EXERCISE 27.1 PAGE NO: 27.13

1. The equation of the directrix of a hyperbola is x – y + 3 = 0. Its focus is (-1, 1) and eccentricity 3. Find the equation of the hyperbola.

Solution:

Given:

The equation of the directrix of a hyperbola => x – y + 3 = 0.

Focus = (-1, 1) and

Eccentricity = 3

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

e = PF/PM

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x+1)^2+(y-1)^2} = 3\left|\frac{(x-y+3)}{\sqrt{1^2+(-1)^2}}\right|$$

$$\sqrt{(x+1)^2+(y-1)^2} = 3\left|\frac{(x-y+3)}{\sqrt{1+1}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x+1)^2+(y-1)^2}\right)^2 = \left(3\left|\frac{(x-y+3)}{\sqrt{1+1}}\right|\right)^2$$

$$(x+1)^2+(y-1)^2 = \frac{3^2(x-y+3)^2}{2}$$

[We know that $(a – b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

So, $2\{x^2 + 1 + 2x + y^2 + 1 – 2y\} = 9\{x^2 + y^2 + 9 + 6x – 6y – 2xy\}$

$2x^2 + 2 + 4x + 2y^2 + 2 – 4y = 9x^2 + 9y^2 + 81 + 54x – 54y – 18xy$

$2x^2 + 4 + 4x + 2y^2 – 4y – 9x^2 - 9y^2 - 81 – 54x + 54y + 18xy = 0$

$– 7x^2 - 7y^2 – 50x + 50y + 18xy – 77 = 0$

$7(x^2 + y^2) – 18xy + 50x – 50y + 77 = 0$

∴The equation of hyperbola is $7(x^2 + y^2) – 18xy + 50x – 50y + 77 = 0$

2. Find the equation of the hyperbola whose

(i) focus is (0, 3), directrix is x + y – 1 = 0 and eccentricity = 2

(ii) focus is (1, 1), directrix is 3x + 4y + 8 = 0 and eccentricity = 2

(iii) focus is (1, 1) directrix is 2x + y = 1 and eccentricity =√3

(iv) focus is (2, -1), directrix is 2x + 3y = 1 and eccentricity = 2
(v) focus is (a, 0), directrix is 2x + 3y = 1 and eccentricity = 2
(vi)focus is (2, 2), directrix is x + y = 9 and eccentricity = 2
Solution:
(i) focus is (0, 3), directrix is x + y – 1 = 0 and eccentricity = 2
Given:
Focus = (0, 3)
Directrix => x + y – 1 = 0
Eccentricity = 2
Now, let us find the equation of the hyperbola
Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.
By using the formula,
e = PF/PM
PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]
So,

$$\sqrt{(x-0)^2+(y-3)^2} = 2\left|\frac{(x+y-1)}{\sqrt{1^2+1^2}}\right|$$

$$\sqrt{(x-0)^2+(y-3)^2} = 2\left|\frac{(x+y-1)}{\sqrt{1+1}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-0)^2+(y-3)^2}\right)^2 = \left(2\left|\frac{(x+y-1)}{\sqrt{1+1}}\right|\right)^2$$

$$(x-0)^2+(y-3)^2 = \frac{2^2(x+y-1)^2}{2}$$

[We know that $(a – b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]
So, $2\{x^2 + y^2 + 9 – 6y\} = 4\{x^2 + y^2 + 1 – 2x – 2y + 2xy\}$
$2x^2 + 2y^2 + 18 – 12y = 4x^2 + 4y^2 + 4 – 8x – 8y + 8xy$
$2x^2 + 2y^2 + 18 – 12y – 4x^2 – 4y^2 – 4 – 8x + 8y – 8xy = 0$
$– 2x^2 – 2y^2 – 8x – 4y – 8xy + 14 = 0$
$-2(x^2 + y^2 – 4x + 2y + 4xy – 7) = 0$
$x^2 + y^2 – 4x + 2y + 4xy – 7 = 0$
∴The equation of hyperbola is $x^2 + y^2 – 4x + 2y + 4xy – 7 = 0$

(ii) focus is (1, 1), directrix is 3x + 4y + 8 = 0 and eccentricity = 2
Focus = (1, 1)
Directrix => 3x + 4y + 8 = 0

Eccentricity = 2
Now, let us find the equation of the hyperbola
Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.
By using the formula,
e = PF/PM
PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]
So,

$$\sqrt{(x-0)^2+(y-3)^2} = 2\left|\frac{(x+y-1)}{\sqrt{1^2+1^2}}\right|$$

$$\sqrt{(x-0)^2+(y-3)^2} = 2\left|\frac{(x+y-1)}{\sqrt{1+1}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-0)^2+(y-3)^2}\right)^2 = \left(2\left|\frac{(x+y-1)}{\sqrt{1+1}}\right|\right)^2$$

$$(x-0)^2+(y-3)^2 = \frac{2^2(x+y-1)^2}{2}$$

[We know that $(a-b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]
$25\{x^2 + 1 - 2x + y^2 + 1 - 2y\} = 4\{9x^2 + 16y^2 + 64 + 24xy + 64y + 48x\}$
$25x^2 + 25 - 50x + 25y^2 + 25 - 50y = 36x^2 + 64y^2 + 256 + 96xy + 256y + 192x$
$25x^2 + 25 - 50x + 25y^2 + 25 - 50y - 36x^2 - 64y^2 - 256 - 96xy - 256y - 192x = 0$
$-11x^2 - 39y^2 - 242x - 306y - 96xy - 206 = 0$
$11x^2 + 96xy + 39y^2 + 242x + 306y + 206 = 0$
∴The equation of hyperbola is$11x^2 + 96xy + 39y^2 + 242x + 306y + 206 = 0$

(iii) focus is (1, 1) directrix is 2x + y = 1 and eccentricity =$\sqrt{3}$
Given:
Focus = (1, 1)
Directrix => 2x + y = 1
Eccentricity =$\sqrt{3}$
Now, let us find the equation of the hyperbola
Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.
By using the formula,
e = PF/PM
PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]
So,

$$\sqrt{(x-1)^2+(y-1)^2} = \sqrt{3}\left|\frac{(2x+y-1)}{\sqrt{2^2+1^2}}\right|$$

$$\sqrt{(x-1)^2+(y-1)^2} = \sqrt{3}\left|\frac{(2x+y-1)}{\sqrt{4+1}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-1)^2+(y-1)^2}\right)^2 = \left(\sqrt{3}\left|\frac{(2x+y-1)}{\sqrt{5}}\right|\right)^2$$

$$(x-1)^2+(y-1)^2 = \frac{3(2x+y-1)^2}{5}$$

[We know that $(a - b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]

$5\{x^2 + 1 - 2x + y^2 + 1 - 2y\} = 3\{4x^2 + y^2 + 1 + 4xy - 2y - 4x\}$

$5x^2 + 5 - 10x + 5y^2 + 5 - 10y = 12x^2 + 3y^2 + 3 + 12xy - 6y - 12x$

$5x^2 + 5 - 10x + 5y^2 + 5 - 10y - 12x^2 - 3y^2 - 3 - 12xy + 6y + 12x = 0$

$-7x^2 + 2y^2 + 2x - 4y - 12xy + 7 = 0$

$7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$

∴The equation of hyperbola is $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0$

(iv) focus is (2, -1), directrix is 2x + 3y = 1 and eccentricity = 2

Given:

Focus = (2, -1)

Directrix => 2x + 3y = 1

Eccentricity = 2

Now, let us find the equation of the hyperbola

Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.

By using the formula,

e = PF/PM

PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]

So,

$$\sqrt{(x-2)^2+(y+1)^2} = 2\left|\frac{(2x+3y-1)}{\sqrt{2^2+3^2}}\right|$$

$$\sqrt{(x-2)^2+(y+1)^2} = 2\left|\frac{(2x+3y-1)}{\sqrt{4+9}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-2)^2+(y+1)^2}\right)^2 = \left(2\left|\frac{(2x+3y-1)}{\sqrt{13}}\right|\right)^2$$

$$(x-2)^2+(y+1)^2=\frac{4(2x+3y-1)^2}{13}$$

[We know that $(a – b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]
$13\{x^2 + 4 – 4x + y^2 + 1 + 2y\} = 4\{4x^2 + 9y^2 + 1 + 12xy – 6y – 4x\}$
$13x^2 + 52 – 52x + 13y^2 + 13 + 26y = 16x^2 + 36y^2 + 4 + 48xy – 24y – 16x$
$13x^2 + 52 – 52x + 13y^2 + 13 + 26y – 16x^2 – 36y^2 – 4 – 48xy + 24y + 16x = 0$
$– 3x^2 – 23y^2 – 36x + 50y – 48xy + 61 = 0$
$3x^2 + 23y^2 + 48xy + 36x – 50y– 61 = 0$
∴The equation of hyperbola is$3x^2 + 23y^2 + 48xy + 36x – 50y– 61 = 0$

(v) focus is (a, 0), directrix is 2x + 3y = 1 and eccentricity = 2
Given:
Focus = (a, 0)
Directrix => 2x + 3y = 1
Eccentricity = 2
Now, let us find the equation of the hyperbola
Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.
By using the formula,
e = PF/PM
PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]
So,

$$\sqrt{(x-a)^2+(y-0)^2}=\frac{4}{3}\left|\frac{(2x-y+a)}{\sqrt{2^2+(-1)^2}}\right|$$

$$\sqrt{(x-a)^2+(y)^2}=\frac{4}{3}\left|\frac{(2x-y+a)}{\sqrt{4+1}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-a)^2+(y)^2}\right)^2=\left(\frac{4}{3}\left|\frac{(2x-y+a)}{\sqrt{5}}\right|\right)^2$$

$$(x-a)^2+(y)^2=\frac{16(2x-y+a)^2}{9\times 5}$$

[We know that $(a – b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]
$45\{x^2 + a^2 – 2ax + y^2\} = 16\{4x^2 + y^2 + a^2 – 4xy – 2ay + 4ax\}$
$45x^2 + 45a^2 – 90ax + 45y^2 = 64x^2 + 16y^2 + 16a^2 – 64xy – 32ay + 64ax$
$45x^2 + 45a^2 – 90ax + 45y^2 – 64x^2 – 16y^2 – 16a^2 + 64xy + 32ay – 64ax = 0$
$19x^2 – 29y^2 + 154ax - 32ay - 64xy - 29a^2 = 0$
∴The equation of hyperbola is$19x^2 – 29y^2 + 154ax - 32ay - 64xy - 29a^2 = 0$

(vi) focus is (2, 2), directrix is x + y = 9 and eccentricity = 2
Given:
Focus = (2, 2)
Directrix => x + y = 9
Eccentricity = 2
Now, let us find the equation of the hyperbola
Let 'M' be the point on directrix and P(x, y) be any point of the hyperbola.
By using the formula,
e = PF/PM
PF = ePM [where, e is eccentricity, PM is perpendicular from any point P on hyperbola to the directrix]
So,

$$\sqrt{(x-2)^2+(y-2)^2} = 2\left|\frac{(x+y-9)}{\sqrt{1^2+1^2}}\right|$$

$$\sqrt{(x-2)^2+(y-2)^2} = 2\left|\frac{(x+y-9)}{\sqrt{1+1}}\right|$$

By squaring on both sides we get

$$\left(\sqrt{(x-2)^2+(y-2)^2}\right)^2 = \left(2\left|\frac{(x+y-9)}{\sqrt{2}}\right|\right)^2$$

$$(x-2)^2+(y-2)^2 = \frac{4(x+y-9)^2}{2}$$

[We know that $(a-b)^2 = a^2 + b^2 + 2ab$ &$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ac$]
$x^2 + 4 - 4x + y^2 + 4 - 4y = 2\{x^2 + y^2 + 81 + 2xy - 18y - 18x\}$
$x^2 - 4x + y^2 + 8 - 4y = 2x^2 + 2y^2 + 162 + 4xy - 36y - 36x$
$x^2 - 4x + y^2 + 8 - 4y - 2x^2 - 2y^2 - 162 - 4xy + 36y + 36x = 0$
$-x^2 - y^2 + 32x + 32y + 4xy - 154 = 0$
$x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$
∴The equation of hyperbola is $x^2 + 4xy + y^2 - 32x - 32y + 154 = 0$

3. Find the eccentricity, coordinates of the foci, equations of directrices and length of the latus-rectum of the hyperbola.
(i) $9x^2 - 16y^2 = 144$
(ii) $16x^2 - 9y^2 = -144$
(iii) $4x^2 - 3y^2 = 36$
(iv) $3x^2 - y^2 = 4$
(v) $2x^2 - 3y^2 = 5$

Solution:

(i) $9x^2 - 16y^2 = 144$

Given:

The equation => $9x^2 - 16y^2 = 144$

The equation can be expressed as:

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1$$

$$\frac{x^2}{16} - \frac{y^2}{9} = 1$$

$$\frac{x^2}{4^2} - \frac{y^2}{3^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a^2 = 16$, $b^2 = 9$ i.e., a = 4 and b = 3

Eccentricity is given by:

$$e = \sqrt{1 + \frac{a^2}{b^2}}$$

$$= \sqrt{1 + \frac{9}{16}}$$

$$= \sqrt{\frac{25}{16}}$$

$$= \frac{5}{4}$$

Foci: The coordinates of the foci are (0, ±be)

(0, ±be) = (0, ±4(5/4))

= (0, ±5)

The equation of directrices is given as:

$$x = \pm \frac{a}{e}$$

$$\Rightarrow x = \pm \frac{4}{\frac{5}{4}}$$

$$\Rightarrow x = \pm \frac{16}{5}$$

$$\Rightarrow 5x = \pm 16$$

$5x \mp 16 = 0$

The length of latus-rectum is given as:
$2b^2/a$
$= 2(9)/4$
$= 9/2$

(ii) $16x^2 - 9y^2 = -144$
Given:
The equation => $16x^2 - 9y^2 = -144$
The equation can be expressed as:

$$\frac{9y^2}{144} - \frac{16x^2}{144} = 1$$

$$\frac{y^2}{16} - \frac{x^2}{9} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{4^2} = -1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, $a^2 = 9$, $b^2 = 16$ i.e., $a = 3$ and $b = 4$

Eccentricity is given by:

$$e = \sqrt{1 + \frac{a^2}{b^2}}$$

$$= \sqrt{1 + \frac{9}{16}}$$

$$= \sqrt{\frac{25}{16}}$$

$$= \frac{5}{4}$$

Foci: The coordinates of the foci are $(0, \pm be)$
$(0, \pm be) = (0, \pm 4(5/4))$
$= (0, \pm 5)$

The equation of directrices is given as:

$y = \pm\frac{b}{e}$

$\Rightarrow y = \pm\frac{4}{\frac{5}{4}}$

$\Rightarrow y = \pm\frac{16}{5}$

$\Rightarrow 5y = \pm 16$

$5y \mp 16 = 0$

The length of latus-rectum is given as:
$2a^2/b$
= 2(9)/4
= 9/2

(iii) $4x^2 - 3y^2 = 36$
Given:
The equation => $4x^2 - 3y^2 = 36$
The equation can be expressed as:

$\frac{4x^2}{36} - \frac{3y^2}{36} = 1$

$\frac{x^2}{9} - \frac{y^2}{12} = 1$

$\frac{x^2}{3^2} - \frac{y^2}{(\sqrt{12})^2} = 1$

The obtained equation is of the form

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Where, $a^2 = 9$, $b^2 = 12$ i.e., a = 3 and b = $\sqrt{12}$

Eccentricity is given by:

$e = \sqrt{1 + \frac{b^2}{a^2}}$

$= \sqrt{1 + \frac{12}{9}}$

$= \sqrt{\frac{21}{9}}$

$= \sqrt{\frac{7}{3}}$

Foci: The coordinates of the foci are (±ae, 0)

$\pm ae = \pm 3 \times \sqrt{\frac{7}{3}}$

$= \pm 3 \times \frac{\sqrt{7}}{\sqrt{3}}$

$= \pm\sqrt{3} \times \sqrt{7}$

$= \pm\sqrt{21}$

(±ae, 0) = (±√21, 0)

The equation of directrices is given as:

$x = \frac{\pm a}{e}$

$x = \pm 3 \times \frac{1}{\frac{\sqrt{7}}{\sqrt{3}}}$

$= \pm \frac{3\sqrt{3}}{\sqrt{7}}$

$\sqrt{7}x \mp 3\sqrt{3} = 0$

The length of latus-rectum is given as:
$2b^2/a$
= 2(12)/3
= 24/3
= 8

(iv) $3x^2 – y^2 = 4$
Given:
The equation => $3x^2 – y^2 = 4$
The equation can be expressed as:

$\frac{3x^2}{4} - \frac{y^2}{4} = 1$

$\frac{x^2}{\frac{4}{3}} - \frac{y^2}{4} = 1$

$\frac{x^2}{\left(\frac{2}{\sqrt{3}}\right)^2} - \frac{y^2}{(2)^2} = 1$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, a = 2/√3 and b = 2

Eccentricity is given by:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$
$$= \sqrt{1 + \frac{4}{\frac{4}{3}}}$$
$$= \sqrt{1+3}$$
$$= \sqrt{4}$$
$$= 2$$

Foci: The coordinates of the foci are (±ae, 0)
(±ae, 0) = ±(2/√3)(2) = ±4/√3
(±ae, 0) = (±4/√3, 0)

The equation of directrices is given as:

$$x = \pm\frac{a}{e}$$
$$\Rightarrow x = \pm\frac{\frac{2}{\sqrt{3}}}{2}$$
$$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$$
$$\Rightarrow \sqrt{3}x = \pm 1$$
$$\sqrt{3}x \mp 1 = 0$$

The length of latus-rectum is given as:
$2b^2/a$
= 2(4)/[2/√3]
= 4√3

(v) $2x^2 - 3y^2 = 5$
Given:
The equation => $2x^2 - 3y^2 = 5$
The equation can be expressed as:

$$\frac{2x^2}{5} - \frac{3y^2}{5} = 1$$

$$\frac{x^2}{\frac{5}{2}} - \frac{y^2}{\frac{5}{3}} = 1$$

$$\frac{x^2}{\left(\frac{\sqrt{5}}{\sqrt{2}}\right)^2} - \frac{y^2}{\left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, a = √5/√2 and b = √5/√3

Eccentricity is given by:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{\frac{5}{3}}{\frac{5}{2}}}$$

$$= \sqrt{1 + \frac{5}{3} \times \frac{2}{5}}$$

$$= \sqrt{1 + \frac{2}{3}}$$

$$= \sqrt{\frac{5}{3}}$$

Foci: The coordinates of the foci are (±ae, 0)

$$\pm ae = \pm\sqrt{\frac{5}{2}} \times \sqrt{\frac{5}{3}}$$

$$= \pm\frac{5}{\sqrt{6}}$$

(±ae, 0) = (±5/√6, 0)

The equation of directrices is given as:

$$x = \pm\frac{a}{e}$$

$$\Rightarrow x = \pm\frac{\frac{\sqrt{5}}{\sqrt{2}}}{\frac{\sqrt{5}}{\sqrt{3}}}$$

$$\Rightarrow x = \pm\frac{1}{\sqrt{6}}$$

$$\Rightarrow \sqrt{6}x = \pm 1$$

$$\sqrt{6}x \mp 1 = 0$$

The length of latus-rectum is given as:

$2b^2/a$

$$= \frac{2\left(\frac{\sqrt{5}}{\sqrt{3}}\right)^2}{\frac{\sqrt{5}}{\sqrt{2}}}$$

$$= \frac{2 \times \frac{5}{3}}{\frac{\sqrt{5}}{\sqrt{2}}}$$

$$= \frac{2\sqrt{10}}{3}$$

4. Find the axes, eccentricity, latus-rectum and the coordinates of the foci of the hyperbola $25x^2 – 36y^2 = 225$.

Solution:

Given:

The equation=> $25x^2 – 36y^2 = 225$

The equation can be expressed as:

$$\frac{25x^2}{225} - \frac{36y^2}{225} = 1$$

$$\frac{x^2}{\left(\frac{15}{5}\right)^2} - \frac{y^2}{\left(\frac{15}{6}\right)^2} = 1$$

$$\frac{x^2}{3^2} - \frac{y^2}{\left(\frac{5}{2}\right)^2} = 1$$

The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, a = 3 and b = 5/2

Eccentricity is given by:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{\frac{25}{4}}{9}}$$

$$= \sqrt{1 + \frac{25}{36}}$$

$$= \sqrt{\frac{61}{36}}$$

$$= \frac{\sqrt{61}}{6}$$

Foci: The coordinates of the foci are (±ae, 0)

(±ae, 0) = ±3 (√61/6) = ± √61/2

(±ae, 0) = (± √61/2, 0)

The equation of directrices is given as:

$$x = \pm \frac{a}{e}$$

$$\Rightarrow x = \pm \frac{3}{\frac{\sqrt{61}}{6}}$$

$$\Rightarrow x = \pm \frac{18}{\sqrt{61}}$$

$$\Rightarrow \sqrt{61}x = \pm 18$$

$$\sqrt{61}x \mp 18 = 0$$

The length of latus-rectum is given as:

$2b^2/a$

$$= \frac{2\left(\frac{5}{2}\right)^2}{3}$$

$$= \frac{2 \times \frac{25}{4}}{3}$$

$$= \frac{25}{6}$$

∴ Transverse axis = 6, conjugate axis = 5, e = √61/6, LR = 25/6, foci = (± √61/2, 0)

5. Find the centre, eccentricity, foci and directions of the hyperbola
(i) $16x^2 – 9y^2 + 32x + 36y – 164 = 0$
(ii) $x^2 – y^2 + 4x = 0$
(iii) $x^2 – 3y^2 – 2x = 8$
Solution:
(i) $16x^2 – 9y^2 + 32x + 36y – 164 = 0$
Given:
The equation => $16x^2 – 9y^2 + 32x + 36y – 164 = 0$
Let us find the centre, eccentricity, foci and directions of the hyperbola
By using the given equation
$16x^2 – 9y^2 + 32x + 36y – 164 = 0$
$16x^2 + 32x + 16 – 9y^2 + 36y – 36 – 16 + 36 – 164 = 0$
$16(x^2 + 2x + 1) – 9(y^2 – 4y + 4) – 16 + 36 – 164 = 0$
$16(x^2 + 2x + 1) – 9(y^2 – 4y + 4) – 144 = 0$
$16(x + 1)^2 – 9(y – 2)^2 = 144$

$$\frac{16(x+1)^2}{144} - \frac{9(y-2)^2}{144} = 1$$

$$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1$$

$$\frac{(x+1)^2}{3^2} - \frac{(y-2)^2}{4^2} = 1$$

Here, center of the hyperbola is (-1, 2)
So, let x + 1 = X and y – 2 = Y
The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, a = 3 and b = 4

Eccentricity is given by:

$e = \sqrt{1 + \frac{b^2}{a^2}}$
$= \sqrt{1 + \frac{16}{9}}$
$= \sqrt{\frac{25}{9}}$
$= \frac{5}{3}$

Foci: The coordinates of the foci are (±ae, 0)
X = ±5 and Y = 0
x + 1 = ±5 and y – 2 = 0
x = ±5 – 1 and y = 2
x = 4, -6 and y = 2
So, Foci: (4, 2) (-6, 2)

Equation of directrix are:
$X = \pm\frac{a}{e}$
$\Rightarrow X = \pm\frac{3}{\frac{5}{3}}$
$\Rightarrow X = \pm\frac{9}{5}$
$\Rightarrow 5X = \pm 9$
$\Rightarrow 5X \mp 9 = 0$
$\Rightarrow 5(x + 1) \mp 9 = 0$
$\Rightarrow 5x + 5 \mp 9 = 0$
$\Rightarrow 5x + 5 - 9 = 0$ and $5x + 5 + 9 = 0$
$5x - 4 = 0$ and $5x + 14 = 0$
∴ The center is (-1, 2), eccentricity (e) = 5/3, Foci = (4, 2) (-6, 2), Equation of directrix = 5x – 4 = 0 and 5x + 14 = 0

(ii) $x^2 - y^2 + 4x = 0$
Given:
The equation => $x^2 - y^2 + 4x = 0$
Let us find the centre, eccentricity, foci and directions of the hyperbola
By using the given equation
$x^2 - y^2 + 4x = 0$

$x^2 + 4x + 4 – y^2 – 4 = 0$
$(x + 2)^2 – y^2 = 4$

$$\frac{(x+2)^2}{4} - \frac{y^2}{4} = 1$$

$$\frac{(x+2)^2}{2^2} - \frac{y^2}{2^2} = 1$$

Here, center of the hyperbola is (2, 0)
So, let x – 2 = X
The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, a = 2 and b = 2

Eccentricity is given by:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$
$$= \sqrt{1 + \frac{4}{4}}$$
$$= \sqrt{1 + 1}$$
$$= \sqrt{2}$$

Foci: The coordinates of the foci are (±ae, 0)
X = ± 2√2 and Y = 0
X + 2 = ± 2√2 and Y = 0
X= ± 2√2 – 2 and Y = 0
So, Foci = (± 2√2 – 2, 0)

Equation of directrix are:

$$X = \pm\frac{a}{e}$$
$$\Rightarrow X = \pm\frac{2}{\sqrt{2}}$$
$$\Rightarrow X = \pm\frac{2}{\sqrt{2}}$$
$$\Rightarrow X = \pm\sqrt{2}$$
$$\Rightarrow X \mp \sqrt{2} = 0$$
$$\Rightarrow x + 2 \mp \sqrt{2} = 0$$
$$x + 2 - \sqrt{2} = 0 \text{ and } x + 2 + \sqrt{2} = 0$$

∴ The center is (-2, 0), eccentricity (e) = $\sqrt{2}$, Foci = ($-2\pm 2\sqrt{2}$, 0), Equation of directrix = $x + 2 = \pm\sqrt{2}$

(iii) $x^2 - 3y^2 - 2x = 8$
Given:
The equation => $x^2 - 3y^2 - 2x = 8$
Let us find the centre, eccentricity, foci and directions of the hyperbola
By using the given equation
$x^2 - 3y^2 - 2x = 8$
$x^2 - 2x + 1 - 3y^2 - 1 = 8$
$(x - 1)^2 - 3y^2 = 9$

$$\frac{(x-1)^2}{9} - \frac{3y^2}{9} = 1$$

$$\frac{(x-1)^2}{3^2} - \frac{y^2}{(\sqrt{3})^2} = 1$$

Here, center of the hyperbola is (1, 0)
So, let x – 1 = X
The obtained equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Where, a = 3 and b = $\sqrt{3}$

Eccentricity is given by:

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$
$$= \sqrt{1 + \frac{3}{9}}$$
$$= \sqrt{1 + \frac{1}{3}}$$
$$= \sqrt{\frac{4}{3}}$$
$$= \frac{2}{\sqrt{3}}$$
$$= \frac{2 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$
$$= \frac{2\sqrt{3}}{3}$$

Foci: The coordinates of the foci are (±ae, 0)

$X = \pm 2\sqrt{3}$ and $Y = 0$
$X - 1 = \pm 2\sqrt{3}$ and $Y = 0$
$X = \pm 2\sqrt{3} + 1$ and $Y = 0$
So, Foci = $(1 \pm 2\sqrt{3}, 0)$

Equation of directrix are:

$$X = \pm\frac{a}{e}$$

$$\Rightarrow X = \pm\frac{3}{\frac{2\sqrt{3}}{3}}$$

$$\Rightarrow X = \pm\frac{9}{2\sqrt{3}}$$

$$X = \pm\frac{9}{2\sqrt{3}} + 1$$

$$x = \pm\frac{9}{2\sqrt{3}}$$

∴ The center is (1, 0), eccentricity (e) = $2\sqrt{3}/3$, Foci = $(1 \pm 2\sqrt{3}, 0)$, Equation of directrix =
$X = 1 \pm 9/2\sqrt{3}$

6. Find the equation of the hyperbola, referred to its principal axes as axes of coordinates, in the following cases:
(i) the distance between the foci = 16 and eccentricity = √2
(ii) conjugate axis is 5 and the distance between foci = 13
(iii) conjugate axis is 7 and passes through the point (3, -2)
Solution:
(i) the distance between the foci = 16 and eccentricity = √2
Given:
Distance between the foci = 16
Eccentricity = √2
Let us compare with the equation of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Distance between the foci is 2ae and $b^2 = a^2(e^2 - 1)$
So,
2ae = 16
ae = 16/2
a√2 = 8
a = 8/√2

$a^2 = 64/2$
$= 32$
We know that, $b^2 = a^2(e^2 – 1)$
So, $b^2 = 32 [(\sqrt{2})^2 – 1]$
$= 32 (2 - 1)$
$= 32$
The Equation of hyperbola is given as
$$\frac{x^2}{32} - \frac{y^2}{32} = 1$$
$x^2 – y^2 = 32$
∴ The Equation of hyperbola is $x^2 – y^2 = 32$

(ii) conjugate axis is 5 and the distance between foci = 13
Given:
Conjugate axis = 5
Distance between foci = 13
Let us compare with the equation of the form
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$
Distance between the foci is 2ae and $b^2 = a^2(e^2 – 1)$
Length of conjugate axis is 2b
So,
$2b = 5$
$b = 5/2$
$b^2 = 25/4$

We know that, $2ae = 13$
$ae = 13/2$
$a^2e^2 = 169/4$
$b^2 = a^2(e^2 – 1)$
$b^2 = a^2e^2 – a^2$
$25/4 = 169/4 – a^2$
$a^2 = 169/4 – 25/4$
$= 144/4$
$= 36$

The Equation of hyperbola is given as
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{36} - \frac{y^2}{\frac{25}{4}} = 1$$
$$\Rightarrow \frac{x^2}{36} - \frac{4y^2}{25} = 1$$
$$\Rightarrow \frac{25x^2 - 144y^2}{900} = 1$$
$$\Rightarrow 25x^2 - 144y^2 = 900$$

∴ The Equation of hyperbola is $25x^2 - 144y^2 = 900$

(iii) conjugate axis is 7 and passes through the point (3, -2)

Given:

Conjugate axis = 7

Passes through the point (3, -2)

Conjugate axis is 2b

So,

2b = 7

b = 7/2

$b^2 = 49/4$

The Equation of hyperbola is given as

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Since it passes through points (3, -2)

$$\Rightarrow \frac{(3)^2}{a^2} - \frac{(-2)^2}{\frac{49}{4}} = 1$$
$$\Rightarrow \frac{9}{a^2} - \frac{4(4)}{49} = 1$$
$$\Rightarrow \frac{9}{a^2} - \frac{16}{49} = 1$$
$$\Rightarrow \frac{9}{a^2} = 1 + \frac{16}{49}$$
$$\Rightarrow \frac{9}{a^2} = \frac{49 + 16}{49}$$
$$\Rightarrow \frac{9}{a^2} = \frac{65}{49}$$
$$\Rightarrow a^2 = \frac{49}{65} \times 9$$

$a^2 = 441/65$

The equation of hyperbola is given as:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$a^2 = 441/65$ and $b^2 = 49/4$

$$\Rightarrow \frac{x^2}{\frac{441}{65}} - \frac{y^2}{\frac{49}{4}} = 1$$

$$\Rightarrow \frac{65x^2}{441} - \frac{4y^2}{49} = 1$$

$$\Rightarrow \frac{65x^2 - 36y^2}{441} = 1$$

$$\Rightarrow 65x^2 - 36y^2 = 441$$

∴ The Equation of hyperbola is $65x^2 – 36y^2 = 441$