EXERCISE 21.1 PAGE NO: 21.10

Find the sum of the following series to n terms:

1. $1^3 + 3^3 + 5^3 + 7^3 + \ldots\ldots..$

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = [1 + (n - 1)2]^3$

$= (2n - 1)^3$

$= (2n)^3 – 3\,(2n)^2.\,1 + 3.1^2.2n-1^3$ [Since, $(a-b)^3 = a^3 – 3a^2b + 3ab^2 - b$]

$= 8n^3 – 12n^2 + 6n – 1$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} [2k-1]^3$$
$$= \sum_{k=1}^{n} [8k^3 - 1 - 6k\,(2k-1)\,]$$
$$= \sum_{k=1}^{n} [8k^3 - 1 - 12k^2 + 6k\,]$$
$$= \sum_{k=1}^{n} [8k^3 - 1 - 12k^2 + 6k\,]$$
$$= 8\sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} 1 - 12\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k$$
$$= \frac{8n^2(n+1)^2}{4} - n - \frac{12n(n+1)(2n+1)}{6} + \frac{6\,n(n+1)}{2}$$

Upon simplification we get,

$= 2n^2\,(n + 1)^2 – n – 2n\,(n + 1)\,(2n + 1) + 3n\,(n + 1)$

$= n\,(n + 1)\,[2n\,(n + 1) – 2\,(2n + 1) + 3] – n$

$= n\,(n + 1)\,[2n^2 – 2n + 1] – n$

$= n\,[2n^3 – 2n^2 + n + 2n^2 – 2n + 1 - 1]$

$= n\,[2n^3 - n]$

$= n^2\,[2n^2 - 1]$

∴ The sum of the series is $n^2\,[2n^2 - 1]$

2. $2^3 + 4^3 + 6^3 + 8^3 + \ldots\ldots\ldots$

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = (2n)^3$

$= 8n^3$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} 8k^3$$
$$= 8\sum_{k=1}^{n} k^3$$
$$= 8\left[\frac{n(n+1)}{2}\right]^2$$
$$= 8 \times \frac{n^2(n+1)^2}{4}$$
$$= 2n^2 (n + 1)^2$$
$$= 2\{n (n + 1)\}^2$$

∴ The sum of the series is $2\{n (n + 1)\}^2$

3. 1.2.5 + 2.3.6 + 3.4.7 + ……..

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = n (n + 1) (n + 4)$
$= n (n^2 + 5n + 4)$
$= n^3 + 5n^2 + 4n$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 5k^2 + \sum_{k=1}^{n} 4k$$
$$= \sum_{k=1}^{n} k^3 + 5\sum_{k=1}^{n} k^2 + 4\sum_{k=1}^{n} k$$

Upon simplification we get,

$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{4n(n+1)}{2}$$
$$= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + 2n(n+1)$$
$$= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} + \frac{5(2n+1)}{3} + 4\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{n^2+n}{2} + \frac{10n+5}{3} + 4\right)$$

$$= \frac{n(n+1)}{2}\left(\frac{3n^2+3n+20n+10+24}{6}\right)$$
$$= \frac{n}{12}(n+1)(3n^2+23n+34)$$

∴ The sum of the series is

$$= \frac{n}{12}(n+1)(3n^2+23n+34)$$

4. 1.2.4 + 2.3.7 + 3.4.10 + … to n terms.

Solution:

Let T_n be the nth term of the given series.

We have:

$T_n = n(n+1)(3n+1)$

$= n(3n^2 + 4n + 1)$

$= 3n^3 + 4n^2 + n$

Now, let S_n be the sum of n terms of the given series.

We have:

$$S_n = \sum_{k=1}^{n} T_k$$
$$= \sum_{k=1}^{n} 3k^3 + \sum_{k=1}^{n} 4k^2 + \sum_{k=1}^{n} k$$
$$= 3\sum_{k=1}^{n} k^3 + 4\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k$$

Upon simplification we get,

$$= \frac{3n^2(n+1)^2}{4} + \frac{4n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$
$$= \frac{3n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{3} + \frac{n(n+1)}{2}$$
$$= \frac{n(n+1)}{2}\left(\frac{3n(n+1)}{2} + \frac{4(2n+1)}{3} + 1\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{3n^2+3n}{2} + \frac{8n+4}{3} + 1\right)$$
$$= \frac{n(n+1)}{2}\left(\frac{9n^2+9n+16n+8+6}{6}\right)$$
$$= \frac{n}{12}(n+1)(9n^2+25n+14)$$

∴ The sum of the series is

$$= \frac{n}{12}(n+1)(9n^2+25n+14)$$

5. 1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + … to n terms

Solution:
Let T_n be the nth term of the given series.
We have:
$T_n = n(n+1)/2$
$= (n^2 + n)/2$
Now, let S_n be the sum of n terms of the given series.
We have:

$$
\begin{aligned}
S_n &= \sum_{k=1}^{n} T_k \\
&= \sum_{k=1}^{n} \left(\frac{k^2+k}{2}\right) \\
&= \frac{1}{2} \sum_{k=1}^{n} \left(k^2 + k\right) \\
&= \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right] \\
&= \frac{n(n+1)}{4} \left(\frac{2n+1}{3} + 1\right) \\
&= \frac{n(n+1)}{4} \left(\frac{2n+4}{3}\right) \\
&= \frac{n(n+1)(2n+4)}{12} \\
&= \frac{n(n+1)(n+2)}{6}
\end{aligned}
$$

∴ The sum of the series is [n(n+1)(n+2)]/6

EXERCISE 21.2

Sum the following series to n terms:

1. 3 + 5 + 9 + 15 + 23 + …………

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots + T_{n-1} + T_n$ … (1)

Equation (1) can be rewritten as:

$S_n = 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)

By subtracting (2) from (1) we get

$S_n = 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots + T_{n-1} + T_n$

$\underline{S_n = \quad 3 + 5 + 9 + 15 + 23 + \ldots\ldots\ldots + T_{n-1} + T_n}$

$0 = 3 + [2 + 4 + 6 + 8 + \ldots + (T_n - T_{n-1})] - T_n$

The difference between the successive terms are 5-3 = 2, 9-5 = 4, 15-9 = 6,

So these differences are in A.P

Now,

$$3+\left[\frac{(n-1)}{2}\{4+(n-2)2\}\right]-T_n=0$$

$$3+\left[\frac{(n-1)}{2}(2n)\right]=T_n$$

$$3+n(n-1)=T_n$$

Now,

$$\begin{aligned}S_n &= \sum_{k=1}^{n} T_k\\ &= \sum_{k=1}^{n}\{3+k(k-1)\}\\ &= \sum_{k=1}^{n} k^2+\sum_{k=1}^{n} 3-\sum_{k=1}^{n} k\\ &= \frac{n(n+1)(2n+1)}{6}+3n-\frac{n(n+1)}{2}\\ &= \frac{n}{3}\left[\frac{(n+1)(2n+1)}{2}+9-\frac{3}{2}(n+1)\right]\\ &= \frac{n[n^2+8]}{3}\\ &= \frac{n}{3}(n^2+8)\end{aligned}$$

∴ The sum of the series is $n/3\ (n^2 + 8)$

2. 2 + 5 + 10 + 17 + 26 + ………..

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n \ldots (1)$

Equation (1) can be rewritten as:

$S_n = 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n \ldots\ldots(2)$

By subtracting (2) from (1) we get

$S_n = 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$

$\underline{S_n = \quad 2 + 5 + 10 + 17 + 26 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$

$0 = 2 + [3 + 5 + 7 + 9 + \ldots + (T_n – T_{n-1})] - T_n$

The difference between the successive terms are 3, 5, 7, 9

So these differences are in A.P

Now,

$$2+\left[\frac{(n-1)}{2}\{6+(n-2)2\}\right]-T_n=0$$

$$2+\left[n^2-1\right]=T_n$$

$$\left[n^2+1\right]=T_n$$

Now,

$$S_n=\sum_{k=1}^{n} T_k$$

$$=\sum_{k=1}^{n}\left(k^2+1\right)$$

$$=\sum_{k=1}^{n} k^2+\sum_{k=1}^{n} 1$$

$$=\frac{n(n+1)(2n+1)}{6}+n$$

$$=\frac{n(n+1)(2n+1)+6n}{6}$$

$$=\frac{n\left(2n^2+3n+7\right)}{6}$$

$$=\frac{n}{6}\left(2n^2+3n+7\right)$$

∴ The sum of the series is $n/6\ (2n^2 + 3n + 7)$

3. 1 + 3 + 7 + 13 + 21 + …

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.

We have,

$S_n = 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n \ldots (1)$

Equation (1) can be rewritten as:

$S_n = 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)
By subtracting (2) from (1) we get
$S_n = 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$
$\underline{S_n = \quad 1 + 3 + 7 + 13 + 21 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$
$0 = 1 + [2 + 4 + 6 + 8 + \ldots + (T_n - T_{n-1})] - T_n$

The difference between the successive terms are 2, 4, 6, 8
So these differences are in A.P
Now,

$$1+\left[\frac{(n-1)}{2}\{4+(n-2)2\}\right]-T_n=0$$
$$1+\left[n^2-n\right]=T_n$$
$$\left[n^2-n+1\right]=T_n$$

Now,

$$S_n=\sum_{k=1}^{n} T_k$$
$$=\sum_{k=1}^{n}\left(k^2-k+1\right)$$
$$=\sum_{k=1}^{n} k^2+\sum_{k=1}^{n} 1-\sum_{k=1}^{n} k$$
$$=\frac{n(n+1)(2n+1)}{6}+n-\frac{n(n+1)}{2}$$
$$=\frac{n(n+1)}{2}\left(\frac{2n-2}{3}\right)+n$$
$$=n\left(\frac{n^2-1+3}{3}\right)$$
$$=\frac{n}{3}(n^2+2)$$

∴ The sum of the series is $n/3\ (n^2 + 2)$

4. 3 + 7 + 14 + 24 + 37 + …
Solution:
Let T_n be the nth term and S_n be the sum to n terms of the given series.
We have,
$S_n = 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ … (1)
Equation (1) can be rewritten as:
$S_n = 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)
By subtracting (2) from (1) we get
$S_n = 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$
$\underline{S_n = \quad 3 + 7 + 14 + 24 + 37 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$
$0 = 3 + [4 + 7 + 10 + 13 + \ldots + (T_n - T_{n-1})] - T_n$

The difference between the successive terms are 4, 7, 10, 13
So these differences are in A.P
Now,

$$3+\left[\frac{(n-1)}{2}\{8+(n-2)3\}\right]-T_n=0$$

$$3+\left[\frac{(n-1)}{2}(3n+2)\right]-T_n=0$$

$$\left[\frac{3n^2-n+4}{2}\right]=T_n$$

$$\left[\frac{3}{2}n^2-\frac{n}{2}+2\right]=T_n$$

Now,

$$\begin{aligned}S_n&=\sum_{k=1}^{n}T_k\\&=\sum_{k=1}^{n}\left(\frac{3}{2}k^2-\frac{k}{2}+2\right)\\&=\frac{3}{2}\sum_{k=1}^{n}k^2+\sum_{k=1}^{n}2-\frac{1}{2}\sum_{k=1}^{n}k\\&=\frac{n(n+1)(2n+1)}{4}+2n-\frac{n(n+1)}{4}\\&=\frac{n(n+1)(2n)+8n}{4}\\&=\frac{(n+1)(2n^2)+8n}{4}\\&=\frac{n}{2}\left[n(n+1)+4\right]\\&=\frac{n}{2}\left[n^2+n+4\right]\end{aligned}$$

∴ The sum of the series is n/2 $[n^2 + n + 4]$

5. 1 + 3 + 6 + 10 + 15 + …

Solution:

Let T_n be the nth term and S_n be the sum to n terms of the given series.
We have,
$S_n = 1 + 3 + 6 + 10 + 15 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ … (1)
Equation (1) can be rewritten as:
$S_n = 1 + 3 + 6 + 10 + 15 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$ ……..(2)
By subtracting (2) from (1) we get
$S_n = 1 + 3 + 6 + 10 + 15 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n$
$\underline{S_n = \quad 1 + 3 + 6 + 10 + 15 + \ldots\ldots\ldots\ldots + T_{n-1} + T_n}$
$0 = 1 + [2 + 3 + 4 + 5 + \ldots + (T_n - T_{n-1})] - T_n$

The difference between the successive terms are 2, 3, 4, 5

So these differences are in A.P
Now,

$$1+\left[\frac{(n-1)}{2}(4+(n-2)1)\right]-T_n=0$$

$$1+\left[\frac{(n-1)}{2}(n+2)\right]-T_n=0$$

$$\left[\frac{n^2+n}{2}\right]=T_n$$

Now,

$$S_n=\sum_{k=1}^{n}T_k$$
$$=\sum_{k=1}^{n}\left(\frac{k^2+k}{2}\right)$$
$$=\frac{1}{2}\sum_{k=1}^{n}k^2+\frac{1}{2}\sum_{k=1}^{n}k$$
$$=\frac{n(n+1)(2n+1)}{12}+\frac{n(n+1)}{4}$$
$$=\frac{n(n+1)}{4}\left(\frac{2n+1}{3}+1\right)$$
$$=\frac{n(n+1)}{4}\left(\frac{2n+4}{3}\right)$$
$$=\frac{n(n+1)}{2}\left(\frac{n+2}{3}\right)$$
$$=\frac{n(n+1)(n+2)}{6}$$
$$=\frac{n}{6}(n+1)(n+2)$$

∴ The sum of the series is n/6 (n+1) (n+2)