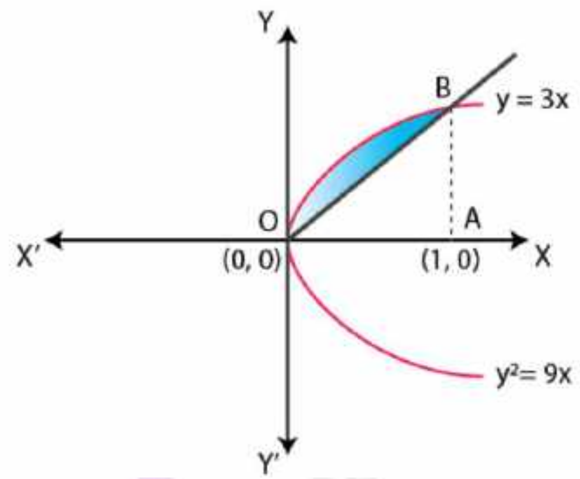
Exercise 8.3 Page No: 176

1. Find the area of the region bounded by the curves $y^2 = 9x$, $y = 3x$.
Solution:



Given curves are $y^2 = 9x$ and $y = 3x$
Now, solving the two equations we have

$(3x)^2 = 9x$

$9x^2 = 9x$

$9x^2 - 9x = 0 \Rightarrow 9x(x - 1) = 0$

Thus, x = 0, 1
So, the area of the shaded region is given by
= ar(region OAB) – ar (Δ OAB)

$$= \text{ar (region OAB)} - \text{ar } (\Delta OAB)$$

$$= -\int_0^1 y_l \, dx = \int_0^1 \sqrt{9x}\, dx - \int_0^1 3x\, dx$$

$$= 3\int_0^1 \sqrt{x}\, dx - 3\int_0^1 x\, dx = 3 \times \frac{2}{3}\left[x^{3/2}\right]_0^1 - 3\left[\frac{x^2}{2}\right]_0^1$$

$$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}\left[(1)^2 - 0\right] = 2(1) - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2} \text{ sq. units}$$

Therefore, the required area is ½ sq.units.

2. Find the area of the region bounded by the parabola $y^2 = 2px$, $x^2 = 2py$.
Solution:

Given parabolas are $y^2 = 2px$ …. (i) and $x^2 = 2py$ …. (ii)
Now, from equation (ii) we have

$y = x^2/2p$

Putting the value of y in equation (i), we have

$(x^2/2p)^2 = 2px$

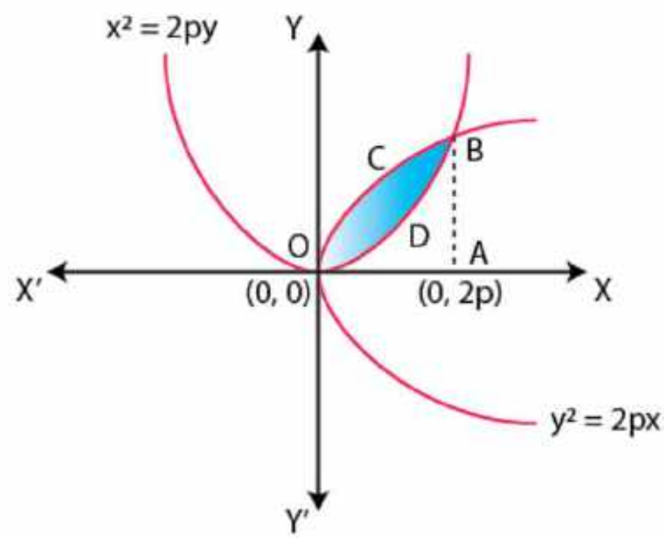
$x^4/4p^2 = 2px$

$x^4 = 8p^3x$

$x^4 - 8p^3x = 0$

$x(x^3 - 8p^3) = 0$

So, x = 0 or $x^3 - 8p^3 = 0 \Rightarrow x = 2p$



Now, the required area is

$$= \text{Area of the region (OCBA – ODBA)}$$

$$= \int_0^{2p} \sqrt{2px}\, dx - \int_0^{2p} \frac{x^2}{2p}\, dx = \sqrt{2p}\int_0^{2p} \sqrt{x}\, dx - \frac{1}{2p}\int_0^{2p} x^2 dx$$

$$= \sqrt{2p} \cdot \frac{2}{3}\left[x^{3/2}\right]_0^{2p} - \frac{1}{2p} \cdot \frac{1}{3}\left[x^3\right]_0^{2p}$$

$$= \frac{2\sqrt{2}}{3}\sqrt{p}\left[(2p)^{3/2} - 0\right] - \frac{1}{6p}\left[(2p)^3 - 0\right]$$

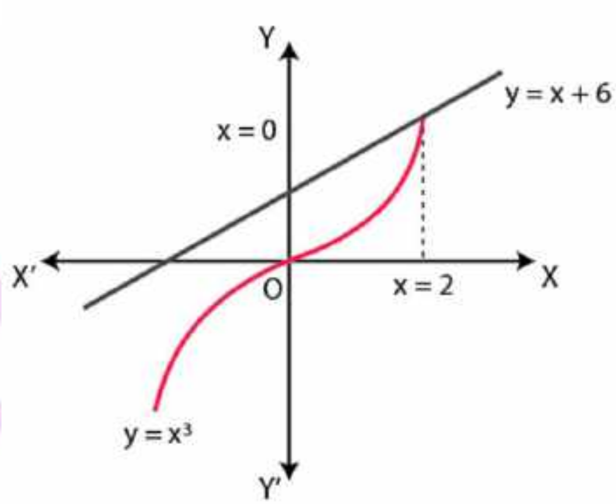
$$= \frac{2\sqrt{2}}{3}\sqrt{p}\,.\,2\sqrt{2}\,p^{\frac{3}{2}} - \frac{1}{6p}.8p^3$$

$$= \frac{8}{3}\cdot p^2 - \frac{8}{6}p^2 = \frac{8}{6}p^2 = \frac{4}{3}p^2 \text{ sq. units}$$

Therefore, the required area is 4/3 p^2 sq. units.

3. Find the area of the region bounded by the curve $y = x^3$ and $y = x + 6$ and $x = 0$.
Solution:

Given curves are $y = x^3$, $y = x + 6$ and $x = 0$
On solving $y = x^3$ and $y = x + 6$, we have
$x^3 = x + 6$
$x^3 - x - 6 = 0$
$x^2(x - 2) + 2x(x - 2) + 3(x - 2) = 0$
$(x - 2)(x^2 + 2x + 3) = 0$
It's seen that $x^2 + 2x + 3 = 0$ has no real roots
So, x = 2 is the only root for the above equation.
Now, the required area of the shaded region is given by



$$= \int_0^2 (x+6)\,dx - \int_0^2 x^3 dx$$

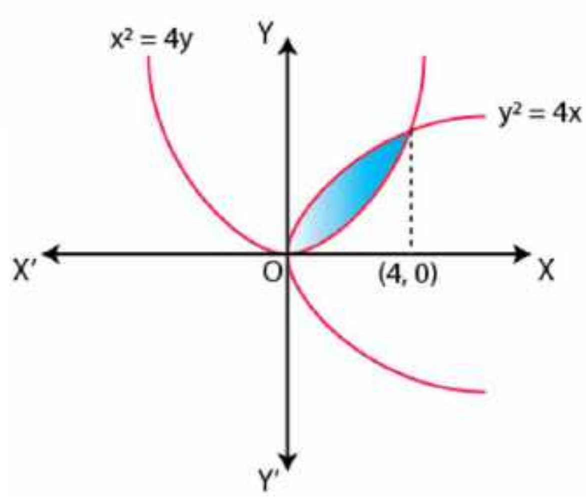
$$= \left[\frac{x^2}{2} + 6x\right]_0^2 - \frac{1}{4}\left[x^4\right]_0^2$$

$$= \left(\frac{4}{2} + 12\right) - (0+0) - \frac{1}{4}[(2)^4 - 0]$$

$$= 14 - \frac{1}{4} \times 16 = 14 - 4 = 10 \text{ sq. units.}$$

4. Find the area of the region bounded by the curve $y^2 = 4x$ and $x^2 = 4y$.
Solution:

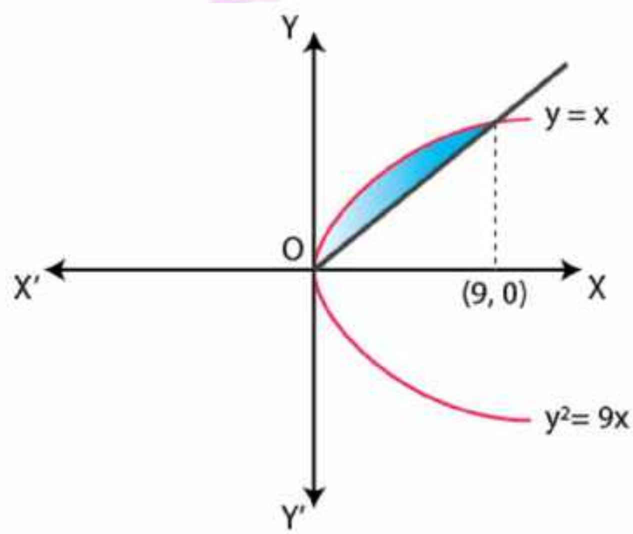
Given curves are $y^2 = 4x$ ... (i) and $x^2 = 4y$ ... (ii)
On solving the equations, we get
From (ii),
$y = x^2/4$
Putting value of y in (i), we have
$(x^2/4)^2 = 4x$
$x^4/16 = 4x$
$x^4 = 64x$
$x^4 - 64x = 0$
So, $x = 0, 4$
Now, the required area is the shaded region



$$\text{Required area} = \int_0^4 \sqrt{4x}\,dx - \int_0^4 \frac{x^2}{4}\,dx = 2\int_0^4 \sqrt{x}\,dx - \frac{1}{4}\int_0^4 x^2\,dx$$
$$= 2.\frac{2}{3}[x^{3/2}]_0^4 - \frac{1}{4}.\frac{1}{3}[x^3]_0^4$$
$$= \frac{4}{3}[(4)^{3/2} - 0] - \frac{1}{12}[(4)^3 - 0] = \frac{4}{3}[8] - \frac{1}{12}[64]$$
$$= \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq. units}$$

5. Find the area of the region included between $y^2 = 9x$ and $y = x$
Solution:

Given curves are $y^2 = 9x$ and $y = x$
Solving the above equations, we have
$x^2 = 9x \Rightarrow x^2 - 9x = 0$
$x(x - 9) = 0$
So, $x = 0, 9$
Now, the required area is



$$= \int_0^9 \sqrt{9x}\,dx - \int_0^9 x\,dx = 3\int_0^9 \sqrt{x}\,dx - \int_0^9 x\,dx$$
$$= 3.\frac{2}{3}[x^{3/2}]_0^9 - \frac{1}{2}[x^2]_0^9$$
$$= 2[(9)^{3/2} - 0] - \frac{1}{2}[(9)^2 - 0]$$
$$= 2(27) - \frac{1}{2}(81) = 54 - \frac{81}{2} = \frac{108 - 81}{2}$$
$$= \frac{27}{2} \text{ sq. units}$$

Therefore, the required area is 27/2 sq. units.

6. Find the area of the region enclosed by the parabola $x^2 = y$ and the line $y = x + 2$
Solution:

Given, equation of parabola $x^2 = y$ and line $y = x + 2$
Solving the above equations, we get

$x^2 = x + 2$
$x^2 - x - 2 = 0$
$x^2 - 2x + x - 2 = 0$
$x(x - 2) + 1(x - 2) = 0$
$(x + 1)(x - 2) = 0$

So, $x = -1, 2$
Now,

Graph of $y = x + 2$

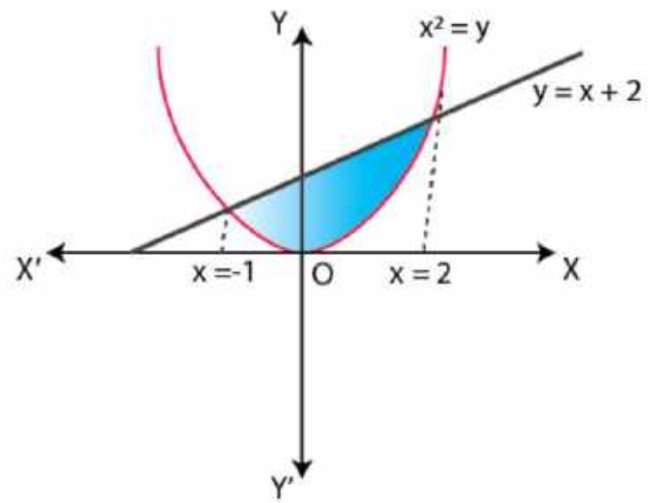
x	0	–2
y	2	0

Area of the required region

$$= \int_{-1}^{2} (x+2)\,dx - \int_{-1}^{2} x^2\,dx = \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} - \frac{1}{3}[x^3]_{-1}^{2}$$

$$= \left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] - \frac{1}{3}[8 - (-1)]$$

$$= \left(6 + \frac{3}{2}\right) - \frac{1}{3}(9) = \frac{15}{2} - 3 = \frac{9}{2} \text{ sq. units}$$



Therefore, the area of the required region is 9/2 sq. units.

7. Find the area of region bounded by the line $x = 2$ and the parabola $y^2 = 8x$
Solution:

Given, equation of line x = 2 and parabola $y^2 = 8x$
Putting value of x in the other equation, we have

$y^2 = 8(2)$

$y^2 = 16$

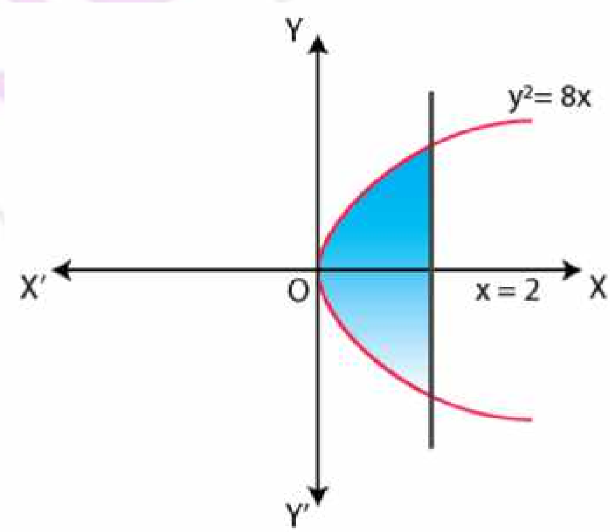
So, $y = \pm 4$

Now, the required area is

$$= 2\int_0^2 \sqrt{8x}\,dx = 2 \times 2\sqrt{2} \int_0^2 \sqrt{x}\,dx$$

$$= 4\sqrt{2} \times \frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= \frac{8\sqrt{2}}{3}\left[(2)^{3/2}\right] = \frac{8\sqrt{2}}{3} \times 2\sqrt{2} = \frac{32}{3} \text{ sq. units}$$



Therefore, the area of the region = 32/3 sq. units

8. Sketch the region $\{(x, 0) : y = \sqrt{(4 - x^2)}\}$ and x-axis. Find the area of the region using integration.
Solution:

Given, $\{(x, 0) : y = \sqrt{(4 - x^2)}\}$
So, $y^2 = 4 - x^2$
$x^2 + y^2 = 4$ which is a circle.
Now, the required area

$$= 2 . \int_0^2 \sqrt{4 - x^2}\, dx$$

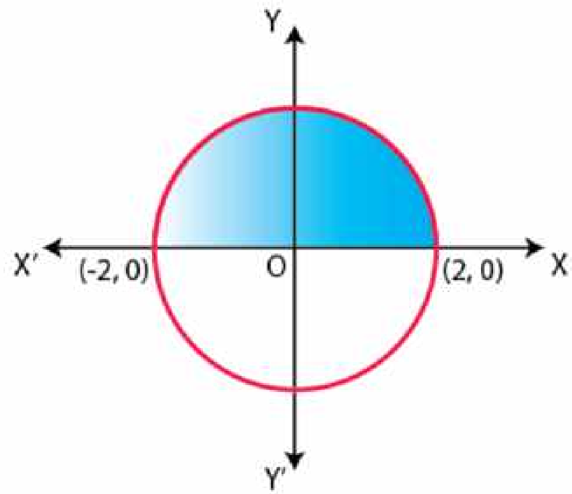
[Since circle is symmetrical about y-axis]

$$= 2 . \int_0^2 \sqrt{(2)^2 - x^2}\, dx$$

$$= 2 . \left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\left[\left(\frac{2}{2}\sqrt{4 - 4} + 2\sin^{-1}(1)\right) - (0 + 0)\right]$$

$$= 2\left[2 . \frac{\pi}{2}\right] = 2\pi \text{ sq. units}$$



9. Calculate the area under the curve $y = 2\sqrt{x}$ included between the lines $x = 0$ and $x = 1$.
Solution:

Given the curves $y = 2\sqrt{x}$, $x = 0$ and $x = 1$.

$$y = 2\sqrt{x} \Rightarrow y^2 = 4x \text{ (Parabola)}$$

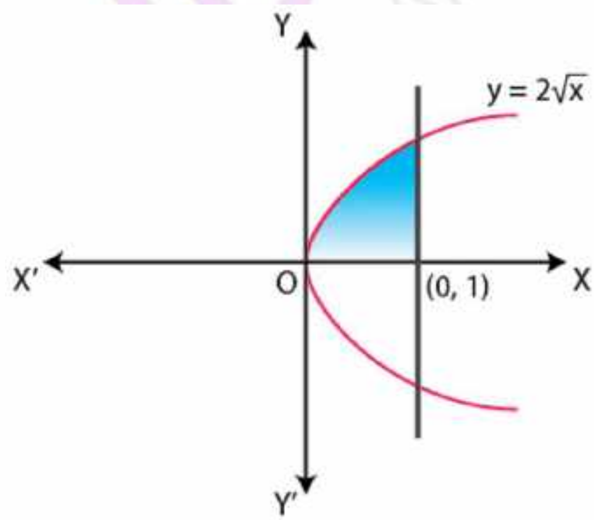
$$\text{Required area} = \int_0^1 (2\sqrt{x})\, dx$$

$$= 2 \times \frac{2}{3}[x^{3/2}]_0^1$$

$$= \frac{4}{3}[(1)^{3/2} - 0]$$

$$= \frac{4}{3} \text{ sq. units}$$

Thus, required area $= \frac{4}{3}$ sq. units.



10. Using integration, find the area of the region bounded by the line $2y = 5x + 7$, x-axis and the lines $x = 2$ and $x = 8$.
Solution:

Given, $2y = 5x + 7$, x-axis, $x = 2$ and $x = 8$
Let's draw the graph of $2y = 5x + 7 \Rightarrow y = (5x + 7)/2$

x	1	-1
y	6	1

Now, let's plot the straight line on a graph with other lines.

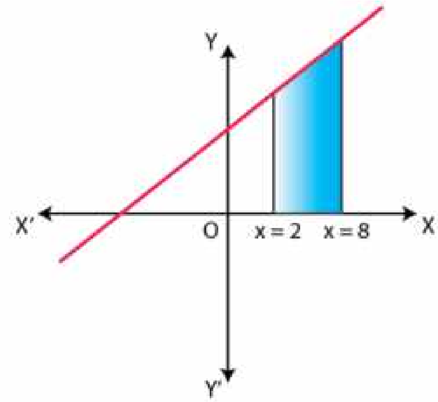
The area of the required region is

$$= \int_2^8 \left(\frac{5x+7}{2}\right) dx = \frac{1}{2}\left[\frac{5}{2}x^2 + 7x\right]_2^8$$

$$= \frac{1}{2}\left[\frac{5}{2}(64-4) + 7(8-2)\right]$$

$$= \frac{1}{2}\left[\frac{5}{2} \times 60 + 7 \times 6\right] = \frac{1}{2}[150 + 42]$$

$$= \frac{1}{2} \times 192 = 96 \text{ sq. units}$$



Therefore, the required area the region = 96 sq. units

11. Draw a rough sketch of the curve $y = \sqrt{(x-1)}$ in the interval [1, 5]. Find the area under the curve and between the lines $x = 1$ and $x = 5$.
Solution:

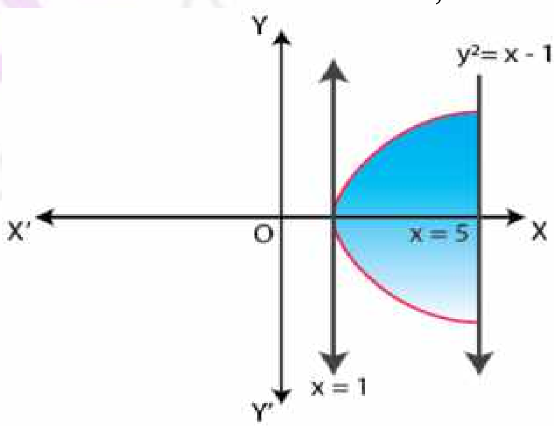
Given curve $y = \sqrt{(x-1)}$
$\Rightarrow y^2 = x - 1$
Plotting the curve and finding the area of the shaded region between the lines $x = 1$ and $x = 5$, we have
Area of the required region

$$= \int_1^5 \sqrt{x-1}\, dx$$

$$= \frac{2}{3}\left[(x-1)^{3/2}\right]_1^5$$

$$= \frac{2}{3}\left[(5-1)^{3/2} - 0\right] = \frac{2}{3} \times (4)^{3/2}$$

$$= \frac{2}{3} \times 8 = \frac{16}{3} \text{ sq. units}$$



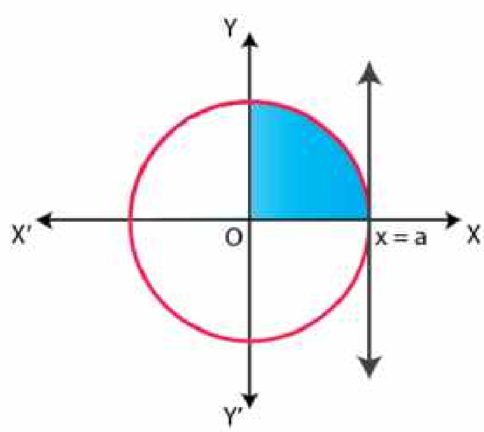
Therefore, the area of the required region = 16/3 sq.units

12. Determine the area under the curve $y = \sqrt{(a^2 - x^2)}$ included between the lines $x = 0$ and $x = a$.
Solution:

Given curve $y = \sqrt{(a^2 - x^2)}$ and lines $x = 0$ and $x = a$
$y = \sqrt{(a^2 - x^2)} \Rightarrow y^2 = a^2 - x^2$
$x^2 + y^2 = a^2$ which is equation of a circle.
Now, the required region is found by plotting the curve and lines.
So, the area of the shaded region is

$$= 2\left[(1)^{3/2} - 0\right] - \frac{3}{2}[(1)^2 - 0]$$

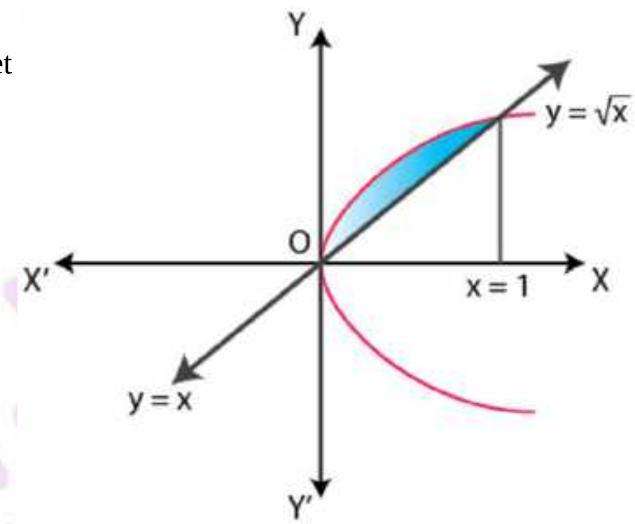
$$= \left[\frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$



$$= \left[\frac{a}{2}\sqrt{a^2-a^2}+\frac{a^2}{2}\sin^{-1}\frac{a}{a}-0-0\right]$$
$$= \frac{a^2}{2}\sin^{-1}(1)=\frac{a^2}{2}\cdot\frac{\pi}{2}=\frac{\pi a^2}{4}$$

Therefore, the required area = $\pi a^2/4$

13. Find the area of the region bounded by $y = \sqrt{x}$ and $y = x$.
Solution:



Given equations of curve $y = \sqrt{x}$ and line $y = x$
Solving the equations $y = \sqrt{x} \Rightarrow y^2 = x$ and $y = x$, we get

$x^2 = x$
$x^2 - x = 0$
$x(x - 1) = 0$

So, $x = 0, 1$

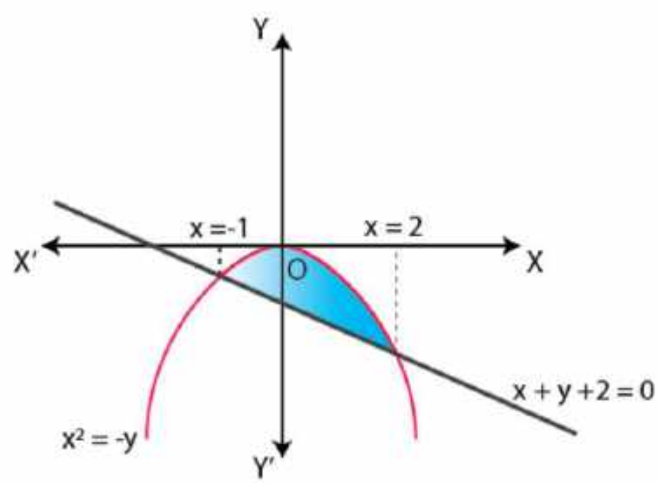
Now, the required area of the shaded region

$$= \int_0^1 \sqrt{x}\,dx - \int_0^1 x\,dx$$
$$= \frac{2}{3}\left[x^{3/2}\right]_0^1 - \frac{1}{2}\left[x^2\right]_0^1$$
$$= \frac{2}{3}[(1)^{3/2}-0] - \frac{1}{2}[(1)^2-0]$$
$$= \frac{2}{3}-\frac{1}{2} \Rightarrow \frac{4-3}{6} \Rightarrow \frac{1}{6}\text{ sq. units}$$

Therefore, the required area = 1/6 sq.units.

14. Find the area enclosed by the curve $y = -x^2$ and the straight-line $x + y + 2 = 0$.
Solution:

Given curve $y = -x^2$ or $x^2 = -y$ and the line $x + y + 2 = 0$
Solving the two equation, we get



$x - x^2 + 2 = 0$
$x^2 - x + 2 = 0$
$x^2 - 2x + x - 2 = 0$
$x(x - 2) + 1(x - 2) = 0$
$(x - 2)(x + 1) = 0$

So, $x = -1, 2$

Now,

The area of the required shaded region

$$= \left|\int_{-1}^{2}(-x-2)\,dx - \int_{-1}^{2} -x^2\,dx\right|$$

$$= \left| -\left[\frac{x^2}{2} + 2x\right]_{-1}^{2} + \frac{1}{3}[x^3]_{-1}^{2} \right|$$

$$= \left| -\left[\left(\frac{4}{2} + 4\right) - \left(\frac{1}{2} - 2\right)\right] + \frac{1}{3}(8+1) \right|$$

$$= \left| -\left(6 + \frac{3}{2}\right) + \frac{1}{3}(9) \right| \Rightarrow \left| -\frac{15}{2} + 3 \right|$$

$$= \left| \frac{-15+6}{2} \right| = \left| \frac{-9}{2} \right| = \frac{9}{2} \text{ sq. units}$$

Therefore, the required area = 9/2 sq.units

15. Find the area bounded by the curve $y = \sqrt{x}$, $x = 2y + 3$ in the first quadrant and x-axis.
Solution:

Given curve $y = \sqrt{x}$ and line $x = 2y + 3$, first quadrant and x-axis.
Solving $y = \sqrt{x}$ and $x = 2y + 3$, we get

$$y = \sqrt{2y+3} \Rightarrow y^2 = 2y + 3$$

$$y^2 - 2y - 3 = 0 \Rightarrow y^2 - 3y + y - 3 = 0$$

$$y(y-3) + 1(y-3) = 0$$

$$(y+1)(y-3) = 0$$

$$\therefore y = -1, 3$$

The area of shaded region

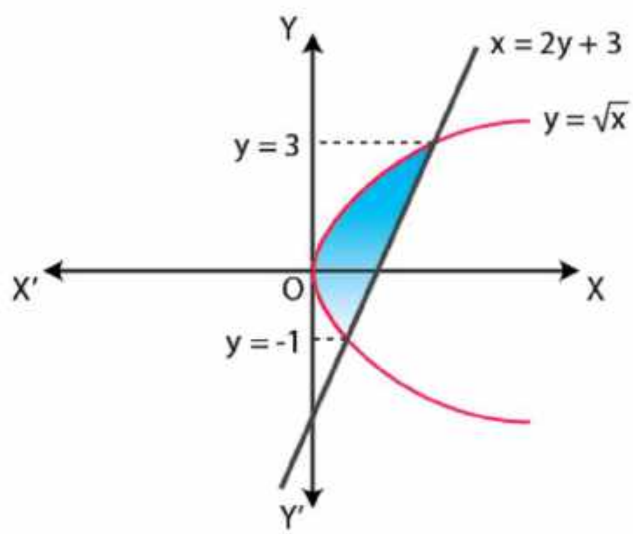
$$= \int_0^3 (2y+3)\,dy - \int_0^3 y^2 dy$$

$$= \left[2\frac{y^2}{2} + 3y\right]_0^3 - \frac{1}{3}\left[y^3\right]_0^3$$

$$= [(9+9) - (0+0)] - \frac{1}{3}[27 - 0]$$

$$= 18 - 9 = 9 \text{ sq. units}$$

Thus, the required area = 9 sq. units.



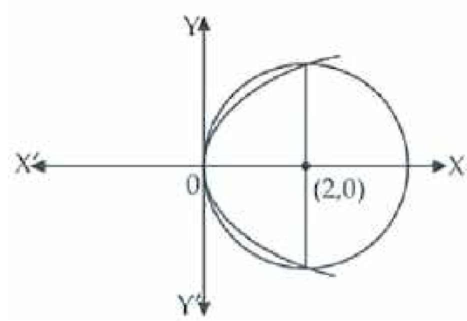
Long Answer (L.A.)
16. Find the area of the region bounded by the curve $y^2 = 2x$ and $x^2 + y^2 = 4x$.
Solution:

Given equation of curves are $y^2 = 2x$ and $x^2 + y^2 = 4x$
Solving the equations, we have

$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 - 4 + y^2 = 0$$
$$(x - 2)^2 + y^2 = 4$$

It's clearly seen that the equation of the circle having its centre (2, 0) and radius 2.
Solving $x^2 + y^2 = 4x$ and $y^2 = 2x$

$x^2 + 2x = 4x$
$x^2 + 2x - 4x = 0$
$x^2 - 2x = 0$
$x(x - 2) = 0$
So, $x = 0, 2$
Now, the area of the required region is given as



Area of the required region

$$= 2\left[\int_0^2 \sqrt{4-(x-2)^2}\,dx - \int_0^2 \sqrt{2x}\,dx\right]$$

[$\because$ Parabola and circle both are symmetrical about x-axis.]

$$= 2\left[\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\frac{x-2}{2}\right]_0^2 - 2.\sqrt{2}.\frac{2}{3}\left[x^{3/2}\right]_0^2$$

$$= 2\left[(0+0)-(0+2\sin^{-1}(-1)\right] - \frac{4\sqrt{2}}{3}\left[2^{3/2}-0\right]$$

$$= -2\times 2.\left(-\frac{\pi}{2}\right) - \frac{4\sqrt{2}}{3}.2\sqrt{2}$$

$$= 2\pi - \frac{16}{3} = 2\left(\pi - \frac{8}{3}\right)\text{sq.units}$$

Thus, the required area = $2\left(\pi - \frac{8}{3}\right)$ sq. units.

17. Find the area bounded by the curve $y = \sin x$ between $x = 0$ and $x = 2\pi$.
Solution:

$$\text{Required area} = \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} |\sin x|\,dx$$

$$= -\left[\cos x\right]_0^{\pi} + \left|(-\cos x)\right|_{\pi}^{2\pi} = -\left[\cos\pi - \cos 0\right] + \left[\cos 2\pi - \cos\pi\right]$$

$$= -[-1-1] + [1+1] = 2 + 2 = 4\text{ sq. units}$$