EXERCISE 20.4 PAGE NO: 20.39

1. Find the sum of the following series to infinity:
(i) $1 – 1/3 + 1/3^2 – 1/3^3 + 1/3^4 + \ldots \infty$
(ii) $8 + 4\sqrt{2} + 4 + \ldots. \infty$
(iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + \ldots. \infty$
(iv) 10 – 9 + 8.1 – 7.29 + …. ∞
Solution:
(i) $1 – 1/3 + 1/3^2 – 1/3^3 + 1/3^4 + \ldots \infty$
Given:
$S_\infty = 1 – 1/3 + 1/3^2 – 1/3^3 + 1/3^4 + \ldots \infty$
Where, a = 1, r = -1/3
By using the formula,
$S_\infty = a/(1 - r)$
= 1 / (1 – (-1/3))
= 1/ (1 + 1/3)
= 1/ ((3+1)/3)
= 1/ (4/3)
= ¾

(ii) $8 + 4\sqrt{2} + 4 + \ldots. \infty$
Given:
$S_\infty = 8 + 4\sqrt{2} + 4 + \ldots. \infty$
Where, a = 8, $r = 4/4\sqrt{2} = 1/\sqrt{2}$
By using the formula,
$S_\infty = a/(1 - r)$
$= 8 / (1 – (1/\sqrt{2}))$
$= 8 / ((\sqrt{2} – 1)/\sqrt{2})$
$= 8\sqrt{2} /(\sqrt{2} – 1)$
Multiply and divide with $\sqrt{2} + 1$ we get,
$= 8\sqrt{2} /(\sqrt{2} – 1) \times (\sqrt{2} + 1)/(\sqrt{2} + 1)$
$= 8 (2 + \sqrt{2})/(2-1)$
$= 8 (2 + \sqrt{2})$

(iii) $2/5 + 3/5^2 + 2/5^3 + 3/5^4 + \ldots. \infty$
The given terms can be written as,
$(2/5 + 2/5^3 + \ldots) + (3/5^2 + 3/5^4 + \ldots)$
(a = 2/5, r = 1/25) and (a = 3/25, r = 1/25)
By using the formula,

$S_\infty = a/(1 - r)$

$$= \left(\frac{\frac{2}{5}}{1-\frac{1}{25}}\right) + \left(\frac{\frac{3}{5}}{1-\frac{1}{25}}\right)$$

$$= \left(\frac{\frac{2}{5}}{\frac{24}{25}}\right) + \left(\frac{\frac{3}{5}}{\frac{24}{25}}\right)$$

$$= \left(\frac{10}{24} + \frac{3}{24}\right)$$

$$= \frac{13}{24}$$

(iv) $10 – 9 + 8.1 – 7.29 + \ldots. \infty$

Given:

$S_\infty = 8 + 4\sqrt{2} + 4 + \ldots. \infty$

Where, a = 10, r = -9/10

By using the formula,

$S_\infty = a/(1 - r)$

= 10 / (1 – (-9/10))

= 10 / (1 + 9/10)

= 10 / ((10+9)/10)

= 10 / (19/10)

= 100/19

= 5.263

2. Prove that :

$(9^{1/3} . 9^{1/9} . 9^{1/27} \ldots.\infty) = 3.$

Solution:

Let us consider the LHS

$(9^{1/3} . 9^{1/9} . 9^{1/27} \ldots.\infty)$

This can be written as

$9^{1/3 + 1/9 + 1/27 + \ldots\infty}$

So let us consider m = 1/3 + 1/9 + 1/27 + … ∞

Where, a = 1/3, r = (1/9) / (1/3) = 1/3

By using the formula,

$S_\infty = a/(1 - r)$

= (1/3) / (1 – (1/3))

= (1/3) / ((3-1)/3)

= (1/3) / (2/3)

= ½

So, $9^m = 9^{1/2} = 3$ = RHS
Hence proved.

3. Prove that :
$(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \ldots \infty) = 2$.
Solution:
Let us consider the LHS
$(2^{1/4} . 4^{1/8} . 8^{1/16} . 16^{1/32} \ldots \infty)$
This can be written as
$2^{1/4} . 2^{2/8} . 2^{3/16} . 2^{1/8} \ldots \infty$
Now,
$2^{1/4 + 2/8 + 3/16 + 1/8 + \ldots \infty}$
So let us consider 2^x, where x = ¼ + 2/8 + 3/16 + 1/8 + … ∞ …. (1)
Multiply both sides of the equation with 1/2, we get
x/2 = ½ (¼ + 2/8 + 3/16 + 1/8 + … ∞)
= 1/8 + 2/16 + 3/32 + … + ∞ …. (2)
Now, subtract (2) from (1) we get,
x – x/2 = (¼ + 2/8 + 3/16 + 1/8 + … ∞) – (1/8 + 2/16 + 3/32 + … + ∞)
By grouping similar terms,
x/2 = ¼ + (2/8 – 1/8) + (3/16 – 2/16) + … ∞
x/2 = ¼ + 1/8 + 1/16 + … ∞
x = ½ + ¼ + 1/8 + 1/16 + … ∞
Where, a = 1/2, r = (1/4) / (1/2) = 1/2
By using the formula,
S_∞ = a/(1 - r)
= (1/2) / (1 – 1/2)
= (1/2) / ((2-1)/2)
= (1/2) / (1/2)
= 1
From equation (1), $2^x = 2^1 = 2$ = RHS
Hence proved.

4. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \ldots$ to ∞ and s_p the sum of the series $1 – r^p + r^{2p} - \ldots$ to ∞, prove that $s_p + S_p = 2 S_{2p}$.
Solution:
Given:
$S_p = 1 + r^p + r^{2p} + \ldots \infty$
By using the formula,
S_∞ = a/(1 - r)

Where, $a = 1$, $r = r^p$
So,
$S_p = 1 / (1 - r^p)$
Similarly, $s_p = 1 - r^p + r^{2p} - \ldots \infty$
By using the formula,
$S_\infty = a/(1 - r)$
Where, $a = 1$, $r = -r^p$
So,
$S_p = 1 / (1 – (-r^p))$
$= 1 / (1 + r^p)$
Now, $S_p + s_p = [1 / (1 - r^p)] + [1 / (1 + r^p)]$
$2S_{2p} = [(1 - r^p) + (1 + r^p)] / (1 – r^{2p})$
$= 2 /(1 – r^{2p})$
$\therefore 2S_{2p} = S_p + S_p$

5. Find the sum of the terms of an infinite decreasing G.P. in which all the terms are positive, the first term is 4, and the difference between the third and fifth term is equal to 32/81.

Solution:

Let 'a' be the first term of GP and 'r' be the common ratio.
We know that nth term of a GP is given by-
$a_n = ar^{n-1}$
As, $a = 4$ (given)
And $a_5 – a_3 = 32/81$ (given)
$4r^4 – 4r^2 = 32/81$
$4r^2(r^2 – 1) = 32/81$
$r^2(r^2 – 1) = 8/81$

Let us denote r^2 with y
$81y(y-1) = 8$
$81y^2 – 81y - 8 = 0$
Using the formula of the quadratic equation to solve the equation, we get

$$y = \frac{-(-81)\pm\sqrt{81^2-4(-8)(81)}}{16}$$
$$= \frac{81\pm\sqrt{6561-2592}}{162}$$
$$= \frac{81\pm63}{162}$$

$y = 18/162 = 1/9$ or

$y = 144/162$
$= 8/9$
So, $r^2 = 1/9$ or $8/9$
$= 1/3$ or $2\sqrt{2}/3$
We know that,
Sum of infinite, $S_\infty = a/(1 - r)$
Where, $a = 4$, $r = 1/3$
$S_\infty = 4 / (1 – (1/3))$
$= 4 / ((3-1)/3)$
$= 4 / (2/3)$
$= 12/2$
$= 6$

Sum of infinite, $S_\infty = a/(1 - r)$
Where, $a = 4$, $r = 2\sqrt{2}/3$
$S_\infty = 4 / (1 – (2\sqrt{2}/3))$
$= 12 / (3 - 2\sqrt{2})$

6. Express the recurring decimal 0.125125125 … as a rational number.
Solution:
Given:
0.125125125
So, $0.125125125 = 0.\overline{125}$
$= 0.125 + 0.000125 + 0.000000125 + \ldots$
This can be written as
$125/10^3 + 125/10^6 + 125/10^9 + \ldots$
$125/10^3 [1 + 1/10^3 + 1/10^6 + \ldots]$
By using the formula,
$S_\infty = a/(1 - r)$
$125/10^3 [1 / (1 – 1/1000)]$
$125/10^3 [1 / ((1000 – 1)/1000))]$
$125/10^3 [1 / (999/1000)]$
$125/1000 (1000/999)$
$125/999$
∴ The decimal 0.125125125 can be expressed in rational number as 125/999