

Exercise 4.3

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Short Answer (S.A.)

Using the properties of determinants in Exercises 1 to 6, evaluate:

1.
$$\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

Solution:

Given,
$$\begin{vmatrix} x^2 - x + 1 & x - 1 \\ x + 1 & x + 1 \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 - C_2$]

$$\begin{aligned} &= \begin{vmatrix} x^2 - 2x + 2 & x - 1 \\ 0 & x + 1 \end{vmatrix} \\ &= (x^2 - 2x + 2) \cdot (x + 1) - (x - 1) \cdot 0 \\ &= x^3 - 2x^2 + 2x + x^2 - 2x + 2 \\ &= x^3 - x^2 + 2 \end{aligned}$$

2.
$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

Solution:

Given,
$$\begin{vmatrix} a+x & y & z \\ x & a+y & z \\ x & y & a+z \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 + C_2 + C_3$]

$$\begin{aligned} &= \begin{vmatrix} a+x+y+z & y & z \\ a+x+y+z & a+y & z \\ a+x+y+z & y & a+z \end{vmatrix} \\ &= (a+x+y+z) \begin{vmatrix} 1 & y & z \\ 1 & a+y & z \\ 1 & y & a+z \end{vmatrix} \end{aligned}$$

[Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$]

$$= (a + x + y + z) \begin{vmatrix} 1 & y & z \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix}$$

$$= (a + x + y + z) \begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2(a + z + x + y)$$

3. $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

Solution:

Given, $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

[Taking x^2 , y^2 and z^2 common from C_1 , C_2 and C_3 , respectively]

$$= x^2y^2z^2 \begin{vmatrix} 0 & x & x \\ y & 0 & y \\ z & z & 0 \end{vmatrix}$$

[Applying $C_2 \rightarrow C_2 - C_3$]

$$= x^2y^2z^2 \begin{vmatrix} 0 & 0 & x \\ y & -y & y \\ z & z & 0 \end{vmatrix} = x^2y^2z^2 (x(yz + yz))$$

$$= x^2y^2z^2 \cdot (2xyz) = 2x^3y^3z^3$$

4. $\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$

Solution:

Given,
$$\begin{vmatrix} 3x & -x+y & -x+z \\ x-y & 3y & z-y \\ x-z & y-z & 3z \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 + C_2 + C_3$]

$$= \begin{vmatrix} x+y+z & -x+y & -x+z \\ x+y+z & 3y & z-y \\ x+y+z & y-z & 3z \end{vmatrix}$$

[Taking $(x + y + z)$ common from column C_1]

$$= (x + y + z) \begin{vmatrix} 1 & -x+y & -x+z \\ 1 & 3y & z-y \\ 1 & y-z & 3z \end{vmatrix}$$

[Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$]

$$= (x + y + z) \begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 2y+x & x-y \\ 0 & x-z & 2z+x \end{vmatrix}$$

[Applying $C_2 \rightarrow C_2 - C_3$]

$$= (x + y + z) \begin{vmatrix} 1 & -x+y & -x+z \\ 0 & 3y & x-y \\ 0 & -3z & 2z+x \end{vmatrix}$$

[Expanding along first column]

$$\begin{aligned} &= (x + y + z) \cdot 1[3y(3z + x) + (3z)(x - y)] \\ &= (x + y + z)(3yz + 3yx + 3xz) \\ &= 3(x + y + z)(xy + yz + zx) \end{aligned}$$

5.
$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

Solution:

Given,
$$\begin{vmatrix} x+4 & x & x \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

$$= \begin{vmatrix} 3x+4 & 3x+4 & 3x+4 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix} \quad [\text{Applying } R_1 \rightarrow R_1 + R_2 + R_3]$$

$$= (3x+4) \begin{vmatrix} 1 & 1 & 1 \\ x & x+4 & x \\ x & x & x+4 \end{vmatrix}$$

$$= (3x+4) \begin{vmatrix} 0 & 0 & 1 \\ -4 & 4 & x \\ 0 & -4 & x+4 \end{vmatrix} = 16(3x+4)$$

[Applying $C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$]

6.
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Solution:

Given,
$$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[Applying $R_1 \rightarrow R_1 + R_2 + R_3$]

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[Taking $(a+b+c)$ common from the first row]

$$= (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$]

$$= (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ 0 & -(a+b+c) & 2b \\ a+b+c & a+b+c & c-a-b \end{vmatrix}$$

Lastly, expanding along R1, we have

$$\begin{aligned} &= (a+b+c) [1 \times 0 + (a+b+c)^2] \\ &= (a+b+c)^3 \end{aligned}$$

Using the properties of determinants in Exercises 7 to 9, prove that:

7.
$$\begin{vmatrix} y^2 z^2 & yz & y+z \\ z^2 x^2 & zx & z+x \\ x^2 y^2 & xy & x+y \end{vmatrix} = 0$$

Solution:

From the given,

[Multiplying R_1, R_2, R_3 by x, y, z respectively]

$$= \frac{1}{xyz} \begin{vmatrix} xy^2 z^2 & xyz & xy + xz \\ x^2 yz^2 & xyz & yz + xy \\ x^2 y^2 z & xyz & xz + yz \end{vmatrix}$$

Next

[Taking (xyz) common from C_1 and C_2]

$$= \frac{1}{xyz} (xyz)^2 \begin{vmatrix} yz & 1 & xy + xz \\ xz & 1 & yz + xy \\ xy & 1 & xz + yz \end{vmatrix}$$

Then,

[Applying $C_3 \rightarrow C_3 + C_1$]

$$= xyz \begin{vmatrix} yz & 1 & xy + yz + zx \\ xz & 1 & xy + yz + zx \\ xy & 1 & xy + yz + zx \end{vmatrix}$$

Lastly,

[Taking $(xy + yz + zx)$ common from C_3]

$$\begin{aligned} &= xyz (xy + yz + zx) \begin{vmatrix} yz & 1 & 1 \\ xz & 1 & 1 \\ xy & 1 & 1 \end{vmatrix} \\ &= 0 \quad [\because C_2 \text{ and } C_3 \text{ are identical}] \end{aligned}$$

8.
$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$

Solution:

Given,
$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 + C_2 + C_3$]

$$\begin{aligned} &= \begin{vmatrix} 2(y+z) & z & y \\ 2(z+x) & z+x & x \\ 2(y+x) & x & x+y \end{vmatrix} \\ &= 2 \begin{vmatrix} y+z & z & y \\ z+x & z+x & x \\ x+y & x & x+y \end{vmatrix} \end{aligned}$$

Now,

[Applying $C_1 \rightarrow C_1 - C_2$]

$$= 2 \begin{vmatrix} y & z & y \\ 0 & z+x & x \\ y & x & x+y \end{vmatrix}$$

Next,

[Applying $C_3 \rightarrow C_3 - C_1$]

$$= 2 \begin{vmatrix} y & z & 0 \\ 0 & z+x & x \\ y & x & x \end{vmatrix}$$

Lastly,

[Applying $R_3 \rightarrow R_3 - R_1$]

$$= 2 \begin{vmatrix} y & z & 0 \\ 0 & z+x & x \\ 0 & x-z & x \end{vmatrix}$$

$$= 2y[(z+x)x - x(x-z)] = 2y[2xz] = 4xyz$$

9.
$$\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} = (a - 1)^3$$

Solution:

Given,
$$\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

[Applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$]

$$= \begin{vmatrix} a^2 - 1 & a - 1 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Now,

[Taking $(a - 1)$ common from R_1 and R_2]

$$(a - 1)^2 \begin{vmatrix} a + 1 & 1 & 0 \\ 2 & 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

Finally,

[Expanding along R_3]

$$= (a - 1)^2 [1 \cdot (a + 1) - 2] = (a - 1)^3$$

10. If $A + B + C = 0$, then prove that

Solution:

Given,
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix}$$

On finding the determinant, we have

$$\begin{aligned} &= 1(1 - \cos^2 A) - \cos C(\cos C - \cos A \cdot \cos B) + \cos B(\cos C \cdot \cos A - \cos^2 B) \\ &= \sin^2 A - \cos^2 C + \cos A \cdot \cos B \cdot \cos C + \cos A \cdot \cos B \cdot \cos C - \cos^2 B \\ &= \sin^2 A - \cos^2 B + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 C \\ &= -\cos(A + B) \cdot \cos(A - B) + 2 \cos A \cdot \cos B \cdot \cos C - \cos^2 C \\ &\quad [\because \cos^2 B - \sin^2 A = \cos(A + B) \cdot \cos(A - B)] \end{aligned}$$

$$\begin{aligned}
 &= -\cos(-C) \cdot \cos(A-B) + \cos C (2 \cos A \cdot \cos B - \cos C) \\
 &= -\cos C (\cos A \cdot \cos B + \sin A \cdot \sin B - 2 \cos A \cdot \cos B + \cos C) \\
 &= \cos C (\cos A \cdot \cos B - \sin A \cdot \sin B - \cos C) \\
 &= \cos C [\cos(A+B) - \cos C] \\
 &= \cos C (\cos C - \cos C) \quad (\text{As } \cos C = \cos(A+B)) \\
 &= 0
 \end{aligned}$$

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length 'a' are (x_1, y_1) ,

$(x_2, y_2), (x_3, y_3)$, then
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

Solution:

We know that, the area of a triangle with vertices $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Also, we know the area of an equilateral triangle with side a is given by

$$\Delta = \frac{\sqrt{3}}{4} a^2$$

Hence,

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{\sqrt{3}}{4} a^2$$

On squaring both the sides, we get

$$\Rightarrow \Delta^2 = \frac{1}{4} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3}{16} a^4$$

or
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}$$

12. Find the value of θ satisfying

$$\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0.$$

Solution:

Given,

$$\begin{vmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{vmatrix} = 0$$

On expanding along C_3 , we have

$$\sin 3\theta \times (28 - 21) - \cos 2\theta \times (-7 - 7) - 2(3 + 4) = 0$$

$$7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$$

$$\sin 3\theta + 2 \cos 2\theta - 2 = 0$$

$$(3 \sin \theta - 4 \sin^3 \theta) + 2(1 - 2 \sin^2 \theta) - 2 = 0$$

$$4 \sin^3 \theta - 4 \sin^2 \theta + 3 \sin \theta = 0$$

$$\sin \theta (4 \sin^2 \theta - 4 \sin \theta + 3) = 0$$

$$\sin \theta (4 \sin^2 \theta - 6 \sin \theta + 2 \sin \theta + 3) = 0$$

$$\sin \theta (2 \sin \theta + 1)(2 \sin \theta - 3) = 0$$

$$\sin \theta = 0 \text{ or } \sin \theta = -1/2 \text{ or } \sin \theta = 3/2$$

$$\theta = n\pi \text{ or } \theta = m\pi + (-1)^n \left(-\frac{\pi}{6} \right); m, n \in \mathbb{Z}$$

$$\sin \theta = \frac{-3}{2} \text{ is not possible}$$

13. If $\begin{bmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{bmatrix}$, then find values of x .

Solution:

Given, $\begin{vmatrix} 4-x & 4+x & 4+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$

[Applying $R_1 \rightarrow R_1 + R_2 + R_3$], we have

$$\Rightarrow \begin{vmatrix} 12+x & 12+x & 12+x \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Now,

[Taking $(12+x)$ common from R_1]

$$\Rightarrow (12+x) \begin{vmatrix} 1 & 1 & 1 \\ 4+x & 4-x & 4+x \\ 4+x & 4+x & 4-x \end{vmatrix} = 0$$

Next,

[Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$]

$$\Rightarrow (12+x) \begin{vmatrix} 0 & 0 & 1 \\ 0 & -2x & 4+x \\ 2x & 2x & 4-x \end{vmatrix} = 0$$

$$\Rightarrow (12+x)(0 - (-2x)(2x)) = 0$$

$$(12+x)(4x^2) = 0$$

Hence, $x = -12, 0$

14. If $a_1, a_2, a_3, \dots, a_r$ are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} \text{ is independent of } r.$$

Solution:

We know that,

$$a_{r+1} = AR^{(r+1)-1} = AR^r;$$

where $a_r = r$ th term of G.P.,

A = First term of G.P.

and R = Common ratio of G.P.

$$\text{Now, } \begin{vmatrix} a_{r+1} & a_{r+5} & a_{r+9} \\ a_{r+7} & a_{r+11} & a_{r+15} \\ a_{r+11} & a_{r+17} & a_{r+21} \end{vmatrix} = \begin{vmatrix} AR^r & AR^{r+4} & AR^{r+8} \\ AR^{r+6} & AR^{r+10} & AR^{r+14} \\ AR^{r+10} & AR^{r+16} & AR^{r+20} \end{vmatrix}$$

[Taking AR^r , AR^{r+6} and AR^{r+10} common from R_1 , R_2 and R_3 , respectively]

$$= AR^r \cdot AR^{r+6} \cdot AR^{r+10} \begin{vmatrix} 1 & AR^4 & AR^8 \\ 1 & AR^4 & AR^8 \\ 1 & AR^6 & AR^{10} \end{vmatrix}$$

$$= 0 \quad [\text{As } R_1 \text{ and } R_2 \text{ are identical}]$$

Hence, the determinant is independent of r .

15. Show that the points $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a) do not lie on a straight line for any value of a .

Solution:

Given points are $(a + 5, a - 4)$, $(a - 2, a + 3)$ and (a, a) .

Now, we have to prove that these points do not lie on a straight line.

So, if we prove that these points form a triangle then it can't lie on a straight line.

$$\text{Area, } \Delta = \frac{1}{2} \begin{vmatrix} a+5 & a-4 & 1 \\ a-2 & a+3 & 1 \\ a & a & 1 \end{vmatrix}$$

[Applying $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$]

$$= \frac{1}{2} \begin{vmatrix} 5 & -4 & 0 \\ -2 & 3 & 0 \\ a & a & 1 \end{vmatrix} = \frac{1}{2} [(1 \cdot (15 - 8))] = \frac{7}{2} \neq 0$$

Hence, the given points form a triangle and can't lie on a straight line.

16. Show that the ΔABC is an isosceles triangle if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

Solution:

Given,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0$$

[Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$]

$$\Rightarrow \begin{vmatrix} 0 & 0 & 1 \\ \cos A - \cos C & \cos B - \cos C & 1 + \cos C \\ \cos^2 A + \cos A - \cos^2 C - \cos C & \cos^2 B + \cos B - \cos^2 C - \cos C & \cos^2 C + \cos C \end{vmatrix} = 0$$

Now,

[Taking $(\cos A - \cos C)$ common from C_1 and $(\cos B - \cos C)$ common from C_2]

$\Rightarrow (\cos A - \cos C)(\cos B - \cos C) \times$

$$\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 + \cos C \\ \cos A + \cos C + 1 & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

[Applying $C_1 \rightarrow C_1 - C_2$]

$$\Rightarrow (\cos A - \cos C)(\cos B - \cos C) \times$$

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 + \cos C \\ \cos A - \cos B & \cos B + \cos C + 1 & \cos^2 C + \cos C \end{vmatrix} = 0$$

So,

$$(\cos A - \cos C)(\cos B - \cos C)(\cos B - \cos A) = 0$$

$$\cos A = \cos C \text{ or } \cos B = \cos C \text{ or } \cos B = \cos A$$

$$A = C \text{ or } B = C \text{ or } B = A$$

Hence, ΔABC is an isosceles triangle.

17. Find A^{-1} if $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that $A^{-1} = (A^2 - 3I)/2$.

Solution:

Given,

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Co-factors are:

$$A_{11} = -1, A_{12} = 1, A_{13} = 1,$$

$$A_{21} = 1, A_{22} = -1, A_{23} = 1,$$

$$A_{31} = 1, A_{32} = 1, A_{33} = -1$$

$$\text{Now, } \text{adj } A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$|A| = 0 - 1(-1) + 1.1 = 2$$

$$\text{Thus, } A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\text{Now, } A^2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

Hence,

$$\begin{aligned}\frac{A^2 - 3I}{2} &= \frac{1}{2} \left\{ \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \right\} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = A^{-1}\end{aligned}$$

Hence proved.

Long Answer (L.A.)

18. If $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1} .

Using A^{-1} , solve the system of linear equations
 $x - 2y = 10$, $2x - y - z = 8$, $-2y + z = 7$.

Solution:

Given,

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$$

Co-factors are:

$$\begin{aligned}A_{11} &= -3, A_{12} = 2, A_{13} = 2, \\ A_{21} &= -2, A_{22} = 1, A_{23} = 1, \\ A_{31} &= -4, A_{32} = 2, A_{33} = 3\end{aligned}$$

Now,

$$\text{adj}A = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix}^T = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

$$|A| = 1(-3) - 2(-2) + 0 = 1$$

Hence,

$$A^{-1} = \frac{\text{adj}A}{|A|} = \begin{bmatrix} -3 & -2 & -4 \\ 2 & 1 & 2 \\ 2 & 1 & 3 \end{bmatrix}$$

Now, the system of linear equations are

$$\begin{aligned}x - 2y &= 10, \\ 2x - y - z &= 8\end{aligned}$$

and, $-2y + z = 7$

Or $AX = B$

$$\begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$$

where, $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix}$

Thus, $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ -2 & 1 & 1 \\ -4 & 2 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ 8 \\ 7 \end{bmatrix} = \begin{bmatrix} -30 + 16 + 14 \\ -20 + 8 + 7 \\ -40 + 16 + 21 \end{bmatrix} = \begin{bmatrix} 0 \\ -5 \\ -3 \end{bmatrix}$$

$$\therefore x = 0, y = -5 \text{ and } z = -3$$

19. Using matrix method, solve the system of equations

$$3x + 2y - 2z = 3, x + 2y + 3z = 6, 2x - y + z = 2.$$

Solution:

Given system of equations are:

$$3x + 2y - 2z = 3$$

$$x + 2y + 3z = 6 \text{ and}$$

$$2x - y + z = 2$$

Or,

$$AX = B$$

So,
$$\begin{bmatrix} 3 & 2 & -2 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

Hence, $X = A^{-1}B$

Now, for A^{-1} the co-factors are

$$A_{11} = 5, A_{12} = 5, A_{13} = -5,$$

$$A_{21} = 0, A_{22} = 7, A_{23} = 7,$$

$$A_{31} = 10, A_{32} = -11 \text{ and } A_{33} = 4$$

So,

$$\text{adj } A = \begin{bmatrix} 5 & 5 & -5 \\ 0 & 7 & 7 \\ 10 & -11 & 4 \end{bmatrix}^T = \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

$$|A| = 3(5) + 2(5) + (-2)(-5) = 35$$

Thus,

$$A^{-1} = \frac{\text{adj } A}{|A|} = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix}$$

Now, $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 5 & 0 & 10 \\ 5 & 7 & -11 \\ -5 & 7 & 4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 15 + 20 \\ 15 + 42 - 22 \\ -15 + 42 + 8 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 35 \\ 35 \\ 35 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, $x = 1$, $y = 1$ and $z = 1$

20. Given $A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$, find BA and use this to solve the system of

equations $y + 2z = 7$, $x - y = 3$, $2x + 3y + 4z = 17$.

Solution:

Given,

$$A = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$$

Now,

$$BA = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I$$

Thus,

$$B^{-1} = \frac{A}{6} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix}$$

Given system of equations are:

$$x - y = 3, 2x + 3y + 4z = 17 \text{ and } y + 2z = 7$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ 17 \\ 7 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 6 + 34 - 28 \\ -12 + 34 - 28 \\ 6 - 17 + 35 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 12 \\ -6 \\ 24 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$$

Therefore,
 $x = 2, y = -1$ and $z = 4$

21. If $a + b + c \neq 0$ and $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$, then prove that $a = b = c$.

Solution:

Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

[Applying $R_1 \rightarrow R_1 + R_2 + R_3$]

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$$

Now,

[Applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$]

$$\Delta = (a+b+c) \begin{vmatrix} 0 & 0 & 1 \\ b-a & c-a & a \\ c-b & a-b & b \end{vmatrix}$$

[Expanding along R_1]

$$\begin{aligned} &= (a+b+c)[1(b-a)(a-b) - (c-a)(c-b)] \\ &= (a+b+c)(ba - b^2 - a^2 + ab - c^2 + cb + ac - ab) \\ &= -(a+b+c)(a^2 + b^2 + c^2 - ab - bc - ca) \\ &= -\frac{1}{2} (a+b+c)[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] \\ &= -\frac{1}{2} (a+b+c)[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ac)] \\ &= -\frac{1}{2} (a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2] \end{aligned}$$

Given, $\Delta = 0$

$$\frac{-1}{2} (a + b + c) [(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \neq 0 \quad [\because a + b + c \neq 0, \text{ given}]$$

$$a - b \neq b - c = c - a = 0$$

$$a = b = c$$

22. Prove that $\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$ is divisible by $a + b + c$ and find the quotient.

Solution:

$$\Delta = \begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ca - b^2 \end{vmatrix}$$

Now, [Applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$]

$$\begin{aligned} \Delta &= \begin{vmatrix} bc - a^2 - ca + b^2 & ca - b^2 - ab + c^2 & ab - c^2 \\ ca - b^2 - ab + c^2 & ab - c^2 - bc + a^2 & bc - a^2 \\ ab - c^2 - bc + a^2 & bc - a^2 - ca + b^2 & ca - b^2 \end{vmatrix} \\ &= \begin{vmatrix} (b - a)(a + b + c) & (c - b)(a + b + c) & ab - c^2 \\ (c - b)(a + b + c) & (a - c)(a + b + c) & bc - a^2 \\ (a - c)(a + b + c) & (b - a)(a + b + c) & ca - b^2 \end{vmatrix} \end{aligned}$$

Next,

[Taking $(a + b + c)$ common from C_1 and C_2 each]

$$\Delta = (a + b + c)^2 \begin{vmatrix} b - a & c - b & ab - c^2 \\ c - b & a - c & bc - a^2 \\ a - c & b - a & ca - b^2 \end{vmatrix}$$

Then,

[Applying $R_1 \rightarrow R_1 + R_2 + R_3$]

$$\Delta = (a + b + c)^2 \begin{vmatrix} 0 & 0 & ab + bc + ca - (a^2 + b^2 + c^2) \\ c - b & a - c & bc - a^2 \\ a - c & b - a & ca - b^2 \end{vmatrix}$$

Lastly,

[Expanding along R_1]

$$\begin{aligned}\Delta &= (a+b+c)^2[ab+bc+ca-(a^2+b^2+c^2)][(c-b)(b-a)-(a-c)^2] \\ &= (a+b+c)^2(ab+bc+ca-a^2-b^2-c^2) \times \\ &\quad (bc-ac-b^2+ab-a^2-c^2+2ac) \\ &= (a+b+c)[(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2]\end{aligned}$$

Therefore, given determinant is divisible by $(a+b+c)$ and quotient is $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)^2$

23. If $x+y+z=0$, prove that

$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

Solution:

Taking,

$$\text{L.H.S.} = \begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix}$$

$$\begin{aligned}& \text{[Expanding]} \\ &= xa(a^2yz - x^2bc) - yb(y^2ac - b^2xz) + zc(c^2xy - z^2ab) \\ &= xyz a^3 - x^3 abc - y^3 abc + b^3 xyz + c^3 xyz - z^3 abc \\ &= xyz(a^3 + b^3 + c^3) - abc(x^3 + y^3 + z^3) \\ &= xyz(a^3 + b^3 + c^3) - abc(3xyz) \quad [\because x+y+z=0 \Rightarrow x^3+y^3+z^3-3xyz] \\ &= xyz(a^3 + b^3 + c^3 - 3abc) \\ &= xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix} = \text{R.H.S.}\end{aligned}$$

Hence proved.

Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. If $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$, then, value of x is

- (A) 3 (B) ± 3 (C) ± 6 (D) 6

Solution:

Option (C) ± 6

From the given,

On equating the determinants, we have

$$2x^2 - 40 = 18 + 14$$

$$2x^2 = 72$$

$$x^2 = 36$$

Thus, $x = \pm 6$

25. The value of determinant

$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$$

- (A) $a^3 + b^3 + c^3$ (B) $3bc$ (C) $a^3 + b^3 + c^3 - 3abc$ (D) none of these

Solution:

Option (C) $a^3 + b^3 + c^3 - 3abc$

Given,

$$\Delta = \begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$$

[Applying $C_1 \rightarrow C_1 - C_3$]

$$= \begin{vmatrix} -b & b+c & a \\ -c & c+a & b \\ -a & a+b & c \end{vmatrix}$$

[Applying $C_2 \rightarrow C_2 + C_1$]

$$= \begin{vmatrix} -b & c & a \\ -c & a & b \\ -a & b & c \end{vmatrix} = - \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}$$

$$= -[b(ac - b^2) - c(c^2 - ab) + a(bc - a^2)]$$

$$= a^3 + b^3 + c^3 - 3abc$$

26. The area of a triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is 9 sq. units. The value of k will be

- (A) 9 (B) 3 (C) -9 (D) 6

Solution:

Option (B) 3

We know that, the area of a triangle with vertices (x_1, y_1) , (x_2, y_2) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Area of triangle with vertices $(-3, 0)$, $(3, 0)$ and $(0, k)$ is

$$\Delta = \frac{1}{2} \begin{vmatrix} -3 & 0 & 1 \\ 3 & 0 & 1 \\ 0 & k & 1 \end{vmatrix} = 9 \quad (\text{given})$$

$$[-3(-k) - 0 + 1(3k)] = \pm 18$$

$$6k = \pm 18$$

Thus, $k = \pm \frac{18}{6} = \pm 3$

27. The determinant $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$ equals

(A) $abc(b-c)(c-a)(a-b)$

(B) $(b-c)(c-a)(a-b)$

(C) $(a+b+c)(b-c)(c-a)(a-b)$

(D) None of these

Solution:

Option (D)

Given, $\begin{vmatrix} b^2 - ab & b - c & bc - ac \\ ab - a^2 & a - b & b^2 - ab \\ bc - ac & c - a & ab - a^2 \end{vmatrix}$

$$= \begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$$

Now, [Taking $(b-a)$ common from C_1 and C_3 each]

$$= (b-a)^2 \begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

$$\begin{aligned}
 & \text{[Applying } C_2 \rightarrow C_2 + C_3] \\
 & = (b-a)^2 \begin{vmatrix} b & b & c \\ a & a & b \\ c & c & a \end{vmatrix} \\
 & = 0 \quad \text{[as } C_1 \text{ and } C_2 \text{ are identical]}
 \end{aligned}$$

28. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in the interval $-\pi/4 \leq x \leq \pi/4$ is
- (A) 0 (B) 2 (C) 1 (D) 3

Solution:

Option (C) 1

Given, $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$

Applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 2 \cos x + \sin x & \cos x & \cos x \\ 2 \cos x + \sin x & \sin x & \cos x \\ 2 \cos x + \sin x & \cos x & \sin x \end{vmatrix} = 0$$

$$(2 \cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

Now,

Applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$(2 \cos x + \sin x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$$(2 \cos x + \sin x)[1 \cdot (\sin x - \cos x)^2] = 0 \quad \text{(expanding along } C_1)$$

$$(2 \cos x + \sin x)(\sin x - \cos x)^2 = 0$$

$$2 \cos x = -\sin x \text{ or } \sin x = \cos x$$

$$\tan x = -2, \text{ which is not possible as for } -\frac{\pi}{4} \leq x \leq \frac{\pi}{4}, \text{ we get } -1 \leq \tan x \leq 1.$$

$$\text{or, } \tan x = 1$$

$$\text{Thus, } x = \frac{\pi}{4}$$

Therefore, only one real root exist.

29. If A, B and C are angles of a triangle, then the determinant to

$$\begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is equal}$$

- (A) 0 (B) -1 (C) 1 (D) None of these

Solution:

Option (C) 0

Given,

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

On expanding the determinant, we get

$$\Delta = -1 + 2 \cos A \cos B \cos C + \cos^2 A + \cos^2 B + \cos^2 C$$

$$\text{Now, } 2 \cos^2 A + 2 \cos^2 B + 2 \cos^2 C$$

$$= 1 + \cos 2A + 1 + \cos 2B + 1 + \cos 2C$$

$$= 3 + (\cos 2A + \cos 2B + \cos 2C)$$

$$= 3 + (\cos 2A + \cos 2B) + \cos 2C$$

$$= 3 + 2 \cos(A+B) \cos(A-B) + 2 \cos^2 C - 1$$

$$= 2 + 2 \cos(\pi - C) \cos(A-B) + 2 \cos^2 C$$

$$= 2 - 2 \cos C \cos(A-B) + 2 \cos^2 C$$

$$= 2 - 2 \cos C [\cos(A-B) - \cos C]$$

$$= 2 - 2 \cos C [\cos(A-B) - \cos \{\pi - (A+B)\}]$$

$$= 2 - 2 \cos C [\cos(A-B) + \cos(A+B)]$$

$$= 2 - 4 \cos A \cos B \cos C$$

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2 \cos A \cos B \cos C$$

Thus, $\Delta = 0$