EXERCISE 21.2

Sum the following series to n terms:

1. 3 + 5 + 9 + 15 + 23 + …………

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ … (1)

Equation (1) can be rewritten as:

$S_n = 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)

By subtracting (2) from (1) we get

$S_n = 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$

$\underline{S_n = \quad 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$

$0 = 3 + [2 + 4 + 6 + 8 + \ldots + (T_n - T_{n-1})] - T_n$

The difference between the successive terms are 5-3 = 2, 9-5 = 4, 15-9 = 6,

So these differences are in A.P

Now,

$$3+\left[\frac{(n-1)}{2}\left\{4+(n-2)2\right\}\right]-T_n=0$$

$$3+\left[\frac{(n-1)}{2}(2n)\right]=T_n$$

$$3+n(n-1)=T_n$$

Now,

$$\begin{aligned}S_n &= \sum_{k=1}^{n} T_k\\ &= \sum_{k=1}^{n}\left\{3+k(k-1)\right\}\\ &= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} 3-\sum_{k=1}^{n} k\\ &= \frac{n(n+1)(2n+1)}{6}+3n-\frac{n(n+1)}{2}\\ &= \frac{n}{3}\left[\frac{(n+1)(2n+1)}{2}+9-\frac{3}{2}(n+1)\right]\\ &= \frac{n\left[n^2+8\right]}{3}\\ &= \frac{n}{3}(n^2+8)\end{aligned}$$

∴ The sum of the series is n/3 $(n^2 + 8)$

2. 2 + 5 + 10 + 17 + 26 + ………..

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n \ldots (1)$

Equation (1) can be rewritten as:

$S_n = 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n \ldots\ldots..(2)$

By subtracting (2) from (1) we get

$S_n = 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$

$\underline{S_n = \quad 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$

$0 = 2 + [3 + 5 + 7 + 9 + \ldots + (T_n - T_{n-1})] - T_n$

The difference between the successive terms are 3, 5, 7, 9

So these differences are in A.P

Now,

$$2+\left[\frac{(n-1)}{2}\left\{6+(n-2)2\right\}\right]-T_n=0$$

$$2+\left[n^2-1\right]=T_n$$

$$\left[n^2+1\right]=T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} T_k \\ &= \sum_{k=1}^{n}\left(k^2+1\right) \\ &= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} 1 \\ &= \frac{n(n+1)(2n+1)}{6}+n \\ &= \frac{n(n+1)(2n+1)+6n}{6} \\ &= \frac{n\left(2n^2+3n+7\right)}{6} \\ &= \frac{n}{6}\left(2n^2+3n+7\right) \end{aligned}$$

∴ The sum of the series is $n/6\ (2n^2 + 3n + 7)$

3. 1 + 3 + 7 + 13 + 21 + …

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n \ldots (1)$

Equation (1) can be rewritten as:

$S_n = 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)

By subtracting (2) from (1) we get

$S_n = 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$

$\underline{S_n = \quad 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$

$0 = 1 + [2 + 4 + 6 + 8 + \ldots + (T_n – T_{n-1})] - T_n$

The difference between the successive terms are 2, 4, 6, 8

So these differences are in A.P

Now,

$$1+\left[\frac{(n-1)}{2}\{4+(n-2)2\}\right]-T_n=0$$

$$1+\left[n^2-n\right]=T_n$$

$$\left[n^2-n+1\right]=T_n$$

Now,

$$\begin{aligned} S_n &= \sum_{k=1}^{n} T_k \\ &= \sum_{k=1}^{n}\left(k^2-k+1\right) \\ &= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} 1-\sum_{k=1}^{n} k \\ &= \frac{n(n+1)(2n+1)}{6}+n-\frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{2}\left(\frac{2n-2}{3}\right)+n \\ &= n\left(\frac{n^2-1+3}{3}\right) \\ &= \frac{n}{3}\left(n^2+2\right) \end{aligned}$$

∴ The sum of the series is $n/3\ (n^2 + 2)$

4. 3 + 7 + 14 + 24 + 37 + …

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ … (1)

Equation (1) can be rewritten as:

$S_n = 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)

By subtracting (2) from (1) we get

$S_n = 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$

$\underline{S_n = \quad 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$

$0 = 3 + [4 + 7 + 10 + 13 + \ldots + (T_n – T_{n-1})] - T_n$

The difference between the successive terms are 4, 7, 10, 13
So these differences are in A.P
Now,

$$3 + \left[\frac{(n-1)}{2}\{8 + (n-2)3\}\right] - T_n = 0$$

$$3 + \left[\frac{(n-1)}{2}(3n+2)\right] - T_n = 0$$

$$\left[\frac{3n^2-n+4}{2}\right] = T_n$$

$$\left[\frac{3}{2}n^2 - \frac{n}{2} + 2\right] = T_n$$

Now,

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n}\left(\frac{3}{2}k^2 - \frac{k}{2} + 2\right)$$
$$= \frac{3}{2}\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} 2 - \frac{1}{2}\sum_{k=1}^{n} k$$
$$= \frac{n(n+1)(2n+1)}{4} + 2n - \frac{n(n+1)}{4}$$
$$= \frac{n(n+1)(2n)+8n}{4}$$
$$= \frac{(n+1)(2n^2)+8n}{4}$$
$$= \frac{n}{2}[n(n+1)+4]$$
$$= \frac{n}{2}[n^2+n+4]$$

∴ The sum of the series is n/2 [n^2 + n + 4]

5. 1 + 3 + 6 + 10 + 15 + …

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.
We have,
S_n = 1 + 3 + 6 + 10 + 15 + …………. + T_{n-1} + T_n … (1)
Equation (1) can be rewritten as:
S_n = 1 + 3 + 6 + 10 + 15 + …………. + T_{n-1} + T_n ……..(2)
By subtracting (2) from (1) we get
S_n = 1 + 3 + 6 + 10 + 15 + …………. + T_{n-1} + T_n
S_n = 1 + 3 + 6 + 10 + 15 + …………. + T_{n-1} + T_n
0 = 1 + [2 + 3 + 4 + 5 + … + (T_n – T_{n-1})] - T_n

The difference between the successive terms are 2, 3, 4, 5

So these differences are in A.P
Now,

$$1+\left[\frac{(n-1)}{2}\left(4+(n-2)1\right)\right]-T_n=0$$

$$1+\left[\frac{(n-1)}{2}(n+2)\right]-T_n=0$$

$$\left[\frac{n^2+n}{2}\right]=T_n$$

Now,

$$\begin{aligned}S_n &= \sum_{k=1}^{n} T_k\\ &= \sum_{k=1}^{n}\left(\frac{k^2+k}{2}\right)\\ &= \frac{1}{2}\sum_{k=1}^{n} k^2+\frac{1}{2}\sum_{k=1}^{n} k\\ &= \frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}\\ &= \frac{n(n+1)}{4}\left(\frac{2n+1}{3}+1\right)\\ &= \frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right)\\ &= \frac{n(n+1)}{2}\left(\frac{n+2}{3}\right)\\ &= \frac{n(n+1)(n+2)}{6}\\ &= \frac{n}{6}(n+1)(n+2)\end{aligned}$$

∴ The sum of the series is n/6 (n+1) (n+2)