EXERCISE 29.11

Evaluate the following limits:

1. $\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi}$

Solution:

Given:

The limit $\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi}$

Let us substitute the value of x = π directly, we get

$Z = \lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi} = \left(1-\frac{\pi}{\pi}\right)^{\pi} = (1-1)^{\pi} = 0^{\pi} = 0$

Since, it is not of indeterminate form.

Z = 0

∴ The value of $\lim_{x\to\pi}\left(1-\frac{x}{\pi}\right)^{\pi} = 0$

2. $\lim_{x\to 0^+}\left\{1+\tan^{\sqrt{x}}\right\}^{1/2x}$

Solution:

Given:

The limit $\lim_{x\to 0^+}\left\{1+\tan^{\sqrt{x}}\right\}^{1/2x}$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a}[1+f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$.

So here,

$f(x) = \tan^2\sqrt{x}$

$g(x) = 2x$

Then,

$$\lim_{x\to 0^+}\left\{1+\tan^{\sqrt{x}}\right\}^{1/2x} = e^{\lim_{x\to 0^+}}\left(\frac{\tan^2\sqrt{x}}{2x}\right)$$

$$= e^{\lim_{x\to 0^+}}\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)\times\left(\frac{\tan\sqrt{x}}{\sqrt{x}}\right)\times\frac{1}{2}$$

$$= e^{1\times 1\times\frac{1}{2}}$$

$$= \sqrt{e}$$

∴ The value of $\lim_{x\to 0^+}\left\{1+\tan^{\sqrt{x}}\right\}^{1/2x} = \sqrt{e}$

3. $\lim_{x\to0} (\cos x)^{1/\sin x}$

Solution:

Given:

The limit $\lim_{x\to0} (\cos x)^{1/\sin x}$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} [1+f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}} \frac{f(x)}{g(x)}$.

So here,

f(x) = cos x - 1

g (x) = sin x

Then,

$$\lim_{x\to0} (\cos x)^{1/\sin x} = e^{\lim_{x\to0}} \left(\frac{\cos x - 1}{\sin x}\right)$$

$$= e^{\lim_{x\to0}} \left(\frac{-2\sin^2 \frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)$$

$$= e^{\lim_{x\to0}} \left(-\tan\frac{x}{2}\right)$$

$$= e^0$$

$$= 1$$

$\therefore$ The value of $\lim_{x\to0} (\cos x)^{1/\sin x} = 1$

4. $\lim_{x\to0} (\cos x + \sin x)^{1/x}$

Solution:

Given:

The limit $\lim_{x\to0} (\cos x + \sin x)^{1/x}$

Let us add and subtract '1' to the given expression, we get

$$\lim_{x\to0} [1 + \cos x + \sin x - 1]^{\frac{1}{x}}$$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} [1+f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}} \frac{f(x)}{g(x)}$.

So here,

f (x) = cos x + sin x - 1

g (x) = x

Then,

$$\lim_{x\to 0} (\cos x + \sin x)^{1/x} = e^{\lim_{x\to 0}\left(\frac{\cos x + \sin x - 1}{x}\right)}$$

Upon computing, we get

$$= e^{\lim_{x\to 0}\left[\frac{\sin x}{x} - \frac{(1-\cos x)}{x}\right]}$$

$$= e^{\lim_{x\to 0}\left(\frac{\sin x}{x} - \frac{2\sin^2\frac{x}{2}}{x}\right)}$$

$$= e^{\lim_{x\to 0}\left(\frac{\sin x}{x} - \frac{2\sin\left(\frac{x}{2}\right)\times\sin\left(\frac{x}{2}\right)}{2\times\frac{x}{2}}\right)}$$

Now, substitute the value of x, we get

$= e^{1-0}$

$= e^1$

$= e$

∴ The value of $\lim_{x\to 0} (\cos x + \sin x)^{1/x} = e$

5. $\lim_{x\to 0} (\cos x + a\sin bx)^{1/x}$

Solution:

Given:

The limit $\lim_{x\to 0} (\cos x + a\sin bx)^{1/x}$

Let us add and subtract '1' to the given expression, we get

$$\lim_{x\to 0} [1 + \cos x + a\sin bx - 1]^{\frac{1}{x}}$$

Let us use the theorem given below

If $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ such that $\lim_{x\to a} \frac{f(x)}{g(x)}$ exists, then $\lim_{x\to a} [1 + f(x)]^{\frac{1}{g(x)}} = e^{\lim_{x\to a}\frac{f(x)}{g(x)}}$.

So here,

f (x) = cos x + a sin bx - 1

g (x) = x

Then,

$$\lim_{x\to 0} (\cos x + a\sin bx)^{1/x} = e^{\lim_{x\to 0}\left[\frac{\cos x + a\sin bx - 1}{x}\right]}$$

Let us compute now, we get

$$= e^{\lim_{x \to 0}} \left[\frac{b \times a \sin bx}{bx} - \frac{(1 - \cos x)}{x} \right]$$

$$= e^{\lim_{x \to 0}} \left(\frac{ab \sin bx}{bx} - \frac{2 \sin^2 \frac{x}{2}}{x} \right)$$

Now, substitute the value of x, we get

$$= e^{ab}$$

∴ The value of $\lim_{x \to 0} (\cos x + a \sin bx)^{1/x} = e^{ab}$