EXERCISE 18.3 **PAGE NO: 18.31**

Find the points of local maxima or local minima and corresponding local maximum
and local minimum values of each of the following functions. Also, find the points of
inflection, if any:

(i) f(x) = x⁴ − 62x² + 120x + 9

Solution:
Given f (x) = $x^4 - 62x^{2+} 120x + 9$
∴ f'(x) = $4x^3 - 124x + 120 = 4(x^3 - 31x + 30)$
f''(x) = $12x^2 - 124 = 4(3x^2 - 31)$
For maxima and minima, f'(x) = 0
$4(x^3 - 31x + 30) = 0$
So roots will be x = 5, 1, − 6
Now, f''(5) = 176 > 0
x = 5 is point of local minima
f''(1) = − 112 < 0
x = 1 is point of local maxima
f''(− 6) = 308 > 0
x = − 6 is point of local minima
Local max value = f (1) = 68
Local min value = f (5) = − 316 and f (− 6) = − 1647

(ii) f (x) = x³ − 6x² + 9x + 15

Solution:
Given f (x) = $x^3 - 6x^2 + 9x + 15$
Differentiating f with respect to x
∴ f'(x) = $3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$
f'' (x) = $6x - 12 = 6(x - 2)$
For maxima and minima, f'(x) = 0
$3(x^2 - 4x + 3) = 0$
So roots will be x = 3, 1
Now, f'' (3) = 6 > 0
x = 3 is point of local minima
f''(1) = − 6 < 0
x = 1 is point of local maxima

Local max value = f (1) = 19 and local min value = f (3) = 15

(iii) f(x) = (x − 1) (x + 2)²

Solution:
Given f (x) = (x − 1) (x + 2)²
∴ f'(x) = (x + 2)² + 2 (x − 1) (x + 2)
= (x + 2) (x + 2 + 2x − 2)
= (x + 2) (3x)
And f''(x) = 3(x + 2) + 3x
= 6x + 6
For maxima and minima, f'(x) = 0
(x + 2) (3x) = 0
So roots will be x = 0, − 2
Now, f'' (0) = 6 > 0
x = 0 is point of local minima
f''(− 2) = − 6 < 0
x = 1 is point of local maxima
Local maxima value = f (− 2) = 0 and local minima value = f (0) = − 4

(iv) f (x) = 2/x − 1/x², x > 0

Solution:
Given $f(x) = \dfrac{2}{x} - \dfrac{1}{x^2}$, x > 0

∴ $f'(x) = -\dfrac{2}{x^2} + \dfrac{4}{x^3}$

And, $f''(x) = +\dfrac{4}{x^3} - \dfrac{12}{x^4}$

For maxima and minima, f'(x) = 0

$0 -\dfrac{2}{x^2} + \dfrac{4}{x^3} = 0$

$-\dfrac{2(x-2)}{x^3} = 0$

x = 2

Now, f'' (2) = $\frac{4}{8} - \frac{12}{6} = \frac{1}{2} - \frac{3}{4} = -\frac{1}{4} < 0$

x = 2 is point of local maxima

Local maxima value = f (2) = ½

(v) f (x) = x e^x

Solution:
Given f(x) = x e^x
f'(x) = e^x + x e^x = e^x(x + 1)
f''(x) = e^x(x + 1) + e^x
= e^x(x + 2)
For maxima and minima,
f'(x) = 0
e^x(x + 1) = 0
x = − 1
Now f'' (− 1) = e^{-1} = 1/e > 0
x = − 1 is point of local minima
Hence, local min = f (− 1) = − 1/e

2. Find the local extremum values of the following functions:
(i) f(x) = (x − 1) (x − 2)^2

Solution:
Given f(x) = (x − 1)(x − 2)^2
f'(x) = (x − 2)^2 + 2(x − 1)(x − 2)
= (x − 2) (x − 2 + 2x − 2)
= (x − 2) (3x − 4)
f''(x) = (3x − 4) + 3(x − 2)
For maxima and minima, 0f'(x) = 0
(x − 2)(3x − 4) = 0
x = 2, 4/3
Now f'' (2) > 0
x = 2 is point of local minima

f''(4/3) = − 2 < 0
x = 4/3 is point of local maxima
Hence local maxima value = f (4/3) = 4/27
Local minima value = f (2) = 0

(ii) $f(x) = x\sqrt{1 - x}, x \leq 1$

Solution:

Given $f(x) = x\sqrt{1 - x}$

$\therefore f'(x) = \sqrt{1 - x} + \dfrac{x}{2\sqrt{1 - x}}(-1)$

$= \dfrac{2(1 - x) - x}{2\sqrt{1 - x}}$

$= \dfrac{2 - 3x}{2\sqrt{1 - x}}$

$f''(x) = \dfrac{2\sqrt{(1-x)}(-3) + \frac{2-3x}{\sqrt{1-x}}}{4(1-x)}$

For maxima and minima, f'(x) = 0

$\dfrac{2 - 3x}{2\sqrt{1 - x}} = 0$

x = 2/3

Now f'' (2/3) < 0

x = 2/3 is point of maxima

Hence local max value = f (2/3) = $\dfrac{2}{3\sqrt{3}}$

(iii) f (x) = − (x − 1)³(x + 1)²

Solution:
Given f (x) = − (x − 1)³(x + 1)²
f'(x) = − 3(x − 1)²(x + 1)² − 2(x − 1)³(x + 1)

= – (x – 1)²(x + 1) (3x + 3 + 2x – 2)

= – (x – 1)²(x + 1) (5x + 1)

f''(x) = – 2(x – 1)(x + 1)(5x + 1) – (x – 1)²(5x + 1) – 5(x – 1)²(x – 1)

For maxima and minima, f'(x) = 0

– (x – 1)²(x + 1) (5x + 1) = 0

x = 1, – 1, – 1/5

Now f'' (1) = 0

x = 1 is inflection point

f''(– 1) = – 4× – 4 = 16 > 0

x = – 1 is point of minima

f'' (– 1/5) = – 5(36/25) × 4/5 = – 144/25 < 0

x = – 1/5 is point of maxima

Hence local max value = f (– 1/5) = 3456/3125

Local min value = f (– 1) = 0

3. The function y = a log x + bx² + x has extreme values at x = 1 and x = 2. Find a and b.

Solution:
Given y = a log x + bx² + x

On differentiating we get

$$\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$$

And $\frac{d^2y}{dx^2} = -\frac{a}{x^2} + 2b$

For maxima and minima, $\frac{dy}{dx} = 0$

$\frac{a}{x} + 2bx + 1 = 0$

Given that extreme value exist at x = 1, 2

a + 2b = – 1 …… (1)

$\frac{a}{2} + 4b = -1$

a + 8b = – 2 …… (2)

Solving (1) and (2) we get

a = – 2/3 b = – 1/6

4. Show that log x/x has a maximum value at x = e.

Solution:

The given function is $f(x) = \dfrac{\log x}{x}$

$$f'(x) = \dfrac{x\left(\dfrac{1}{x}\right) - \log x}{x^2} = \dfrac{1 - \log x}{x^2}$$

Now f'(x) = 0

1 – Log x = 0

Log x = 1

Log x = log e

x = e

Now $f''(x) = \dfrac{x^2\left(-\dfrac{1}{x}\right) - (1 - \log x)(2x)}{x^4}$

$$= \dfrac{-x - 2x(1 - \log x)}{x^4}$$

$$= \dfrac{-3 + 2\log x}{x^3}$$

Now f''(x) = $\dfrac{-3 + 2\log e}{e^3} = \dfrac{-3 + 2}{e^3} = -\dfrac{1}{e^3} < 0$

Therefore, by second derivation test f is the maximum at x = e.