

EXERCISE

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SHORT ANSWER TYPE:

1. Find the term independent of x , $x \neq 0$, in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$.

Solution:

$$\text{Given } \left(\frac{3x^2}{2} - \frac{1}{3x}\right)^{15}$$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{15}C_r \left(\frac{3x^2}{2}\right)^{15-r} \left(-\frac{1}{3x}\right)^r$$

$$T_{r+1} = {}^{15}C_r (-1)^r 3^{15-2r} 2^{r-15} x^{30-3r}$$

For the term independent of x , we have

$$30 - 3r = 0$$

Which implies $r = 10$

By substituting the value of r in above obtained expression we get

$$\begin{aligned} T_{10+1} &= {}^{15}C_{10} 3^{-5} 2^{-5} \\ &= {}^{15}C_{10} \left(\frac{1}{6}\right)^5 \end{aligned}$$

2. If the term free from x in the expansion of $\sqrt{x} - \frac{k}{x^2}$ is 405, find the value of k .

Solution:

$$\text{Given } \sqrt{x} - \frac{k}{x^2}$$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \left(\frac{-k}{x^2}\right)^r = {}^{10}C_r (x)^{\frac{1}{2}(10-r)} (-k)^r x^{-2r}$$

$$= {}^{10}C_r (x)^{5-\frac{r}{2}-2r} (-k)^r = {}^{10}C_r x^{\frac{10-5r}{2}} (-k)^r$$

For the term free from x we have

$$(10 - 5r)/2 = 0$$

Which implies $r = 2$

So, the term free from x is

$$T_{2+1} = {}^{10}C_2 (-k)^2$$

$${}^{10}C_2 (-k)^2 = 405$$

$$\frac{10 \times 9 \times 8!}{2! \times 8!} (-k)^2 = 405$$

$$45k^2 = 405 \Rightarrow k^2 = 9 \therefore k = \pm 3$$

3. Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

Solution:

Given $(1 - 3x + 7x^2)(1 - x)^{16}$

$$= (1 - 3x + 7x^2)({}^{16}C_0 - {}^{16}C_1 x^1 + {}^{16}C_2 x^2 + \dots + {}^{16}C_{16} x^{16})$$

$$= (1 - 3x + 7x^2)(1 - 16x + 120x^2 + \dots)$$

Coefficient of x = -19

4. Find the term independent of x in the expansion of $(3x - \frac{2}{x^2})^{15}$

Solution:

Given $(3x - \frac{2}{x^2})^{15}$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{15}C_r (3x)^{15-r} \left(\frac{-2}{x^2}\right)^r = {}^{15}C_r 3^{15-r} x^{15-3r} (-2)^r$$

For the term independent of x, we have

$$15 - 3r = 0$$

Which implies $r = 5$

The term independent of x is

$$\begin{aligned}
 T_{5+1} &= {}^{15}C_5 3^{15-5} (-2)^5 \\
 &= \frac{15 \times 14 \times 13 \times 12 \times 11 \times 10!}{5 \times 4 \times 3 \times 2 \times 1 \times 10!} \cdot 3^{10} \cdot 2^5 \\
 &= -3003 \times 3^{10} \times 2^5
 \end{aligned}$$

5. Find the middle term (terms) in the expansion of

(i) $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$ (ii) $3x - \frac{x^3}{6}$

Solution:

a. Given $\left(\frac{x}{a} - \frac{a}{x}\right)^{10}$

Here index $n = 10$ which is even number.

So, there is one middle term which is $(10/2 + 1)^{\text{th}}$ term that is 6th term

$$\begin{aligned}
 \therefore T_6 = T_{5+1} &= {}^{10}C_5 \left(\frac{x}{a}\right)^{10-5} \left(\frac{-a}{x}\right)^5 \\
 &= -{}^{10}C_5 \left(\frac{x}{a}\right)^5 \left(\frac{a}{x}\right)^5 \\
 &= -\frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5! \times 5 \times 4 \times 3 \times 2 \times 1} \left(\frac{x}{a}\right)^5 \left(\frac{x}{a}\right)^{-5} = -252
 \end{aligned}$$

b. Given $\left(3x - \frac{x^3}{6}\right)^9$

Here index $n = 9$ which is odd

So, there is one middle term which is $(9/2 + 1)^{\text{th}}$ term that is 5th term and 6th term

$$\therefore T_5 = T_{4+1} = {}^9C_4 (3x)^{9-4} \left(-\frac{x^3}{6}\right)^4$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{4 \times 3 \times 2 \times 1 \times 5!} 3^5 x^5 x^{12} 6^{-4}$$

$$= \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} \times \frac{3^5}{3^4 \times 2^4} x^{17} = \frac{189}{8} x^{17}$$

And 6th term,

$$T_6 = T_{5+1} = {}^9C_5 (3x)^{9-5} \left(-\frac{x^3}{6}\right)^5$$

$$= \frac{9 \times 8 \times 7 \times 6 \times 5!}{5! \times 4 \times 3 \times 2 \times 1} \cdot 3^4 \cdot x^4 \cdot x^{15} \cdot 6^{-5} = -\frac{21}{16} x^{19}$$

6. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

Solution:

Given $(x - x^2)^{10}$

$$T_{r+1} = {}^{10}C_r x^{10-r} (-x^2)^r = (-1)^r {}^{10}C_r x^{10-r} x^{2r} = (-1)^r {}^{10}C_r x^{10+r}$$

For the coefficient of x^{15} , we have

$$10 + r = 15 \Rightarrow r = 5$$

$$T_{5+1} = (-1)^5 {}^{10}C_5 x^{15}$$

$$\text{Coefficient of } x^{15} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5 \times 4 \times 3 \times 2 \times 1 \times 5!} = -252$$

7. Find the coefficient of $1/x^{17}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$

Solution:

$$\text{Given } \left(x^4 - \frac{1}{x^3}\right)^{15}$$

From the standard formula of T_{r+1} we can write given expression as

$$T_{r+1} = {}^{15}C_r (x^4)^{15-r} \left(-\frac{1}{x^3}\right)^r = {}^{15}C_r x^{60-4r} (-1)^r x^{-3r} = {}^{15}C_r x^{60-7r} (-1)^r$$

For the coefficient x^{-17} , we have

$$60 - 7r = -17$$

Therefore, $r = 11$

Then above expression becomes

$$T_{11+1} = {}^{15}C_{11} x^{60-77} (-1)^{11}$$

$$\text{Coefficient of } x^{-17} = \frac{-15 \times 14 \times 13 \times 12 \times 11!}{11! \times 4 \times 3 \times 2 \times 1}$$

$$= -15 \times 7 \times 13 = -1365$$

8. Find the sixth term of the expansion $\left(y^{\frac{1}{2}} + x^{\frac{1}{3}}\right)^n$ if the binomial coefficient of the third term from the end is 45.

Solution:

Given $(y^{\frac{1}{2}} + x^{\frac{1}{3}})^n$.

Also given that binomial coefficient of third term from the end = 45

Therefore,

$${}^nC_{n-2} = 45$$

The above expression can be written as

$${}^nC_2 = 45$$

$$\Rightarrow \frac{n(n-1)(n-2)!}{2!(n-2)!} = 45$$

$$\Rightarrow n(n-1) = 90$$

$$\Rightarrow n^2 - n - 90 = 0$$

$$\Rightarrow (n-10)(n+9) = 0$$

Therefore $n = 10$

$$\text{Now, sixth term} = {}^{10}C_5 (y^{\frac{1}{2}})^{10-5} (x^{\frac{1}{3}})^5 = 252 y^{5/2} \cdot x^{5/3}$$

9. Find the value of r , if the coefficients of $(2r + 4)^{\text{th}}$ and $(r - 2)^{\text{th}}$ terms in the expansion of $(1 + x)^{18}$ are equal.

Solution:

Given $(1 + x)^{18}$

Now, $(2r + 4)^{\text{th}}$ term,

That is $T_{(2r+3)+1}$

$$T_{(2r+3)+1} = {}^{18}C_{2r+3} (x)^{2r+3}$$

And $(r - 2)^{\text{th}}$ term, that is $T_{(r-3)+1}$

$$T_{(r-3)+1} = {}^{18}C_{r-3} x^{r-3}$$

Now according to the question,

$${}^{18}C_{2r+3} = {}^{18}C_{r-3}$$

$$2r + 3 + r - 3 = 18$$

$$3r = 18 \quad \therefore r = 6$$

10. If the coefficient of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P. Show that $2n^2 - 9n + 7 = 0$.

Solution:

Given $(1 + x)^{2n}$

Now, coefficient of 2nd, 3rd and 4th terms are ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$, respectively.

Given that, ${}^{2n}C_1$, ${}^{2n}C_2$ and ${}^{2n}C_3$ are in A.P.

Then,

$$2 \cdot {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$$

$$2 \left[\frac{2n(2n-1)(2n-2)!}{2 \times 1 \times (2n-2)!} \right] = 2n + \frac{2n(2n-1)(2n-2)(2n-3)!}{3!(2n-3)!}$$

$$n(2n-1) = n + \frac{n(2n-1)(n-1)}{3}$$

$$3(2n-1) = 3 + (2n^2 - 3n + 1)$$

$$6n - 3 = 2n^2 - 3n + 4 \quad \Rightarrow \quad 2n^2 - 9n + 7 = 0$$

11. Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Solution:

Given expression is $(1 + x + x^2 + x^3)^{11}$
 $= [(1 + x) + x^2(1 + x)]^{11} = [(1 + x)(1 + x^2)]^{11} = (1 + x)^{11} \cdot (1 + x^2)^{11}$
 $= ({}^{11}C_0 + {}^{11}C_1x + {}^{11}C_2x^2 + {}^{11}C_3x^3 + {}^{11}C_4x^4 + \dots)({}^{11}C_0 + {}^{11}C_1x^2 + {}^{11}C_2x^4 + \dots)$
 Coefficient of $x^4 = {}^{11}C_0 \times {}^{11}C_4 + {}^{11}C_1 \times {}^{11}C_2 + {}^{11}C_2 \times {}^{11}C_0$
 $= 330 + 605 + 55 = 990$

LONG ANSWER TYPE:

12. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2} + 2\right)^8$ is 1120, find p .

Solution:

Given expansion is $\left(\frac{p}{2} + 2\right)^8$

Since index is $n = 8$, there is only one middle term, i.e., $\left(\frac{8}{2} + 1\right)^{\text{th}} = 5^{\text{th}}$ term

$$T_5 = T_{4+1} = {}^8C_4 \left(\frac{p}{2}\right)^{8-4} \cdot 2^4$$

$$\Rightarrow 1120 = {}^8C_4 p^4$$

By substituting the values, we get

$$\Rightarrow 1120 = \frac{8 \times 7 \times 6 \times 5 \times 4!}{4! \times 4 \times 3 \times 2 \times 1} p^4$$

$$\Rightarrow 1120 = 7 \times 2 \times 5 \times p^4$$

$$\Rightarrow p^4 = \frac{1120}{70}$$

$$\Rightarrow p^4 = 16$$

$$\Rightarrow p^2 = 4 \Rightarrow p = \pm 2$$

13. Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is

$$\frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{n!} \times (-2)^n$$

Solution:

Given, expression is $\left(x - \frac{1}{x}\right)^{2n}$.

Since the index is $2n$, which is even. So, there is only one middle term, i.e.,

$\left(\frac{2n}{2} + 1\right)$ th term = $(n + 1)$ th term

$$\begin{aligned} T_{n+1} &= {}^{2n}C_n (x)^{2n-n} \left(-\frac{1}{x}\right)^n = {}^{2n}C_n (-1)^n = (-1)^n \frac{(2n!)}{n! \cdot n!} \\ &= (-1)^n \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \dots (2n-1) \cdot (2n)}{n! \cdot n!} = (-1)^n \frac{[1 \cdot 3 \cdot 5 \dots (2n-1)] \cdot [2 \cdot 4 \cdot 6 \dots (2n)]}{(1 \cdot 2 \cdot 3 \dots n) \cdot n!} \\ &= (-1)^n \frac{[1 \cdot 3 \cdot 5 \dots (2n-1)] \cdot 2^n [1 \cdot 2 \cdot 3 \dots n]}{(1 \cdot 2 \cdot 3 \dots n) \cdot n!} = (-2)^n \frac{[1 \cdot 3 \cdot 5 \dots (2n-1)] \cdot 2^n}{n!} \end{aligned}$$

14. Find n in the binomial $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, if the ratio of 7th term from the beginning to the 7th term from the end is $\frac{1}{6}$.

Solution:

Given expression is $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$

Now, 7th term from beginning, $T_7 = T_{6+1} = {}^nC_6 (\sqrt[3]{2})^{n-6} \left(\frac{1}{\sqrt[3]{3}}\right)^6$

And 7th term from end is same as 7th term from the beginning of $\left(\frac{1}{\sqrt[3]{3}} + \sqrt[3]{2}\right)^n$

$$T_7 = {}^nC_6 \left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} (\sqrt[3]{2})^6$$

Given that

$$\frac{{}^n C_6 (\sqrt[3]{2})^{n-6} \left(\frac{1}{\sqrt[3]{3}}\right)^6}{{}^n C_6 \left(\frac{1}{\sqrt[3]{3}}\right)^{n-6} (\sqrt[3]{2})^6} = \frac{1}{6}$$

$$\frac{(\sqrt[3]{2})^{n-12}}{\left(\frac{1}{\sqrt[3]{3}}\right)^{n-12}} = \frac{1}{6} \Rightarrow (\sqrt[3]{2} \sqrt[3]{3})^{n-12} = 6^{-1} \Rightarrow 6^{\frac{n-12}{3}} = 6^{-1}$$

$$\frac{n-12}{3} = -1 \Rightarrow n = 9$$

15. In the expansion of $(x + a)^n$ if the sum of odd terms is denoted by O and the sum of even term by E. Then prove that

(i) $O^2 - E^2 = (x^2 - a^2)^n$

(ii) $4OE = (x + a)^{2n} - (x - a)^{2n}$

Solution:

(i) We know that

$$(x + a)^n = {}^n C_0 x^n + {}^n C_1 x^{n-1} a + {}^n C_2 x^{n-2} a^2 + {}^n C_3 x^{n-3} a^3 + \dots$$

Sum of odd terms,

$$O = {}^n C_0 x^n + {}^n C_2 x^{n-2} a^2 + \dots$$

And also sum of even terms

$$E = {}^n C_1 x^{n-1} a + {}^n C_3 x^{n-3} a^3 + \dots$$

$$\text{Since } (x + a)^n = O + E$$

$$(x - a)^n = O - E$$

Therefore,

$$(O + E)(O - E) = (x + a)^n (x - a)^n$$

(ii) $4OE = (O + E)^2 - (O - E)^2$

$$= (x + a)^{2n} - (x - a)^{2n}$$

16. If x^p occurs in the expansion of $x^2 + \frac{1}{x}^{2n}$
Prove that its coefficient is

$$\frac{\overbrace{2n}^{|2n|}}{\frac{4n-p}{3} \mid \frac{2n+p}{3}}$$

Solution:

Given expression is $\left(x^2 + \frac{1}{x}\right)^{2n}$

Using the standard formula above expression can be written as

$$T_{r+1} = {}^{2n}C_r (x^2)^{2n-r} \left(\frac{1}{x}\right)^r = {}^{2n}C_r x^{4n-3r}$$

If x^p occurs in the expansion,

$$\text{Let } 4n - 3r = p$$

$$3r = 4n - p \Rightarrow r = \frac{4n - p}{3}$$

$$\begin{aligned} \text{Coefficient of } x^p &= {}^{2n}C_r = \frac{(2n)!}{r!(2n-r)!} = \frac{(2n)!}{\left(\frac{4n-p}{3}\right)! \left(2n - \frac{4n-p}{3}\right)!} \\ &= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)! \left(\frac{2n+p}{3}\right)!} \end{aligned}$$

17. Find the term independent of x in the expansion of

$$(1 + x + 2x^3) \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$$

Solution:

Given

$$(1 + x + 2x^3) \left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$$

Consider

$$\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$$

Using standard formula above expression can be written as

$$T_{r+1} = {}^9C_r \left(\frac{3}{2}x^2\right)^{9-r} \left(-\frac{1}{3x}\right)^r = {}^9C_r \left(\frac{3}{2}\right)^{9-r} \left(-\frac{1}{3}\right)^r x^{18-3r}$$

Hence the general term in the expression of given expansion is

$${}^9C_r \left(\frac{3}{2}\right)^{9-r} \left(-\frac{1}{3}\right)^r x^{18-3r} + {}^9C_r \left(\frac{3}{2}\right)^{9-r} \left(-\frac{1}{3}\right)^r x^{19-3r} \\ + 2 \cdot {}^9C_r \left(\frac{3}{2}\right)^{9-r} \left(-\frac{1}{3}\right)^r x^{21-3r}$$

For independent term of x , substitute $18 - 3r = 0$
 $19 - 3r = 0$ and $21 - 3r = 0$

We get $r = 6$ and $r = 7$

Hence second term is not independent of x

Therefore, term independent of x is

$${}^9C_6 \left(\frac{3}{2}\right)^{9-6} \left(-\frac{1}{3}\right)^6 + 2 \cdot {}^9C_7 \left(\frac{3}{2}\right)^{9-7} \left(-\frac{1}{3}\right)^7 \\ = \frac{9 \times 8 \times 7 \times 6!}{6! \times 3 \times 2} \cdot \frac{1}{2^3 \cdot 3^3} - 2 \cdot \frac{9 \times 8 \times 7!}{7! \times 2 \times 1} \cdot \frac{3^2}{2^2} \cdot \frac{1}{3^7} \\ = \frac{84}{8} \cdot \frac{1}{3^3} - \frac{36}{4} \cdot \frac{2}{3^5} = \frac{21-4}{54} = \frac{17}{54}$$

OBJECTIVE TYPE QUESTIONS:

Choose the correct answer from the given options in each of the Exercises 18 to 24 (M.C.Q.).

18. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is

- (A) 50 (B) 202 (C) 51 (D) none of these

Solution:

(C) 51

Explanation:

Given $(x + a)^{100} + (x - a)^{100}$

$$\begin{aligned}
 &= ({}^{100}C_0x^{100} + {}^{100}C_1x^{99}a + {}^{100}C_2x^{98}a^2 + \dots) \\
 &\quad + ({}^{100}C_0x^{100} - {}^{100}C_1x^{99}a + {}^{100}C_2x^{98}a^2 + \dots) \\
 &= 2({}^{100}C_0x^{100} + {}^{100}C_2x^{98}a^2 + \dots + {}^{100}C_{100}a^{100})
 \end{aligned}$$

So, there are 51 terms

Hence option c is the correct answer.

19. Given the integers $r > 1$, $n > 2$, and coefficients of $(3r)^{\text{th}}$ and $(r + 2)^{\text{nd}}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then

(A) $n = 2r$ (B) $n = 3r$ (C) $n = 2r + 1$ (D) none of these

Solution:

(A) $n = 2r$

Explanation:

Given $(1 + x)^{2n}$

$$\begin{aligned}
 T_{3r} &= T_{(3r-1)+1} = {}^{2n}C_{3r-1} x^{3r-1} \\
 T_{r+2} &= T_{(r+1)+1} = {}^{2n}C_{r+1} x^{r+1} \\
 {}^{2n}C_{3r-1} &= {}^{2n}C_{r+1} \\
 3r-1 + r+1 &= 2n \\
 n &= 2r
 \end{aligned}$$

Hence option A is the correct answer.

20. The two successive terms in the expansion of $(1 + x)^{24}$ whose coefficients are in the ratio 1: 4 are

(A) 3rd and 4th (B) 4th and 5th (C) 5th and 6th (D) 6th and 7th

Solution:

(C) 5th and 6th

Explanation:

Let the two successive terms in the expansion of $(1 + x)^{24}$ be $(r + 1)^{\text{th}}$ and $(r + 2)^{\text{th}}$ term

Now,

$$T_{r+1} = {}^{24}C_r x^r \text{ and } T_{r+2} = {}^{24}C_{r+1} x^{r+1}$$

Given

$$\frac{{}^{24}C_r}{{}^{24}C_{r+1}} = \frac{1}{4}$$

$$\Rightarrow \frac{\frac{(24)!}{r!(24-r)!}}{\frac{(24)!}{(r+1)!(24-r-1)!}} = \frac{1}{4} \Rightarrow \frac{(r+1)r!(23-r)!}{r!(24-r)(23-r)!} = \frac{1}{4} \Rightarrow \frac{r+1}{24-r} = \frac{1}{4}$$

$$4r + 4 = 24 - r$$

Which implies $r = 4$

$$T_{4+1} = T_5 \text{ and } T_{4+2} = T_6$$

Hence the correct option is c

