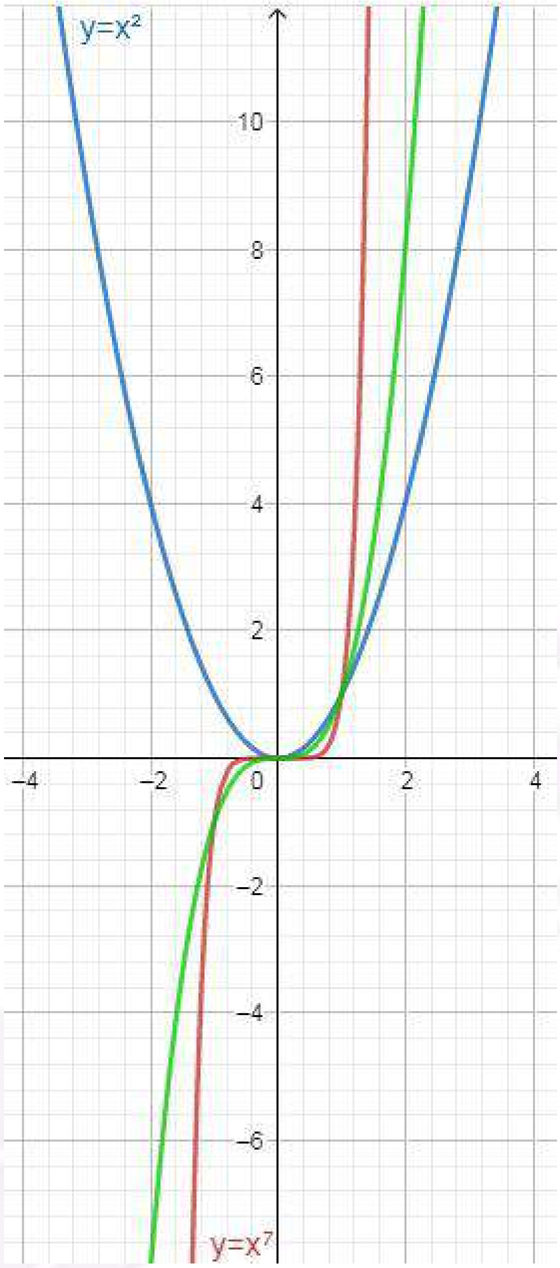


Graph of a Quadratic polynomial that doesn't touch the x-axis (a<0)

Graph of the polynomial x^n

For a polynomial of the form $y = x^n$ where n is a whole number:

- as n increases, the graph becomes steeper or draws closer to the **Y**-axis.
- If n is odd, the graph lies in the first and third quadrants

- If n is even, the graph lies in the first and second quadrants.
- The graph of $y = -x^n$ is the reflection of the graph of $y = x^n$ on the x-axis
- 

Graph of polynomials with different degrees.

Zeroes of a Polynomial

Zeros of a Polynomial

A zero of a polynomial **p(x)** is the value of x for which the value of p(x) is 0. If k is a zero of p(x), then **p(k)=0.**

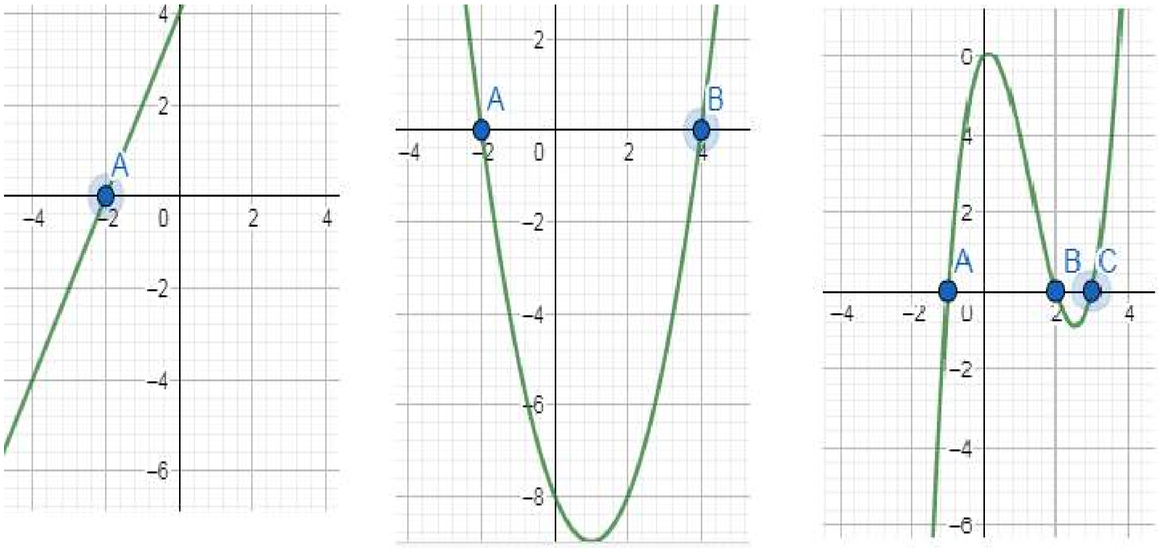
For example, consider a polynomial $p(x) = x^2 - 3x + 2$.
When $x = 1$, the value of $p(x)$ will be equal to
$p(1) = 1^2 - 3 \times 1 + 2$

$= 1 - 3 + 2$
$= 0$

Since $p(x) = 0$ at $x = 1$, we say that 1 is a zero of the polynomial $x^2 - 3x + 2$

Geometrical Meaning of Zeros of a Polynomial

Geometrically, zeros of a polynomial are the points where its graph cuts the x-axis.



(i) One zero *(ii) Two zeros* *(iii) Three zeros*

Here A, B and C correspond to the zeros of the polynomial represented by the graphs.

Number of Zeros

In general a polynomial of degree n has at most n zeros.

1. A linear polynomial has one zero,
2. A quadratic polynomial has at most two zeros.
3. A cubic polynomial has at most 3 zeros.

Factorization of Polynomials

Factorisation of Quadratic Polynomials

Quadratic polynomials can be factorized by splitting the middle term.

For example, consider the polynomial $2x^2 - 5x + 3$

Splitting the middle term.

The middle term in the polynomial $2x^2 - 5x + 3$ is -5. This must be expressed as a sum of two terms such that the product of their coefficients is equal to the product of 2 and 3 (coefficient of x^2 and the constant term)

-5 can be expressed as $(-2) + (-3)$, as $-2 \times -3 = 6 = 2 \times 3$

Thus, $2x^2 - 5x + 3 = 2x^2 - 2x - 3x + 3$

Now, identify the common factors in individual groups

$2x^2 - 2x - 3x + 3 = 2x(x-1) - 3(x-1)$

Taking $(x-1)$ as the common factor, this can be expressed as

$2x(x-1) - 3(x-1) = (x-1)(2x-3)$

Relationship between Zeroes and Coefficients

Relationship between Zeroes and Coefficients of a Polynomial

If α and β are the roots of a quadratic polynomial $ax^2 + bx + c$, then,
$\alpha + \beta = -\frac{b}{a}$
$\alpha\beta = \frac{c}{a}$
Sum of zeroes = $-\frac{coefficient\ of\ x}{coefficient\ of\ x^2}$

Product of zeroes = $\frac{constant\ term}{coefficient\ of\ x^2}$

If α ,β and γ are the roots of a cubic polynomial $ax^3 + bx^2 + cx + d$, then,
$\alpha + \beta + \gamma = -\frac{b}{a}$
$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
$\alpha\beta\gamma = \frac{-d}{a}$

Division Algorithm

Division Algorithm for a Polynomial

To divide one polynomial by another, follow the steps given below.

Step 1: arrange the terms of the dividend and the divisor in the decreasing order of their degrees.

Step 2: To obtain the first term of the quotient, divide the highest degree term of the dividend by the highest degree term of the divisor Then carry out the division process.

Step 3: The remainder from the previous division becomes the dividend for the next step. Repeat this process until the degree of the remainder is less than the degree of the divisor.

$$
\begin{array}{r|l}
 & x - 2 \\
\hline
-x^2 + x - 1 & -x^3 + 3x^2 - 3x + 5 \\
 & -x^3 + x^2 - x \\
 & + \quad - \quad + \\
\hline
 & 2x^2 - 2x + 5 \\
 & 2x^2 - 2x + 2 \\
 & - \quad + \quad - \\
\hline
 & 3
\end{array}
$$

Algebraic Identities

Algebraic Identities

1. $(a+b)^2 = a^2 + 2ab + b^2$
2. $(a-b)^2 = a^2 - 2ab + b^2$
3. $(x+a)(x+b) = x^2 + (a+b)x + ab$
4. $a^2 - b^2 = (a+b)(a-b)$
5. $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
6. $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
7. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$
8. $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$