## Exercise 11.3



**Short Answer (S.A.)**

**1. Find the position vector of a point A in space such that $\overrightarrow{OA}$ is inclined at 60° to OX and at 45° to OY and $|\overrightarrow{OA}|$ = 10 units.**

**Solution:**

We know that, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$\cos^2 60^o + \cos^2 45^o + \cos^2 \gamma = 1$

$(1/2)^2 + (1/\sqrt{2})^2 + \cos^2 \gamma = 1$

$\frac{1}{4} + \frac{1}{2} + \cos^2 \gamma = 1$

$\cos^2 \gamma = 1 - \frac{3}{4} = \frac{1}{4}$

So, $\cos \gamma = \pm \frac{1}{2} \Rightarrow \cos \gamma = \frac{1}{2}$ [Rejecting $\cos \gamma = -\frac{1}{2}$, as $\gamma < 90^o$]

Now,

$$\overline{OA} = |\overline{OA}|\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right) = 10\left(\frac{1}{2}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} + \frac{1}{2}\hat{k}\right)$$

$$= 5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k}$$

Thus, the position vector of A is $(5\hat{i} + 5\sqrt{2}\hat{j} + 5\hat{k})$.

**2. Find the vector equation of the line which is parallel to the vector $3\hat{i} - 2\hat{j} + 6\hat{k}$ and which passes through the point (1, –2, 3).**

**Solution:**

We know that the equation of line is

$$\vec{r} = \vec{a} + \vec{b}\lambda$$

Here, $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\vec{b} = 3\hat{i} - 2\hat{j} + 6\hat{k}$

$\therefore$ Equation of line is $\vec{r} = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) = (\hat{i} - 2\hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

$\Rightarrow (x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

Thus, the required equation is

$(x-1)\hat{i} + (y+2)\hat{j} + (z-3)\hat{k} = \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$

**3. Show that the lines**

$$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$$

and $\frac{x-4}{5} = \frac{y-1}{2} = z$ intersect.

**Also, find their point of intersection.**

**Solution:**

Given equation are,

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-4}{5}=\frac{y-1}{2}=z$$

Let, $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}=\lambda$

$\Rightarrow x=2\lambda+1,\ y=3\lambda+2$ and $z=4\lambda+3$

And, $\frac{x-4}{5}=\frac{y-1}{2}=\frac{z}{1}=\mu$

$\Rightarrow x=5\mu+4,\ y=2\mu+1$ and $z=\mu$

If the two lines intersect each other at one point,

then $2\lambda+1=5\mu+4 \Rightarrow 2\lambda-5\mu=3$ ...(*i*)

$3\lambda+2=2\mu+1 \Rightarrow 3\lambda-2\mu=-1$ ...(*ii*)

and $4\lambda+3=\mu \Rightarrow 4\lambda-\mu=-3$ ...(*iii*)

Solving eqns. (*i*) and (*ii*) we get

$2\lambda-5\mu=3$ [multiply by 3]

$3\lambda-2\mu=-1$ [multiply by 2]

$$\Rightarrow 6\lambda-15\mu=9$$
$$6\lambda-4\mu=-2$$
$$(-)\quad(+)\quad(+)$$
$$-11\mu=11 \quad \therefore \mu=-1$$

Putting the value of $\mu$ in eq. (*i*) we get,

$2\lambda-5(-1)=3$

$2\lambda+5=3$

$2\lambda=-2 \Rightarrow \lambda=-1$

$4(-1)-(-1)=-3$

$-4+1=-3$

$-3=-3$ (satisfied)

So, Coordinates of the point of intersection are

$x=5(-1)+4=-5+4=-1$

$y=2(-1)+1=-2+1=-1$

$z=-1$

Thus, the given lines intersect each other at $(-1,-1,-1)$.

**4. Find the angle between the lines**

$\vec{r}=3\hat{i}-2\hat{j}+6\hat{k}+\lambda(2\hat{i}+\hat{j}+2\hat{k})$ and $\vec{r}=(2\hat{j}-5\hat{k})+\mu(6\hat{i}+3\hat{j}+2\hat{k})$

**Solution:**

Here, $\vec{b}_1=2\hat{i}+\hat{j}+2\hat{k}$ and $\vec{b}_2=6\hat{i}+3\hat{j}+2\hat{k}$

So,
$$\cos\theta=\frac{\vec{b}_1.\vec{b}_2}{|\vec{b}_1||\vec{b}_2|}=\frac{(2\hat{i}+\hat{j}+2\hat{k}).(6\hat{i}+3\hat{j}+2\hat{k})}{\sqrt{(2)^2+(1)^2+(2)^2}.\sqrt{(6)^2+(3)^2+(2)^2}}$$
$$=\frac{12+3+4}{\sqrt{4+1+4}.\sqrt{36+9+4}}=\frac{19}{\sqrt{9}.\sqrt{49}}=\frac{19}{3\cdot 7}=\frac{19}{21}$$

$$\therefore \theta = \cos^{-1}\left(\frac{19}{21}\right)$$

Thus, the required angle is $\cos^{-1}\left(\frac{19}{21}\right)$.

**5. Prove that the line through A (0, –1, –1) and B (4, 5, 1) intersects the line through C (3, 9, 4) and D (– 4, 4, 4).**
**Solution:**

Given points, A (0, –1, –1) and B (4, 5, 1)
C (3, 9, 4) and D (– 4, 4, 4).
Cartesian form of equation AB is

$$\frac{x-0}{4-0} = \frac{y+1}{5+1} = \frac{z+1}{1+1} \Rightarrow \frac{x}{4} = \frac{y+1}{6} = \frac{z+1}{2}$$

and its vector form is $\vec{r} = (-\hat{j} - \hat{k}) + \lambda(4\hat{i} + 6\hat{j} + 2\hat{k})$
Similarly, equation of CD is

$$\frac{x-3}{-4-3} = \frac{y-9}{4-9} = \frac{z-4}{4-4} \Rightarrow \frac{x-3}{-7} = \frac{y-9}{-5} = \frac{z-4}{0}$$

and its vector form is $\vec{r} = (3\hat{i} + 9\hat{j} + 4\hat{k}) + \mu(-7\hat{i} - 5\hat{j})$

Now, here $\vec{a}_1 = -\hat{j} - \hat{k},\ \vec{b}_1 = 4\hat{i} + 6\hat{j} + 2\hat{k}$
$\vec{a}_2 = 3\hat{i} + 9\hat{j} + 4\hat{k},\ \vec{b}_2 = -7\hat{i} - 5\hat{j}$

Shortest distance between AB and CD

$$\text{S.D.} = \left|\frac{(\vec{a}_2 - \vec{a}_1)\cdot(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|}\right|$$

$$\vec{a}_2 - \vec{a}_1 = (3\hat{i} + 9\hat{j} + 4\hat{k}) - (-\hat{j} - \hat{k}) = 3\hat{i} + 10\hat{j} + 5\hat{k}.$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 6 & 2 \\ -7 & -5 & 0 \end{vmatrix}$$

$$= \hat{i}(0+10) - \hat{j}(0+14) + \hat{k}(-20+42)$$

$$= 10\hat{i} - 14\hat{j} + 22\hat{k}$$

$$|\vec{b}_1 \times \vec{b}_2| = \sqrt{(10)^2 + (-14)^2 + (22)^2}$$

$$= \sqrt{100 + 196 + 484} = \sqrt{780}$$

$$\therefore \text{S.D} = \frac{(3\hat{i} + 10\hat{j} + 5\hat{k})\cdot(10\hat{i} - 14\hat{j} + 22\hat{k})}{\sqrt{780}}$$

$$= \frac{30 - 140 + 110}{\sqrt{780}} = 0$$

Thus, the two lines intersect each other.

**6. Prove that the lines *x* = *py* + *q*, *z* = *ry* + *s* and *x* = *p′y* + *q′*, *z* = *r′y* + *s′* are perpendicular if *pp′* + *rr′* + 1 = 0.**
**Solution:**

Given, $x = py + q \Rightarrow y = \frac{x-q}{p}$

and $z = ry + s \Rightarrow y = \frac{z-s}{r}$

So, the equation becomes

$\frac{x-q}{p} = \frac{y}{1} = \frac{z-s}{r}$ in which d'ratios are $a_1 = p, b_1 = 1, c_1 = r$

Similarly $x = p'y + q' \Rightarrow y = \frac{x-q'}{p'}$

and $z = r'y + s' \Rightarrow y = \frac{z-s'}{r'}$

Hence, the equation becomes

$\frac{x-q'}{p'} = \frac{y}{1} = \frac{z-s'}{r'}$ in which $a_2 = p', b_2 = 1, c_2 = r'$

If the lines are perpendicular to each other, then

$a_1a_2 + b_1b_2 + c_1c_2 = 0$

$pp' + 1.1 + rr' = 0$

Thus, $pp' + rr' + 1 = 0$ is the required condition.

**7. Find the equation of a plane which bisects perpendicularly the line joining the points A (2, 3, 4) and B (4, 5, 8) at right angles.**
**Solution:**

Given coordinates are A (2, 3, 4) and B (4, 5, 8)
Now, the coordinates of the mid-point C are (2+4/2, 3+5/2, 4+8/2) = (3, 4, 6)
And, the direction ratios of the normal to the plane = direction ratios of AB
= 4 – 2, 5 – 3, 8 – 4 = (2, 2, 4)

Equation of the plane is
$a(x – x_1) + b(y – y_1) + c(z – z_1) = 0$
2(x - 3) + 2(y - 4) + 4(z - 6) = 0
2x – 6 + 2y – 8 + 4z – 24 = 0
2x + 2y + 4z = 38
x + y + 2z = 19

Thus, the required equation of plane is x + y + 2z = 19 or $\vec{r}(\hat{i} + \hat{j} + 2\hat{k}) = 19.$

**8. Find the equation of a plane which is at a distance 3√3 units from origin and the normal to which is equally inclined to coordinate axis.**
**Solution:**

As the normal to the plane is equally inclined to the axes we have,
cos α = cos β = cos γ

So, $\cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$
$3 \cos^2 \alpha = 1 \Rightarrow \cos \alpha = \pm 1/\sqrt{3}$
And, $\cos \alpha = \cos \beta = \cos \gamma = \pm 1/\sqrt{3}$

Now, the normal is $\vec{N} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$

$\therefore$ Equation of the plane is $\vec{r}.\vec{N} = d$

$$\vec{r}.\frac{\vec{N}}{|\vec{N}|} = d$$

$$\frac{\vec{r}.\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right)}{1} = 3\sqrt{3}$$

$$\vec{r}.\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) = 3\sqrt{3}$$

$$(x\hat{i} + y\hat{j} + z\hat{k}).\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$$

$$x + y + z = 3\sqrt{3}.\sqrt{3}$$

$$\Rightarrow x + y + z = 9$$

Thus, the equation of the plane is $x + y + z = 9$

**9. If the line drawn from the point (–2, – 1, – 3) meets a plane at right angle at the point (1, – 3, 3), find the equation of the plane.**
**Solution:**

Given, points (–2, – 1, – 3) and (1, – 3, 3)
Direction ratios of the normal to the plane are (1 + 2, -3 + 1, 3 + 3) = (3, -2, 6)
Now, the equation of plane passing through one point $(x_1, y_1, z_1)$ is
$a(x – x_1) + b(y – y_1) + c(z – z_1) = 0$
$3(x - 1) - 2(y + 3) + 6(z - 3) = 0$
$3x – 3 – 2y – 6 + 6z – 18 = 0$
$3x – 2y + 6z – 27 = 0 \Rightarrow 3x – 2y + 6z = 27$
Thus, the required equation of plane is $3x – 2y + 6z = 27$.

**10. Find the equation of the plane through the points (2, 1, 0), (3, –2, –2) and (3, 1, 7).**
**Solution:**

Given points are (2, 1, 0), (3, –2, –2) and (3, 1, 7)
As the equation of the plane passing through the points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3, z_3)$ is
$(x - 2) (-21) – (y - 1)(7 + 2) + z (3) = 0$
$-21 (x - 2) – 9(y - 1) + 3z = 0$
$-21x + 42 – 9y + 9 + 3z = 0$
$-21x – 9y + 3z + 51 = 0 \Rightarrow 7x + 3y – z – 17 = 0$
Thus, the required equation of plane is $7x + 3y – z – 17 = 0$.

**11. Find the equations of the two lines through the origin which intersect the line**

$\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ **at angles of π/3 each.**

**Solution:**

Any point in the given line is
x – 3/ 2 = y – 3/1 = z/1 = λ
x = 2λ + 3, y = λ + 3 and z = λ
Let it be the coordinates of P
So, the direction ratios of OP are (2λ + 3 - 0), (λ + 3 - 0) and (λ - 0) ⇒ 2λ + 3, λ + 3, λ
But the direction ratios of the line PQ are 2, 1, 1
Now, we know that

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,.\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\cos\frac{\pi}{3} = \frac{2(2\lambda+3) + 1(\lambda+3) + 1.\lambda}{\sqrt{(2)^2 + (1)^2 + (1)^2}\,.\sqrt{(2\lambda+3)^2 + (\lambda+3)^2 + \lambda^2}}$$

$$\frac{1}{2} = \frac{4\lambda + 6 + \lambda + 3 + \lambda}{\sqrt{6}\,.\sqrt{4\lambda^2 + 9 + 12\lambda + \lambda^2 + 9 + 6\lambda + \lambda^2}}$$

$$\frac{\sqrt{6}}{2} = \frac{6\lambda + 9}{\sqrt{6\lambda^2 + 18\lambda + 18}} = \frac{6\lambda + 9}{\sqrt{6}\sqrt{\lambda^2 + 3\lambda + 3}}$$

$$\frac{6}{2} = \frac{3(2\lambda+3)}{\sqrt{\lambda^2 + 3\lambda + 3}} \Rightarrow 3 = \frac{3(2\lambda+3)}{\sqrt{\lambda^2 + 3\lambda + 3}}$$

$$1 = \frac{2\lambda + 3}{\sqrt{\lambda^2 + 3\lambda + 3}} \Rightarrow \sqrt{\lambda^2 + 3\lambda + 3} = 2\lambda + 3$$

$\lambda^2 + 3\lambda + 3 = 4\lambda^2 + 9 + 12\lambda$ (On squaring on both sides)
$3\lambda^2 + 9\lambda + 6 = 0$
$\lambda^2 + 3\lambda + 2 = 0$
(λ + 1)( λ + 2) = 0
λ = -1, -2
So, the direction are:
[2(-1) + 3, -1 + 3, -1] = (-2, 2, -1) when λ = -1 and
[2(-2) + 3, -2 + 3, -2] = (-1, 1, -2) when λ = -2.
Thus, the required equation of planes are
x/1 = y/2 = z/-1 and x/-1 = y/1 = z/-2

**12. Find the angle between the lines whose direction cosines are given by the equations *l* + *m* + *n* = 0, $l^2 + m^2 - n^2 = 0$.**
**Solution:**

Given equations are,
*l* + *m* + *n* = 0 ….. (i)

$l^2 + m^2 - n^2 = 0$ …. (ii)
From equation (i), we have n = - (l + m)
Putting the value of n is equation (ii), we get
$l^2 + m^2 + [-(l + m)]^2 = 0$
$l^2 + m^2 - l^2 - m^2 - 2lm = 0$
$-2lm = 0$
$lm = 0 \Rightarrow (- m - n)m = 0$ [Since, l = - m - n]
$(m + n)m = 0 \Rightarrow m = 0$ or $m = -n$
$\Rightarrow l = 0$ or $l = -n$
Now, the direction cosines of the two lines are
0, -n, n and -n, 0, n ⇒ 0, -1, 1 and -1, 0, 1

$$\cos\theta = \frac{(0\hat{i} - \hat{j} + \hat{k}).(-\hat{i} + 0\hat{j} + \hat{k})}{\sqrt{(-1)^2 + (1)^2}\sqrt{(-1)^2 + (1)^2}} = \frac{1}{\sqrt{2}.\sqrt{2}} = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

Thus, the required angle π/3.

**13. If a variable line in two adjacent positions has direction cosines *l, m, n* and *l* + δ*l, m* + δ*m, n* + δ*n*, show that the small angle δθ between the two positions is given by**
**$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2$**
**Solution:**

Given that l, m, n and *l* + δ*l*, *m* + δ*m*, *n* + δ*n* are the direction cosines of a variable line in two positions
$l^2 + m^2 + n^2 = 1$ ….. (i) and
$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$ …. (ii)

$$l^2 + \delta l^2 + 2l.\delta l + m^2 + \delta m^2 + 2m.\delta m + n^2 + \delta n^2 + 2n.\delta n = 1$$
$$(l^2 + m^2 + n^2) + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) = 1$$
$$1 + (\delta l^2 + \delta m^2 + \delta n^2) + 2(l.\delta l + m.\delta m + n.\delta n) = 1$$
$$\Rightarrow l.\delta l + m.\delta m + n.\delta n = -\frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$

Let $\vec{a}$ and $\vec{b}$ be the unit vectors along a line with d'cosines $l, m, n$ and $(l + \delta l), (m + \delta m), (n + \delta n)$.

$$\therefore \vec{a} = l\hat{i} + m\hat{j} + n\hat{k} \text{ and } \vec{b} = (l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}$$

$$\cos\delta\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\delta\theta = \frac{(l\hat{i} + m\hat{j} + n\hat{k}).\left[(l + \delta l)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k}\right]}{1.1} \quad \left[\because |\vec{a}| = |\vec{b}| = 1\right]$$

$$\cos\delta\theta = l(l + \delta l) + m(m + \delta m) + n(n + \delta n)$$
$$\cos\delta\theta = l^2 + l.\delta l + m^2 + m.\delta m + n^2 + n.\delta n$$
$$\cos\delta\theta = (l^2 + m^2 + n^2) + (l.\delta l + m.\delta m + n.\delta n)$$

$$\cos \delta\theta = 1 - \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$
$$1 - \cos \delta\theta = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$
$$2\sin^2 \frac{\delta\theta}{2} = \frac{1}{2}(\delta l^2 + \delta m^2 + \delta n^2)$$
$$4\sin^2 \frac{\delta\theta}{2} = \delta l^2 + \delta m^2 + \delta n^2$$
$$4\left(\frac{\delta\theta}{2}\right)^2 = \delta l^2 + \delta m^2 + \delta n^2 \qquad \left[\because \frac{\delta\theta}{2} \text{ is very small so, } \sin\frac{\delta\theta}{2} = \frac{\delta\theta}{2}\right]$$
$$(\delta\theta)^2 = \delta l^2 + \delta m^2 + \delta n^2$$

Hence proved.

**14. O is the origin and A is (*a*, *b*, *c*). Find the direction cosines of the line OA and the equation of plane through A at right angle to OA.**
**Solution:**

Given, O (0, 0, 0) and A(a, b, c)
So, the direction ratios of OA = a – 0, b – 0, c – 0 = a, b, c
And, the direction cosines of line OA

$$= \frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$$

Now, the direction ratios of the normal to the plane are (a, b, c).
We know that, the equation of the plan passing through the point A(a, b, c) is
a(x – a) + b(y - b) + c(z - c) = 0
$ax - a^2 + by - b^2 + cz - c^2 = 0$
$ax + by + cz = a^2 + b^2 + c^2$
Thus, the required equation of the plane is $ax + by + cz = a^2 + b^2 + c^2$

**15. Two systems of rectangular axis have the same origin. If a plane cuts them at distances *a, b, c* and *a′*, *b′*, *c′*, respectively, from the origin, prove that**

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$

**Solution:**

Let's take OX, OY, OZ and ox, oy, oz to be two rectangular systems.
And, the equations of two planes are

$$\frac{X}{a}+\frac{Y}{b}+\frac{Z}{c} = 1 \quad ...(i) \quad \text{and} \quad \frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'} = 1 \qquad ...(ii)$$

Length of perpendicular from origin to plane (*i*) is

$$= \left|\frac{\frac{0}{a}+\frac{0}{b}+\frac{0}{c}-1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}\right| = \frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}$$

Length of perpendicular from origin to plane (*ii*)

$$= \left| \frac{\frac{0}{a'} + \frac{0}{b'} + \frac{0}{c'} - 1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}} \right| = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

As per the condition of the question

$$\frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

Thus, $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$

**Long Answer (L.A.)**

**16. Find the foot of perpendicular from the point (2, 3, –8) to the line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$. Also, find the perpendicular distance from the given point to the line.**
**Solution:**

Given,

$\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$ is the equation of the line.

So, $\frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3} = \lambda$

Now, the coordinates of any point Q on the line are
x = -2λ + 4, y = 6λ and z = -3λ + 1 and the given point is P(2, 3, -8)
The direction ratios of PQ are -2λ + 4 – 2, 6λ – 3, -3λ + 1 + 8 i.e. -2λ + 2, 6λ – 3, -3λ + 9
And the direction ratios of the given line are -2, 6, -3.
If PQ ⊥ line, then
-2(-2λ + 2) + 6(6λ - 3) – 3(-3λ + 9) = 0
4λ – 4 + 36λ – 18 + 9λ – 27 = 0
49λ – 49 = 0
λ = 1
Now, the foot of the perpendicular is -2(1) + 4, 6(1), -3(1) + 1 i.e. 2, 6, -2
Hence, the distance PQ is

$$= \sqrt{(2-2)^2 + (3-6)^2 + (-8+2)^2}$$
$$= \sqrt{9+36} = \sqrt{45} = 3\sqrt{5}$$

Thus, the required coordinates of the foot of perpendicular are 2, 6, -2 and the required distance is 3√5 units.

**17. Find the distance of a point (2, 4, –1) from the line**

$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}$

**Solution:**

Given equation of line,

$$\frac{x+5}{1}=\frac{y+3}{4}=\frac{z-6}{-9}=\lambda \text{ and any point } P(2, 4, -1)$$

Let Q be any point on the given line

So, Coordinates of Q are $x=\lambda-5,\ y=4\lambda-3,\ z=-9\lambda+6$

D'ratios of PQ are $\lambda-5-2,\ 4\lambda-3-4,\ -9\lambda+6+1$ i.e., $\lambda-7,\ 4\lambda-7,\ -9\lambda+7$

and the d'ratios of the line are 1, 4, –9

If PQ $\perp$ line then

$$1(\lambda-7)+4(4\lambda-7)-9(-9\lambda+7)=0$$
$$\lambda-7+16\lambda-28+81\lambda-63=0$$
$$98\lambda-98=0 \quad \Rightarrow \quad \lambda=1$$

So, the coordinates of Q are $1-5,\ 4\times1-3,\ -9\times1+6$ i.e., $-4, 1, -3$

$$\therefore PQ=\sqrt{(-4-2)^2+(1-4)^2+(-3+1)^2}$$
$$=\sqrt{(-6)^2+(-3)^2+(-2)^2}=\sqrt{36+9+4}=\sqrt{49}=7$$

Thus, the required distance is 7 units.

**18. Find the length and the foot of perpendicular from the point (1, 3/2, 2) to the plane 2*x* – 2*y* + 4*z* + 5 = 0.**

**Solution:**

Given plane is 2x – 2y + 4z + 5 = 0 and point (1, 3/2, 2)

The direction ratios of the normal to the plane are 2, -2, 4

So, the equation of the line passing through (1, 3/2, 2) and direction ratios are equal to the direction ratios of the normal to the plane i.e. 2, -2, 4 is

$$\frac{x-1}{2}=\frac{y-\frac{3}{2}}{-2}=\frac{z-2}{4}=\lambda$$

Now, any point in the plane is 2λ + 1, -2λ + 3/2, 4λ + 2

Since, the point lies in the plane, then

2(2λ + 1) – 2(-2λ + 3/2) + 4(4λ + 2) + 5 = 0

4λ + 2 + 4λ – 3 + 16λ + 8 + 5 = 0

24λ + 12 = 0 λ = ½

So, the coordinates of the point in the plane are

2(-1/2) + 1, -2(-1/2) + 3/2, 4(-1/2) + 2 = 0, 5/2, 0

Thus, the foot of the perpendicular is (0, 5/2, 0) and the required length

$$=\sqrt{(1-0)^2+\left(\frac{3}{2}-\frac{5}{2}\right)^2+(2-0)^2}$$
$$=\sqrt{1+1+4}=\sqrt{6} \text{ units}$$

**19. Find the equations of the line passing through the point (3,0,1) and parallel to the planes $x + 2y = 0$ and $3y – z = 0$.**
**Solution:**

Given point is (3, 0, 1) and the equation of planes are
x + 2y = 0 …. (i) and 3y – z = 0 …. (ii)
Equation of any line l passing through (3, 0, 1) is
l: (x - 3)/a = (y - 0)/b = (z - 1)/c
Now, the direction ratios of the normal to plane (i) and plane (ii) are (1, 2, 0) and (0, 3, 1).
As the line is parallel to both the planes, we have
1.a + 2.b + 0.c = 0 ⇒ a + 2b + 0c = 0 and
0.a + 3.b – 1.c = 0 ⇒ 0a + 3b – c = 0

So, $$\frac{a}{-2-0} = \frac{-b}{-1-0} = \frac{c}{3-0} = \lambda$$
$\therefore a = -2\lambda, b = \lambda, c = 3\lambda$
So, the equation of line is
$$\frac{x-3}{-2\lambda} = \frac{y}{\lambda} = \frac{z-1}{3\lambda}$$
Thus, the required equation is
$$\frac{x-3}{-2} = \frac{y}{1} = \frac{z-1}{3}$$
or in vector form is
$$(x-3)\hat{i} + y\hat{j} + (z-1)\hat{k} = \lambda(-2\hat{i} + \hat{j} + 3\hat{k})$$

**20. Find the equation of the plane through the points (2, 1, –1) and (–1, 3, 4) and perpendicular to the plane $x – 2y + 4z = 10$.**
**Solution:**

We know that, equation of the plane passing through two points ($x_1$, $y_1$, $z_1$) and ($x_2$, $y_2$, $z_2$) with its normal’s direction ratios is
$a(x – x_1) + b(y – y_1) + c(z – z_1) = 0$ …. (i)
Now, if the plane is passing through two points (2, 1, -1) and (-1, 3, 4) then
$a(x_2 - x_1) + b(y_2 – y_1) + c(z_2 – z_1) = 0$
a(-1 - 2) + b(3 - 1) + c(4 + 1) = 0
-3a + 2b + 5c = 0 …. (ii)
As the required plane is perpendicular to the given plane x – 2y + 4z = 10, then
1.a – 2.b + 4.c = 10 …. (iii)
On solving (ii) and (iii) we get,
$$\frac{a}{8+10} = \frac{-b}{-12-5} = \frac{c}{6-2} = \lambda$$
So, a = 18λ, b = 17λ and c = 4λ
Thus, the required plane is
18λ (x - 2) + 17λ (y - 1) + 4λ (z + 1) = 0
18x – 36 + 17y – 17 + 4z + 4 = 0

⇒ 18x + 17y + 4z – 49 = 0

**21. Find the shortest distance between the lines given by** $\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k}$ **and** $\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

**Solution:**

Given equations of lines are

$$\vec{r} = (8+3\lambda)\hat{i} - (9+16\lambda)\hat{j} + (10+7\lambda)\hat{k} \quad ...(i)$$

and $$\vec{r} = 15\hat{i} + 29\hat{j} + 5\hat{k} + \mu(3\hat{i} + 8\hat{j} - 5\hat{k}) \quad ...(ii)$$

Equation (*i*) can be re-written as

$$\vec{r} = 8\hat{i} - 9\hat{j} + 10\hat{k} + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k}) \quad ...(iii)$$

Here, $\vec{a}_1 = 8\hat{i} - 9\hat{j} + 10\hat{k}$ and $\vec{a}_2 = 15\hat{i} + 29\hat{j} + 5\hat{k}$

$\vec{b}_1 = 3\hat{i} - 16\hat{j} + 7\hat{k}$ and $\vec{b}_2 = 3\hat{i} + 8\hat{j} - 5\hat{k}$

$$\vec{a}_2 - \vec{a}_1 = 7\hat{i} + 38\hat{j} - 5\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$

$$= \hat{i}(80 - 56) - \hat{j}(-15 - 21) + \hat{k}(24 + 48)$$

$$= 24\hat{i} + 36\hat{j} + 72\hat{k}$$

Thus,

Shortest distance, SD $= \left| \dfrac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{|\vec{b}_1 \times \vec{b}_2|} \right|$

$$= \left| \frac{(7\hat{i} + 38\hat{j} - 5\hat{k}).(24\hat{i} + 36\hat{j} + 72\hat{k})}{\sqrt{(24)^2 + (36)^2 + (72)^2}} \right|$$

$$= \left| \frac{168 + 1368 - 360}{\sqrt{576 + 1296 + 5184}} \right| = \left| \frac{168 + 1008}{\sqrt{7056}} \right| = \frac{1176}{84} = 14 \text{ units}$$

**22. Find the equation of the plane which is perpendicular to the plane 5*x* + 3*y* + 6*z* + 8 = 0 and which contains the line of intersection of the planes *x* + 2*y* + 3*z* – 4 = 0 and 2*x* + *y* – *z* + 5 = 0.**

**Solution:**

The given planes are

$P_1$: 5x + 3y + 6z + 8 = 0

$P_2$: x + 2y + 3z – 4 = 0

$P_3$: 2x + y – z + 5 = 0

Now, the equation of the plane passing through the line of intersection of $P_1$ and $P_3$ is

(x + 2y + 3z – 4) + λ(2x + y – z + 5) = 0

(1 + 2λ)x + (2 + λ)y + (3 - λ)z – 4 + 5λ = 0 …. (i)

From the question its understood that plane (i) is perpendicular to $P_1$, then

5(1 + 2λ) + 3(2 + λ) + 6(3 - λ) = 0

5 + 10λ + 6 + 3λ + 18 – 6λ = 0
7λ + 29 = 0
λ = -29/7
Putting the value of ; in equation (i), we get

$$\left[1+2\left(\frac{-29}{7}\right)\right]x+\left[2-\frac{29}{7}\right]y+\left[3+\frac{29}{7}\right]z-4+5\left(\frac{-29}{7}\right)=0$$

$$\Rightarrow \frac{-15}{7}x-\frac{15}{7}y+\frac{50}{7}z-4-\frac{145}{7}=0$$

-15x – 15y + 50z – 28 – 145 = 0
-15x – 15y + 50z – 173 = 0 ⇒ 51x + 15y – 50z + 173 = 0
Thus, the required equation of plane is 51x + 15y – 50z + 173 = 0.

**23. The plane *ax* + *by* = 0 is rotated about its line of intersection with the plane *z* = 0 through an angle *α*. Prove that the equation of the plane in its new position is**

$$ax+by\pm(\sqrt{a^2+b^2}\tan\alpha)z=0.$$

**Solution:**

Given planes are: ax + by = 0 …. (i) and z = 0 …. (ii)
Now, the equation of any plan passing through the line of intersection of plane (i) and (ii) is
(ax + by) + kz = 0 ⇒ ax + by + kz = 0 …. (iii)

Dividing both sides by $\sqrt{a^2+b^2+k^2}$, we get

$$\frac{a}{\sqrt{a^2+b^2+k^2}}x+\frac{b}{\sqrt{a^2+b^2+k^2}}y+\frac{k}{\sqrt{a^2+b^2+k^2}}z=0$$

So, direction cosines of the normal to the plane are

$$\frac{a}{\sqrt{a^2+b^2+k^2}},\ \frac{b}{\sqrt{a^2+b^2+k^2}},\ \frac{k}{\sqrt{a^2+b^2+k^2}}$$

and the direction cosines of the plane (*i*) are

$$\frac{a}{\sqrt{a^2+b^2}},\ \frac{b}{\sqrt{a^2+b^2}},\ 0$$

As, α is the angle between the planes (*i*) and (*iii*), we get

$$\Rightarrow \cos\alpha=\frac{a.a+b.b+k.0}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{a^2+b^2}{\sqrt{a^2+b^2+k^2}.\sqrt{a^2+b^2}}$$

$$\cos\alpha=\frac{\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+k^2}} \quad\Rightarrow\ \cos^2\alpha=\frac{a^2+b^2}{a^2+b^2+k^2}$$

$$(a^2+b^2+k^2)\cos^2\alpha=a^2+b^2$$

$$a^2\cos^2\alpha+b^2\cos^2\alpha+k^2\cos^2\alpha=a^2+b^2$$

$$k^2\cos^2\alpha=a^2-a^2\cos^2\alpha+b^2-b^2\cos^2\alpha$$

$$k^2\cos^2\alpha = \alpha^2(1-\cos^2\alpha) + b^2(1-\cos^2\alpha)$$
$$k^2\cos^2\alpha = a^2\sin^2\alpha + b^2\sin^2\alpha$$
$$k^2\cos^2\alpha = (a^2+b^2)\sin^2\alpha$$
$$\Rightarrow k^2 = (a^2+b^2)\frac{\sin^2\alpha}{\cos^2\alpha} \quad \Rightarrow k = \pm\sqrt{a^2+b^2}\,.\tan\alpha$$

Putting the value of $k$ in eq. (*iii*) we get

$ax + by \pm \left(\sqrt{a^2+b^2}\,.\tan\alpha\right)z = 0$ which is the required equation of plane.

- Hence proved.

**24. Find the equation of the plane through the intersection of the planes $\vec{r}\,.(\hat{i}+3\hat{j})-6=0$ and $\vec{r}\,.(3\hat{i}-\hat{j}-4\hat{k})=0,$ whose perpendicular distance from origin is unity.**

**Solution:**

Given planes are

$$\vec{r}.(\hat{i}+3\hat{j})-6=0 \quad \Rightarrow \quad x+3y-6=0 \qquad ...(i)$$

and $$\vec{r}.(3\hat{i}-\hat{j}-4\hat{k})=0 \quad \Rightarrow \quad 3x-y-4z=0 \qquad ...(ii)$$

Equation of the plane passing through the line of intersection of plane (*i*) and (*ii*) is

$$(x+3y-6)+k(3x-y-4z)=0 \qquad ...(iii)$$
$$(1+3k)x+(3-k)y-4kz-6=0$$

Perpendicular distance from origin

$$\Rightarrow \left|\frac{-6}{\sqrt{(1+3k)^2+(3-k)^2+(-4k)^2}}\right| = 1$$

$$\frac{36}{1+9k^2+6k+9+k^2-6k+16k^2} = 1 \text{ [Squaring both sides]}$$

$$\frac{36}{26k^2+10} = 1 \quad \Rightarrow 26k^2+10=36$$

$$26k^2 = 26 \quad \Rightarrow \quad k^2 = 1 \quad \therefore k = \pm 1$$

Putting the value of $k$ in eq. (*iii*) we get,

$$(x+3y-6) \pm (3x-y-4z) = 0$$

$\Rightarrow x+3y-6+3x-y-4z=0$ and $x+3y-6-3x+y+4z=0$

$\Rightarrow 4x+2y-4z-6=0$ and $-2x+4y+4z-6=0$

Thus, the required equations of planes are;

4x + 2y – 4z – 6 = 0 and -2x – 4y + 4z – 6 = 0.