Exercise 10.3

1. Find the unit vector in the direction of sum of vectors $\vec{a}=2\hat{i}-\hat{j}+\hat{k}$ and $\vec{b}=2\hat{j}+\hat{k}$.
Solution:

Given vectors are,

$$\vec{a} = 2\hat{i}-\hat{j}+\hat{k} \text{ and } \vec{b}=2\hat{j}+\hat{k}$$

$$\vec{a}+\vec{b} = (2\hat{i}-\hat{j}+\hat{k})+(2\hat{j}+\hat{k}) = 2\hat{i}+\hat{j}+2\hat{k}$$

So,

Unit vector in the direction of $\vec{a}+\vec{b} = \frac{\vec{a}+\vec{b}}{|\vec{a}+\vec{b}|}$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{(2)^2+(1)^2+(2)^2}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{4+1+4}}$$

$$= \frac{2\hat{i}+\hat{j}+2\hat{k}}{\sqrt{9}} = \frac{2\hat{i}+\hat{j}+2\hat{k}}{3} = \frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$$

Thus, the required unit vector is $\frac{2}{3}\hat{i}+\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}$.

2. If $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$, find the unit vector in the direction of

(i) $6\vec{b}$ (ii) $2\vec{a}-\vec{b}$

Solution:

Given, $\vec{a}=\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-2\hat{k}$

(i) $6\vec{b}=6(2\hat{i}+\hat{j}-2\hat{k})=12\hat{i}+6\hat{j}-12\hat{k}$

So, Unit vector in the direction of $6\vec{b}=\frac{6\vec{b}}{|6\vec{b}|}$

$$= \frac{12\hat{i}+6\hat{j}-12\hat{k}}{\sqrt{(12)^2+(6)^2+(-12)^2}} = \frac{12\hat{i}+6\hat{j}-12\hat{k}}{\sqrt{144+36+144}}$$

$$= \frac{12\hat{i}+6\hat{j}-12\hat{k}}{\sqrt{324}} = \frac{12\hat{i}+6\hat{j}-12\hat{k}}{18}$$

$$= \frac{6}{18}(2\hat{i}+\hat{j}-2\hat{k}) = \frac{1}{3}(2\hat{i}+\hat{j}-2\hat{k})$$

Thus, the required unit vector is $\frac{1}{3}(2\hat{i}+\hat{j}-2\hat{k})$.

(ii) $2\vec{a}-\vec{b} = 2(\hat{i}+\hat{j}+2\hat{k})-(2\hat{i}+\hat{j}-2\hat{k})$

$$= 2\hat{i}+2\hat{j}+4\hat{k}-2\hat{i}-\hat{j}+2\hat{k} = \hat{j}+6\hat{k}$$

So,

Unit vector in the direction of $2\vec{a}-\vec{b}$

$$= \frac{2\vec{a}-\vec{b}}{|2\vec{a}-\vec{b}|} = \frac{\hat{j}+6\hat{k}}{\sqrt{(1)^2+(6)^2}} = \frac{\hat{j}+6\hat{k}}{\sqrt{1+36}}$$

$$= \frac{\hat{j}+6\hat{k}}{\sqrt{37}} = \frac{1}{\sqrt{37}}[\hat{j}+6\hat{k}]$$

Thus, the required unit vector is $\frac{1}{\sqrt{37}}[\hat{j}+6\hat{k}]$.

3. Find a unit vector in the direction of $\overrightarrow{PQ}$, where P and Q have co-ordinates (5, 0, 8) and (3, 3, 2), respectively.
Solution:

Given coordinates are P(5, 0, 8) and Q(3, 3, 2).

So, $\overrightarrow{PQ} = (3-5)\hat{i} + (3-0)\hat{j} + (2-8)\hat{k} = -2\hat{i} + 3\hat{j} - 6\hat{k}$

And,

Unit vector in the direction of $\overrightarrow{PQ} = \frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}$

$$= \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{(-2)^2+(3)^2+(-6)^2}} = \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{4+9+36}} = \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{\sqrt{49}}$$

$$= \frac{-2\hat{i}+3\hat{j}-6\hat{k}}{7} = \frac{1}{7}(-2\hat{i}+3\hat{j}-6\hat{k})$$

Thus, the required unit vector is $\frac{1}{7}(-2\hat{i}+3\hat{j}-6\hat{k})$.

4. If $\vec{a}$ and $\vec{b}$ are the position vectors of A and B, respectively, find the position vector of a point C in BA produced such that BC = 1.5 BA.
Solution:

Given,

BC = 1.5 BA

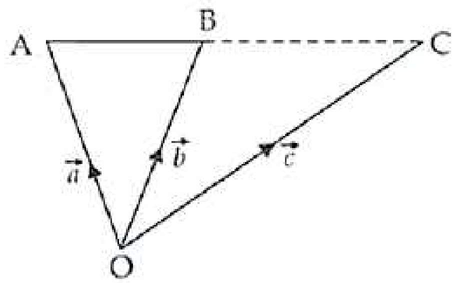
$$\Rightarrow \frac{BC}{BA} = 1.5 = \frac{3}{2}$$

$$\frac{\vec{c}-\vec{b}}{\vec{a}-\vec{b}} = \frac{3}{2}$$

$$2\vec{c} - 2\vec{b} = 3\vec{a} - 3\vec{b}$$

$$2\vec{c} = 3\vec{a} - 3\vec{b} + 2\vec{b} \Rightarrow 2\vec{c} = 3\vec{a} - \vec{b}$$

$$\therefore \vec{c} = \frac{3\vec{a}-\vec{b}}{2}$$



Thus, the required vector is $\vec{c} = \frac{3\vec{a}-\vec{b}}{2}$.

5. Using vectors, find the value of *k* such that the points (*k*, – 10, 3), (1, –1, 3) and (3, 5, 3) are collinear.
Solution:

Let the given points be A(k, – 10, 3), B(1, –1, 3) and C(3, 5, 3).

$$\overrightarrow{AB} = (1-k)\hat{i} + (-1+10)\hat{j} + (3-3)\hat{k}$$

$$\overrightarrow{AB} = (1-k)\hat{i} + 9\hat{j} + 0\hat{k}$$

So, $|\overrightarrow{AB}| = \sqrt{(1-k)^2 + (9)^2} = \sqrt{(1-k)^2 + 81}$

$$\overrightarrow{BC} = (3-1)\hat{i} + (5+1)\hat{j} + (3-3)\hat{k} = 2\hat{i} + 6\hat{j} + 0\hat{k}$$

So, $|\overrightarrow{BC}| = \sqrt{(2)^2 + (6)^2} = \sqrt{4+36} = \sqrt{40} = 2\sqrt{10}$

$$\overrightarrow{AC} = (3-k)\hat{i} + (5+10)\hat{j} + (3-3)\hat{k} = (3-k)\hat{i} + 15\hat{j} + 0\hat{k}$$

So, $|\overrightarrow{AC}| = \sqrt{(3-k)^2 + (15)^2} = \sqrt{(3-k)^2 + 225}$

If A, B and C are collinear, then

$$|\overrightarrow{AB}| + |\overrightarrow{BC}| = |\overrightarrow{AC}|$$

$$\sqrt{(1-k)^2 + 81} + \sqrt{40} = \sqrt{(3-k)^2 + 225}$$

Squaring both sides, we have

$$\left[\sqrt{(1-k)^2 + 81} + \sqrt{40}\right]^2 = \left[\sqrt{(3-k)^2 + 225}\right]^2$$

$$(1-k)^2 + 81 + 40 + 2\sqrt{40}\,\sqrt{(1-k)^2 + 81} = (3-k)^2 + 225$$

$$1 + k^2 - 2k + 121 + 2\sqrt{40}\,\sqrt{1 + k^2 - 2k + 81} = 9 + k^2 - 6k + 225$$

$$\Rightarrow 122 - 2k + 2\sqrt{40}\,\sqrt{k^2 - 2k + 82} = 234 - 6k$$

Now, on divding by 2, we get

$$61 - k + \sqrt{40}\,\sqrt{k^2 - 2k + 82} = 117 - 3k$$

$$\sqrt{40}\,\sqrt{k^2 - 2k + 82} = 117 - 61 - 3k + k$$

$$\sqrt{40}\,\sqrt{k^2 - 2k + 82} = 56 - 2k \Rightarrow 2\sqrt{10}\,\sqrt{k^2 - 2k + 82} = 56 - 2k$$

$$\Rightarrow \sqrt{10}\,\sqrt{k^2 - 2k + 82} = 28 - k \qquad \text{(Dividing by 2)}$$

Squaring both sides, we get

$$10(k^2 - 2k + 82) = 784 + k^2 - 56k$$

$$10k^2 - 20k + 820 = 784 + k^2 - 56k$$

$$10k^2 - k^2 - 20k + 56k + 820 - 784 = 0$$

$$9k^2 + 36k + 36 = 0 \Rightarrow k^2 + 4k + 4 = 0 \Rightarrow (k+2)^2 = 0$$

$$k + 2 = 0 \Rightarrow k = -2$$

Thus, the required value is $k = -2$

6. A vector $\vec{r}$ is inclined at equal angles to the three axes. If the magnitude of $\vec{r}$ is 2√3 units, find $\vec{r}$.

Solution:

As the vector $\vec{r}$ makes equal angles with the axes, their direction cosines should also be same
So, l = m = n
And we know that,

$l^2 + m^2 + n^2 = 1 \Rightarrow l^2 + l^2 + l^2 = 1$

$3l^2 = 1$

$l = \pm 1/\sqrt{3}$

Now, $\hat{r} = \pm\frac{1}{\sqrt{3}}\hat{i} \pm \frac{1}{\sqrt{3}}\hat{j} \pm \frac{1}{\sqrt{3}}\hat{k} \Rightarrow \hat{r} = \pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})$

And, w.k.t $\vec{r} = (\hat{r})|\vec{r}|$

$= \pm\frac{1}{\sqrt{3}}(\hat{i}+\hat{j}+\hat{k})\,2\sqrt{3} = \pm 2(\hat{i}+\hat{j}+\hat{k})$

Thus, the required value of $\vec{r}$ is $\pm 2(\hat{i}+\hat{j}+\hat{k})$.

7. A vector $\vec{r}$ has magnitude 14 and direction ratios 2, 3, – 6. Find the direction cosines and components of $\vec{r}$, given that $\vec{r}$ makes an acute angle with *x*-axis.

Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors such that $\vec{a} = 2k, \vec{b} = 3k$ and $\vec{c} = -6k$

If l, m and n are the direction cosines of vector $\vec{r}$, then

$$l = \frac{\vec{a}}{|\vec{r}|} = \frac{2k}{14} = \frac{k}{7}$$

$$m = \frac{\vec{b}}{|\vec{r}|} = \frac{3k}{14} \quad \text{and} \quad n = \frac{\vec{c}}{|\vec{r}|} = \frac{-6k}{14} = \frac{-3k}{7}$$

We know that $l^2 + m^2 + n^2 = 1$

So, $$\frac{k^2}{49} + \frac{9k^2}{196} + \frac{9k^2}{49} = 1$$

$$\frac{4k^2 + 9k^2 + 36k^2}{196} = 1 \Rightarrow 49k^2 = 196 \Rightarrow k^2 = 4$$

$\therefore\ k = \pm 2$ and $l = \frac{k}{7} = \frac{2}{7}$

$$m = \frac{3k}{14} = \frac{3\times 2}{14} = \frac{3}{7} \quad \text{and} \quad n = \frac{-3k}{7}\frac{-3\times 2}{7} = \frac{-6}{7}$$

Now, $$\hat{r} = \pm\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right)$$

$$\hat{r} = \hat{r}|\vec{r}|$$

$$\Rightarrow \vec{r} = \pm\left(\frac{2}{7}\hat{i} + \frac{3}{7}\hat{j} - \frac{6}{7}\hat{k}\right)\cdot 14 = \pm(4\hat{i} + 6\hat{j} - 12\hat{k})$$

Thus, the required direction cosines are $\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}$ and the components of $\vec{r}$ are $4\hat{i}, 6\hat{j}$ and $-12\hat{k}$.

8. Find a vector of magnitude 6, which is perpendicular to both the vectors $2\hat{i}-\hat{j}+2\hat{k}$ and $4\hat{i}-\hat{j}+3\hat{k}$.

Solution:

Let $\vec{a} = 2\hat{i} - \hat{j} + 2\hat{k}$ and $\vec{b} = 4\hat{i} - \hat{j} + 3\hat{k}$

We know that unit vector perpendicular to $\vec{a}$ and $\vec{b} = \frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 4 & -1 & 3 \end{vmatrix}$$

$$= \hat{i}(-3+2) - \hat{j}(6-8) + \hat{k}(-2+4) = -\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\therefore |\vec{a}\times\vec{b}| = \sqrt{(-1)^2 + (2)^2 + (2)^2} = \sqrt{1+4+4} = \sqrt{9} = 3$$

So, $\frac{(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} = \frac{-\hat{i}+2\hat{j}+2\hat{k}}{3} = \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})$

Now the vector of magnitude $6 = \frac{1}{3}(-\hat{i}+2\hat{j}+2\hat{k})\cdot 6$

$$= 2(-\hat{i}+2\hat{j}+2\hat{k}) = -2\hat{i}+4\hat{j}+4\hat{k}$$

Thus, the required vector is $-2\hat{i}+4\hat{j}+4\hat{k}$.

9. Find the angle between the vectors $2\hat{i}-\hat{j}+\hat{k}$ and $3\hat{i}+4\hat{j}-\hat{k}$.

Solution:

Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$ and $\vec{b} = 3\hat{i} + 4\hat{j} - \hat{k}$ and let θ be the angle between $\vec{a}$ and $\vec{b}$.

Now,

$$\therefore \cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{(2\hat{i}-\hat{j}+\hat{k})\cdot(3\hat{i}+4\hat{j}-\hat{k})}{\sqrt{4+1+1}\cdot\sqrt{9+16+1}}$$

$$= \frac{6-4-1}{\sqrt{6}\cdot\sqrt{26}} \Rightarrow \frac{1}{2\sqrt{3}\cdot\sqrt{13}} = \frac{1}{2\sqrt{39}}$$

$$\therefore \theta = \cos^{-1}\frac{1}{2\sqrt{39}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{156}\right)$$

Thus, the required value of θ is $\cos^{-1}\left(\frac{1}{156}\right)$.

10. If $\vec{a}+\vec{b}+\vec{c}=0$, show that $\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$. Interpret the result geometrically?

Solution:

Given that $\vec{a}+\vec{b}+\vec{c}=0$

So, $\vec{a}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{a}\times 0$

$\vec{a}\times\vec{a}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=0$

$\vec{o}+\vec{a}\times\vec{b}+\vec{a}\times\vec{c}=0 \qquad (\vec{a}\times\vec{a}=0)$

$\vec{a}\times\vec{b}-\vec{c}\times\vec{a}=0 \qquad (\vec{a}\times\vec{c}=-\vec{c}\times\vec{a})$

$\Rightarrow \vec{a}\times\vec{b}=\vec{c}\times\vec{a} \qquad \ldots(i)$

Now, $\vec{a}+\vec{b}+\vec{c}=0$

$\vec{b}\times(\vec{a}+\vec{b}+\vec{c}) = \vec{b}\times 0$

$\vec{b}\times\vec{a}+\vec{b}\times\vec{b}+\vec{b}\times\vec{c}=0$

$\vec{b}\times\vec{a}+\vec{o}+\vec{b}\times\vec{c}=0 \qquad (\because \quad \vec{b}\times\vec{b}=0)$

$-(\vec{a}\times b)+\vec{b}\times\vec{c}=0$

$\therefore\ \vec{b}\times\vec{c}=\vec{a}\times\vec{b} \qquad \ldots(ii)$

From eq. (*i*) and (*ii*) we get

$\vec{a}\times\vec{b}=\vec{b}\times\vec{c}=\vec{c}\times\vec{a}$.

- Hence proved.

11. Find the sine of the angle between the vectors $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$.

Solution:

Given that $\vec{a}=3\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}=2\hat{i}-2\hat{j}+4\hat{k}$

We know that $|\vec{a}\times\vec{b}| = |\vec{a}||\vec{b}|\sin\theta$

So,

$$\vec{a}\times\vec{b} = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 3 & 1 & 2\\ 2 & -2 & 4\end{vmatrix}$$

$$= \hat{i}(4+4)-\hat{j}(12-4)+\hat{k}(-6-2)$$

$$= 8\hat{i}-8\hat{j}-8\hat{k}$$

$$|\vec{a}\times\vec{b}| = \sqrt{(8)^2+(-8)^2+(-8)^2}$$

$$= \sqrt{64+64+64} = \sqrt{192} = \sqrt{64\times 3} = 8\sqrt{3}$$

$$|\vec{a}| = \sqrt{(3)^2+(1)^2+(2)^2} = \sqrt{9+1+4} = \sqrt{14}$$

$$|\vec{b}| = \sqrt{(2)^2+(-2)^2+(4)^2} = \sqrt{4+4+16}$$

$$= \sqrt{24} = 2\sqrt{6}$$

Now, $$\sin\theta = \frac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|} = \frac{8\sqrt{3}}{\sqrt{14}\cdot 2\sqrt{6}}$$

$$= \frac{4\sqrt{3}}{\sqrt{84}} = \frac{4\sqrt{3}}{2\sqrt{21}} = \frac{2}{\sqrt{7}}$$

Thus, $\sin\theta = 2/\sqrt{7}$

12. If A, B, C, D are the points with position vectors $\hat{i}+\hat{j}-\hat{k}$, $2\hat{i}-\hat{j}+3\hat{k}$, $2\hat{i}-3\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$, respectively, find the projection of $\overrightarrow{AB}$ along $\overrightarrow{CD}$.
Solution:

We have,

Position vector of A = $\hat{i}+\hat{j}-\hat{k}$

Position vector of B = $2\hat{i}-\hat{j}+3\hat{k}$

Position vector of C = $2\hat{i}-3\hat{k}$

Position vector of D = $3\hat{i}-2\hat{j}+\hat{k}$

$$\overrightarrow{AB} = \text{P.V of B} - \text{P.V of A}$$
$$= (2\hat{i}-\hat{j}+3\hat{k})-(\hat{i}+\hat{j}-\hat{k}) = \hat{i}-2\hat{j}+4\hat{k}$$
$$\overrightarrow{CD} = \text{P.V. of D} - \text{P.V. of C}$$
$$= (3\hat{i}-2\hat{j}+\hat{k})-(2\hat{i}-3\hat{k}) = \hat{i}-2\hat{j}+4\hat{k}$$
$$\text{Projection of } \overrightarrow{AB} \text{ on } \overrightarrow{CD} = \frac{\overrightarrow{AB}\cdot\overrightarrow{CD}}{|\overrightarrow{CD}|}$$
$$= \frac{(\hat{i}-2\hat{j}+4\hat{k})\cdot(\hat{i}-2\hat{j}+4\hat{k})}{\sqrt{(1)^2+(-2)^2+(4)^2}}$$
$$= \frac{1+4+16}{\sqrt{1+4+16}} = \frac{21}{\sqrt{21}} = \sqrt{21}$$

Thus, the required projection = $\sqrt{21}$.

13. Using vectors, find the area of the triangle ABC with vertices A(1, 2, 3), B(2, – 1, 4) and C(4, 5, – 1).
Solution:

Given vertices A(1, 2, 3), B(2, – 1, 4) and C(4, 5, – 1).

$$\overrightarrow{AB} = (2-1)\hat{i}+(-1-2)\hat{j}+(4-3)\hat{k}$$
$$\overrightarrow{AB} = \hat{i}-3\hat{j}+\hat{k}$$
$$\overrightarrow{AC} = (4-1)\hat{i}+(5-2)\hat{j}+(-1-3)\hat{k} = 3\hat{i}+3\hat{j}-4\hat{k}$$

Now,
$$\text{Area of } \Delta ABC = \frac{1}{2}|\overrightarrow{AB}\times\overrightarrow{AC}| = \frac{1}{2}\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -3 & 1\\ 3 & 3 & -4\end{vmatrix}$$
$$= \frac{1}{2}[\hat{i}(12-3)-\hat{j}(-4-3)+\hat{k}(3+9)]$$
$$= \frac{1}{2}|9\hat{i}+7\hat{j}+12\hat{k}| = \frac{1}{2}\sqrt{(9)^2+(7)^2+(12)^2}$$
$$= \frac{1}{2}\sqrt{81+49+144} = \frac{1}{2}\sqrt{274}$$

Thus, the required area is $\frac{\sqrt{274}}{2}$.

14. Using vectors, prove that the parallelogram on the same base and between the same parallels are equal in area.
Solution:

Let's consider ABCD and ABFE be two parallelograms on the same base AB and between same parallel lines AB and DF.

Let $\overline{AB} = \vec{a}$ and $\overline{AD} = \vec{b}$

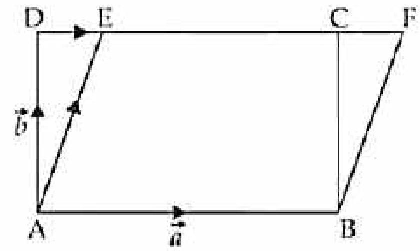
$\therefore$ Area of parallelogram ABCD = $|\vec{a} \times \vec{b}|$

Now, Area of parallelogram ABFE = $|\overline{AB} \times \overline{AE}|$

$= |\vec{a} \times (\overline{AD} + \overline{DE})| = |\vec{a} \times (\vec{b} \times K\vec{a})|$

$= |(\vec{a} \times \vec{b}) + K(\vec{a} \times \vec{a})| = |\vec{a} \times \vec{b}| + 0 \quad [\because \vec{a} \times \vec{a} = 0]$

$= |\vec{a} \times \vec{b}|$



- Hence proved.

Long Answer (L.A.)

15. Prove that in any triangle ABC, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ **, where *a, b, c* are the magnitudes of the sides opposite to the vertices A, B, C, respectively.**
Solution:

In triangle ABC, the components of c are c cos A and c sin A.

$\therefore \overline{CD} = b - c\cos A$

In ΔBDC,

$a^2 = CD^2 + BD^2$

$a^2 = (b - c\cos A)^2 + (c\sin A)^2$

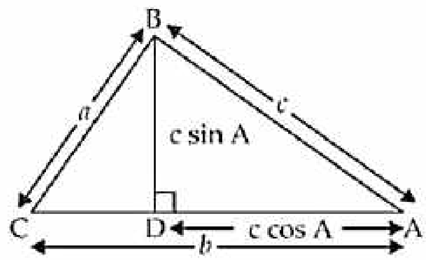
$a^2 = b^2 + c^2\cos^2 A - 2bc\cos A + c^2\sin^2 A$

$a^2 = b^2 + c^2(\cos^2 A + \sin^2 A) - 2bc\cos A$

$a^2 = b^2 + c^2 - 2bc\cos A \quad \Rightarrow 2bc\cos A = b^2 + c^2 - a^2$

Thus,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$



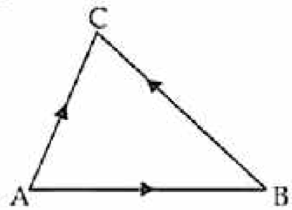
- Hence proved.

16. If $\vec{a}, \vec{b}, \vec{c}$ **determine the vertices of a triangle, show that** $\frac{1}{2}[\vec{b} \times \vec{c} + \vec{c} \times \vec{a} + \vec{a} \times \vec{b}]$ **gives the vector area of the triangle. Hence deduce the condition that the three points** $\vec{a}, \vec{b}, \vec{c}$ **are collinear. Also find the unit vector normal to the plane of the triangle.**
Solution:

As, $\vec{a}, \vec{b}$ and $\vec{c}$ are the vertices of ΔABC

So, $\overrightarrow{AB} = \vec{b} - \vec{a}, \overrightarrow{BC} = \vec{c} - \vec{b}$

And, $\overrightarrow{AC} = \vec{c} - \vec{a}$



$\therefore$ Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$= \frac{1}{2}\left|(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})\right|$

$= \frac{1}{2}\left|\vec{b} \times \vec{c} - \vec{b} \times \vec{a} - \vec{a} \times \vec{c} + \vec{a} \times \vec{a}\right|$ $\left[\because \begin{array}{l} \vec{a} \times \vec{b} = -\vec{b} \times \vec{a} \\ \vec{c} \times \vec{a} = -\vec{a} \times \vec{c} \\ \vec{a} \times \vec{a} = \vec{0} \end{array}\right]$

$= \frac{1}{2}\left|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right|$

If three vectors are collinear, area of $\Delta ABC = 0$

So, $\frac{1}{2}\left|\vec{b} \times \vec{c} + \vec{a} \times \vec{b} + \vec{c} \times \vec{a}\right| = 0$

$\therefore \left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right| = 0$

which is the condition of collinearity of $\vec{a}, \vec{b}$ and $\vec{c}$.

Let $\hat{n}$ be the unit vector normal to the plane of the ΔABC

So, $\hat{n} = \frac{\overrightarrow{AB} \times \overrightarrow{AC}}{\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|}$

$= \frac{\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}}{\left|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}\right|}$

17. Show that area of the parallelogram whose diagonals are given by $\vec{a}$ and $\vec{b}$ is $\frac{\left|\vec{a} \times \vec{b}\right|}{2}$. Also find the area of the parallelogram whose diagonals are $2\hat{i} - \hat{j} + \hat{k}$ and $\hat{i} + 3\hat{j} - \hat{k}$.

Solution:

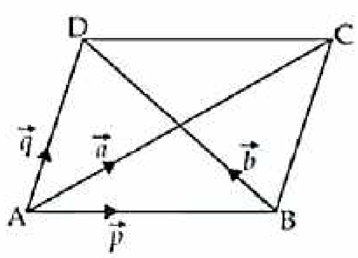
Let's take ABCD to be a parallelogram such that

$\overrightarrow{AB} = \vec{p}, \overrightarrow{AD} = \vec{q} = \overrightarrow{BC}$

So by law of triangle, we get

$\overrightarrow{AC} = \vec{a} = \vec{p} + \vec{q}$ ...(i)

and $\overrightarrow{BD} = \vec{b} = -\vec{p} + \vec{q}$ ...(ii)



Adding eq. (i) and (ii) we get,

$\vec{a} + \vec{b} = 2\vec{q} \Rightarrow \vec{q} = \left(\frac{\vec{a} + \vec{b}}{2}\right)$

Subtracting eq. (ii) from eq. (i) we get

$\vec{a} - \vec{b} = 2\vec{p} \Rightarrow \vec{p} = \left(\frac{\vec{a} - \vec{b}}{2}\right)$

So,

$$\vec{p}\times\vec{q} = \frac{1}{4}(\vec{a}+\vec{b})\times(\vec{a}-\vec{b}) = \frac{1}{4}(\vec{a}\times\vec{a}-\vec{a}\times\vec{b}+\vec{b}\times\vec{a}-\vec{b}\times\vec{b})$$

$$= \frac{1}{4}(-\vec{a}\times\vec{b}+\vec{b}\times\vec{a}) \qquad \left[\because \begin{array}{l}\vec{a}\times\vec{a}=0\\ \vec{b}\times\vec{b}=0\end{array}\right]$$

$$= \frac{1}{4}(\vec{a}\times\vec{b}+\vec{a}\times\vec{b}) = \frac{1}{4}\cdot 2(\vec{a}\times\vec{b}) = \frac{|\vec{a}\times\vec{b}|}{2}$$

And, the area of the parallelogram ABCD $= |\vec{p}\times\vec{q}| = \frac{1}{2}|\vec{a}\times\vec{b}|$

Now area of parallelogram whose diagonals are $2\hat{i}-\hat{j}+\hat{k}$ and $\hat{i}+3\hat{j}-\hat{k} = \frac{1}{2}|(2\hat{i}-\hat{j}+\hat{k})\times(\hat{i}+3\hat{j}-\hat{k})|$

$$= -\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 1 & 1\\ 1 & 3 & 1\end{vmatrix}$$

$$= \frac{1}{2}|\hat{i}(1-3)-\hat{j}(-2-1)+\hat{k}(6+1)| = \frac{1}{2}|-2\hat{i}+3\hat{j}+7\hat{k}|$$

$$= \frac{1}{2}\sqrt{(-2)^2+(3)^2+(7)^2} = \frac{1}{2}\sqrt{4+9+49}$$

$$= \frac{1}{2}\sqrt{62} \text{ sq. units}$$

Thus, the required area is $\frac{1}{2}\sqrt{62}$ sq. units.

18. If $\vec{a}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$, **find a vector** $\vec{c}$ **such that** $\vec{a}\times\vec{c}=\vec{b}$ and $\vec{a}.\vec{c}=3$.

Solution:

Let $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$

Also given that $\vec{a} = \hat{i}+\hat{j}+\hat{k}$ and $\vec{b}=\hat{j}-\hat{k}$

As, $\vec{a}\times\vec{c} = \vec{b}$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 1 & 1\\ c_1 & c_2 & c_3\end{vmatrix} = \hat{j}-\hat{k}$$

$$= \hat{i}(c_3-c_2)-\hat{j}(c_3-c_1)+\hat{k}(c_2-c_1) = \hat{j}-\hat{k}$$

On comparing the like terms, we get

$c_3-c_2=0$ ...(i)

$c_1-c_3=1$ ...(ii)

and $c_2-c_1=-1$ ...(iii)

Now for $\vec{a}\cdot\vec{c} = 3$

$$(\hat{i}+\hat{j}+\hat{k})\cdot(c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) = 3$$

$$\therefore\ c_1+c_2+c_3=3 \qquad ...(iv)$$

Adding eq. (*ii*) and eq. (*iii*) we get,

$$c_2 - c_3 = 0 \quad ...(v)$$

From (*iv*) and (*v*) we get

$$c_1 + 2c_2 = 3 \quad ...(vi)$$

From (*iii*) and (*vi*) we get

$$\begin{array}{r} c_1 + 2c_2 = 3 \\ + \quad -c_1 + c_2 = -1 \\ \hline 3c_2 = 2 \end{array}$$

We have, $c_2 = \frac{2}{3}$

$$c_3 - c_2 = 0 \Rightarrow c_3 - \frac{2}{3} = 0$$

$$\therefore c_3 = \frac{2}{3}$$

Now, $c_2 - c_1 = -1 \Rightarrow \frac{2}{3} - c_1 = -1$

$$\Rightarrow c_1 = 1 + \frac{2}{3} = \frac{5}{3}$$

$$\therefore \vec{c} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Thus, $\vec{c} = \frac{1}{3}(5\hat{i} + 2\hat{j} + 2\hat{k})$.

Objective Type Questions

Choose the correct answer from the given four options in each of the Exercises from 19 to 33 (M.C.Q)

19. The vector in the direction of the vector $\hat{i} - 2\hat{j} + 2\hat{k}$ that has magnitude 9 is

(A) $\hat{i} - 2\hat{j} + 2\hat{k}$ (B) $\frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$

(C) $3(\hat{i} - 2\hat{j} + 2\hat{k})$ (D) $9(\hat{i} - 2\hat{j} + 2\hat{k})$

Solution:

The correct option is (C).

Let $\vec{a} = \hat{i} - 2\hat{j} + 2\hat{k}$

Unit vector in the direction of $\vec{a} = \frac{\vec{a}}{|\vec{a}|}$

$$= \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{(1)^2 + (-2)^2 + (2)^2}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{\sqrt{1+4+4}} = \frac{\hat{i} - 2\hat{j} + 2\hat{k}}{3}$$

$$\therefore \text{Vector of magnitude } 9 = \frac{9(\hat{i} - 2\hat{j} + 2\hat{k})}{3} = 3(\hat{i} - 2\hat{j} + 2\hat{k})$$

20. The position vector of the point which divides the join of points $2\vec{a}-3\vec{b}$ and $\vec{a}+\vec{b}$ in the ratio 3 : 1 is

(A) $\frac{3\vec{a}-2\vec{b}}{2}$ (B) $\frac{7\vec{a}-8\vec{b}}{4}$ (C) $\frac{3\vec{a}}{4}$ (D) $\frac{5\vec{a}}{4}$

Solution:

The correct option is (D).
The given vectors are in the ratio 3: 1
So, the position vector of the required point c which divides the join of the given vectors $\vec{a}$ and $\vec{b}$ is

$$\vec{c} = \frac{m_1x_2 + m_2x_1}{m_1 + m_2}$$
$$= \frac{1\cdot(2\vec{a}-3\vec{b})+3(\vec{a}+\vec{b})}{3+1} = \frac{2\vec{a}-3\vec{b}+3\vec{a}+3\vec{b}}{4}$$
$$= \frac{5\vec{a}}{4} = \frac{5}{4}\vec{a}$$

21. The vector having initial and terminal points as (2, 5, 0) and (–3, 7, 4), respectively is

(A) $-\hat{i}+12\hat{j}+4\hat{k}$ (B) $5\hat{i}+2\hat{j}-4\hat{k}$

(C) $-5\hat{i}+2\hat{j}+4\hat{k}$ (D) $\hat{i}+\hat{j}+\hat{k}$

Solution:

The correct option is (C).
Let A and B be two points whose coordinates are given as (2, 5, 0) and (-3, 7, 4)
So, we have

$$\overrightarrow{AB} = (-3-2)\hat{i}+(7-5)\hat{j}+(4-0)\hat{k}$$
$$\overrightarrow{AB} = -5\hat{i}+2\hat{j}+4\hat{k}$$

22. The angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes √3 and 4, respectively, and $\vec{a}.\vec{b}=2\sqrt{3}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{5\pi}{2}$

Solution:

The correct option is (B).

Here, given that $|\vec{a}| = \sqrt{3}, |\vec{b}| = 4$ and $\vec{a} \cdot \vec{b} = 2\sqrt{3}$

So, from scalar product, we know that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}| \cos\theta$$

$$2\sqrt{3} = \sqrt{3} \cdot 4 \cdot \cos\theta$$

$$\cos\theta = \frac{2\sqrt{3}}{\sqrt{3} \cdot 4} = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$