Quadratic Equation

Introduction to Quadratic Equations

Quadratic Polynomial

A polynomial of the form $ax^2 + bx + c$, where a,b and c are real numbers and $a \neq 0$ is called a quadratic polynomial.

Quadratic Equation

When we equate a quadratic polynomial to a constant, we get a quadratic equation.

Any equation of the form p(x)=c, where p(x) is a polynomial of degree 2 and c is a constant, is a quadratic equation.

Standard form of a Quadratic Equation

The standard form of a quadratic equation is $ax^2 + bx + c = 0$, where a,b and c are real numbers and $a \neq 0$.
'a' is the coefficient of x^2. It is called the quadratic coefficient. 'b' is the coefficient of x. It is called the linear coefficient. 'c' is the constant term.

Solving QE by Factorisation

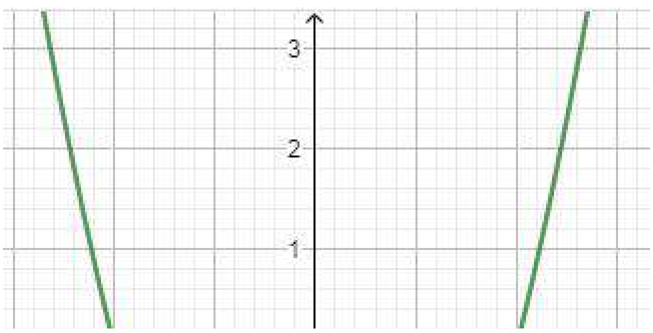
Roots of a Quadratic equation

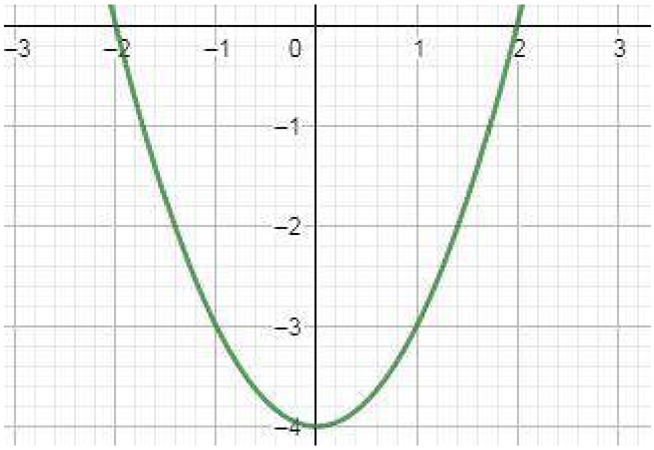
The values of x for which a quadratic equation is satisfied are called the roots of the quadratic equation.

If α is a root of the quadratic equation $ax^2 + bx + c = 0$,then, $a\alpha^2 + b\alpha + c = 0$.

A quadratic equation can have two distinct roots, two equal roots or real roots may not exist.

Graphically, the roots of a quadratic equation are the points where the graph of the quadratic polynomial cuts the x-axis.





Graph of a Quadratic Equation

In the above figure, -2 and 2 are the roots of the quadratic equation $x^2 - 4 = 0$
Note:

- If the graph of the quadratic polynomial cuts the x-axis at two distinct points, then it has real and distinct roots.
- If the graph of the quadratic polynomial touches the x-axis, then it has real and equal roots.
- If the graph of the quadratic polynomial does not cut or touch the x-axis then it does not have any real roots.

Solving a Quadratic Equation by Factorization method

Consider a quadratic equation $2x^2 - 5x + 3 = 0$
$\Rightarrow 2x^2 - 2x - 3x + 3 = 0$
This step is splitting the middle term
We split the middle term by finding two numbers (-2 and -3) such that their sum is equal to the coefficient of x and their product is equal to the product of the coefficient of x^2 and the constant.
(-2) + (-3) = (-5)
And (-2) $\times$ (-3) = 6
$2x^2 - 2x - 3x + 3 = 0$
$2x(x - 1) - 3(x - 1) = 0$
$(x - 1)(2x - 3) = 0$
In this step, we have expressed the quadratic polynomial as a product of its factors.
Thus, x = 1 and x = $\frac{3}{2}$ are the roots of the given quadratic equation.
This method of solving a quadratic equation is called the factorisation method.

Solving QE by Completing the Square

Solving a Quadratic Equation by Completion of squares method

In the method of completing the squares, the quadratic equation is expressed in the form $(x \pm k)^2 = p^2$.

Consider the quadratic equation $2x^2 - 8x = 10$
(i) Express the quadratic equation in standard form.
$2x^2 - 8x - 10 = 0$

(ii) Divide the equation by the coefficient of x^2 to make the coefficient of x^2 equal to 1.
$x^2 - 4x - 5 = 0$

(iii) Add square of half of the coefficient of x to both sides of the equation to get an expression of the form $x^2 \pm 2kx + k^2$.
$(x^2 - 4x + 4) - 5 = 0 + 4$

(iv) Isolate the above expression, $(x \pm k)^2$ on the LHS to obtain an equation of the form $(x \pm k)^2 = p^2$
$(x - 2)^2 = 9$

(v) Take the positive and negative square roots.
$x - 2 = \pm 3$
$x = -1 \text{ or } x = 5$

Solving QE Using Quadratic Formula

Quadratic Formula

Quadratic Formula is used to directly obtain the roots of a quadratic equation from the standard form of the equation.

For the quadratic equation $ax^2 + bx + c = 0$,
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

By substituting the values of a,b and c, we can directly get the roots of the equation.

Discriminant

For a quadratic equation of the form $ax^2 + bx + c = 0$, the expression $b^2 - 4ac$ is called the **discriminant**, (denoted by **D**), of the quadratic equation.
The **discriminant** determines the **nature of roots** of the quadratic equation based on the **coefficients** of the quadratic polynomial.

Solving using Quadratic Formula when D>0

Solve $2x^2 - 7x + 3 = 0$ using the quadratic formula.
(i) Identify the coefficients of the quadratic polynomial. a = 2,b = -7,c = 3
(ii) Calculate the discriminant, $b^2 - 4ac$
$D = (-7)^2 - 4 \times 2 \times 3 = 25$
D> 0, therefore, the roots are distinct.
(iii) Substitute the coefficients in the quadratic formula to find the roots
$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 2 \times 3}}{2 \times 2}$
$x = \frac{7 \pm 5}{4}$
$x = 3$ and $x = \frac{1}{2}$ are the roots.

Nature of Roots

Nature Of Roots

Based on the value of the discriminant, $D = b^2 - 4ac$, the roots of a quadratic equation can be of three types.

Case 1: If **D>0**, the equation has two **distinct real roots**.

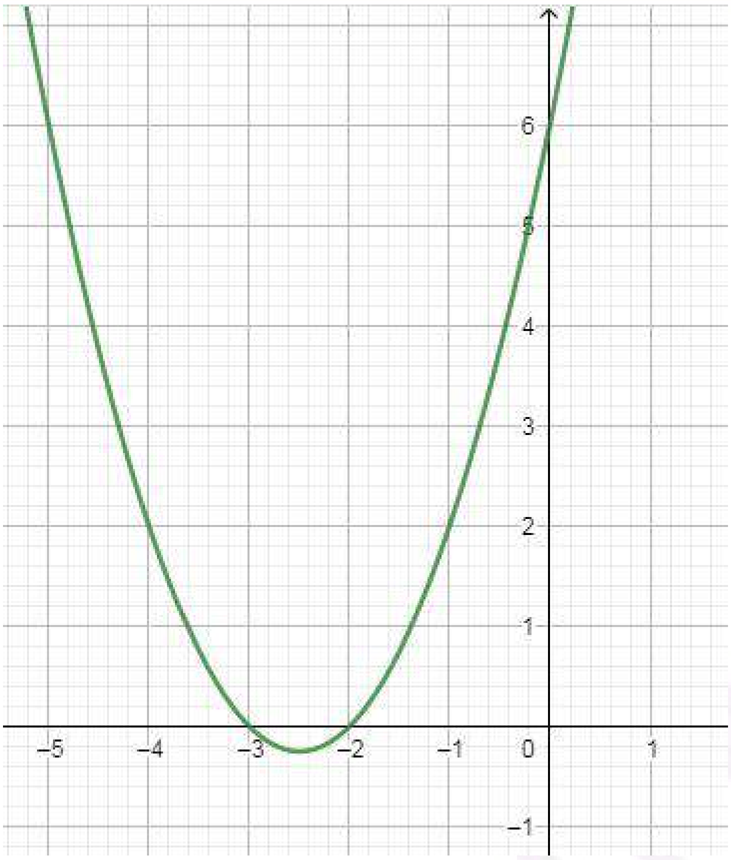
Case 2: If **D=0**, the equation has two **equal real roots**.

Case 3: If **D<0**, the equation has **no real roots**.

Be More Curious

Graphical Representation of a Quadratic Equation

The graph of a quadratic polynomial is a parabola. The roots of a quadratic equation are the points where the parabola cuts the x-axis i.e. the points where the value of the quadratic polynomial is zero.

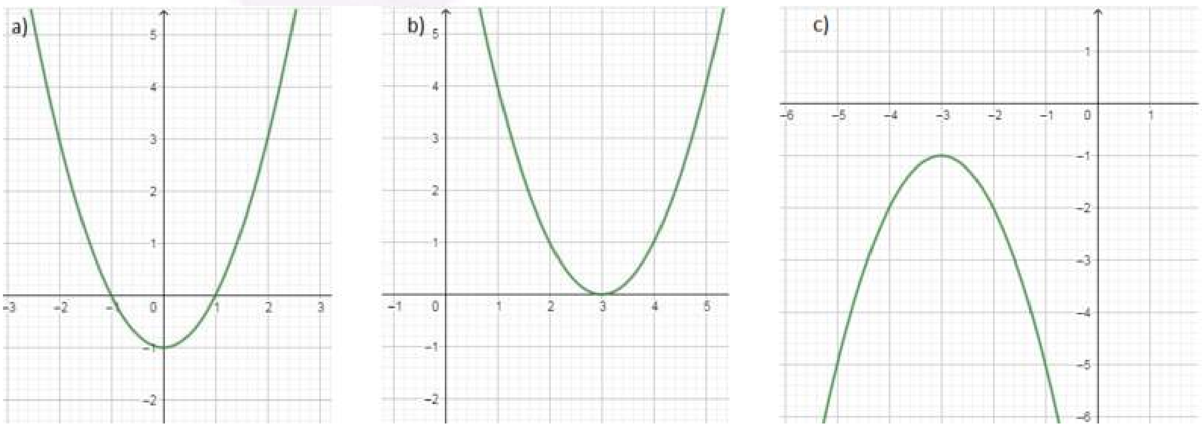


In the above figure, -2 and -3 are the roots of the quadratic equation
$x^2 + 5x + 6 = 0$.

For a quadratic polynomial $ax^2 + bx + c$,

If **a>0**, the parabola opens **upwards**.
If **a<0**, the parabola opens **downwards.**
If **a = 0**, the polynomial will become a first-degree polynomial and its graph is linear.

The discriminant, $D = b^2 - 4ac$



Nature of graph for different values of D.

If **D>0**, the parabola cuts the x-axis at exactly two distinct points. The roots are distinct.
This case is shown in the above figure in a,
where the quadratic polynomial cuts the x-axis at **two distinct points.**

If **D=0**, the parabola just touches the x-axis at one point and the rest of the parabola lies above or below the x-axis. In this case, the roots are equal.
This case is shown in the above figure in b, where the quadratic polynomial touches the x-axis at **only a single point**.

If **D<0**, the parabola lies entirely above or below the x-axis and there is no point of contact with the x-axis. In this case, there are no real roots.
This case is shown in the above figure in c, where the quadratic polynomial **neither cuts nor touches** the x-axis.

Formation of a quadratic equation from it roots

To find out the standard form of a quadratic equation when the roots are given:
Let α and β be the roots of the quadratic equation $ax^2 + bx + c = 0$. Then,
$(x - \alpha)(x - \beta) = 0$
On expanding, we get,
$x^2 - (\alpha + \beta)x + \alpha\beta = 0$, which is the standard form of the quadratic equation. Here,
$a = 1, b = -(\alpha + \beta)\ and\ c = \alpha\beta.$

Sum and Product of roots of a Quadratic equation

Let $\alpha\ and\ \beta$ be the roots of the quadratic equation $ax^2 + bx + c = 0$. Then,
Sum of roots $= \alpha + \beta = \frac{-b}{a}$
Product of roots $= \alpha\beta = \frac{c}{a}$