EXERCISE 30.1 PAGE NO: 30.3

1. Find the derivative of f(x) = 3x at x = 2

Solution:

Given:

f(x) = 3x

By using the derivative formula,

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$ {Where, h is a small positive number}

Derivative of f(x) = 3x at x = 2 is given as

$$f'(2) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{3(2+h)-3\times 2}{h}$$

$$= \lim_{h\to 0}\frac{3h+6-6}{h} = \lim_{h\to 0}\frac{3h}{h}$$

$$= \lim_{h\to 0} 3 = 3$$

Hence,

Derivative of f(x) = 3x at x = 2 is 3

2. Find the derivative of f(x) = x^2 – 2 at x = 10

Solution:

Given:

f(x) = $x^2 - 2$

By using the derivative formula,

Derivative of $x^2 - 2$ at x = 10 is given as

$$f'(10) = \lim_{h\to 0}\frac{f(10+h)-f(10)}{h}$$

$$= \lim_{h\to 0}\frac{(10+h)^2-2-(10^2-2)}{h}$$

$$= \lim_{h\to 0}\frac{100+h^2+20h-2-100+2}{h} = \lim_{h\to 0}\frac{h^2+20h}{h}$$

$$= \lim_{h\to 0}\frac{h(h+20)}{h} = \lim_{h\to 0}(h+20)$$

= 0 + 20 = 20

Hence,

Derivative of $f(x) = x^2 - 2$ at $x = 10$ is 20

3. Find the derivative of f(x) = 99x at x = 100.

Solution:

Given:

$f(x) = 99x$

By using the derivative formula,

$$f'(a) = \lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$ {Where h is a very small positive number}

Derivative of 99x at x = 100 is given as

$$f'(100) = \lim_{h\to0}\frac{f(100+h)-f(100)}{h}$$
$$= \lim_{h\to0}\frac{99(100+h)-99\times100}{h}$$
$$= \lim_{h\to0}\frac{9900+99h-9900}{h} = \lim_{h\to0}\frac{99h}{h}$$
$$= \lim_{h\to0}99 = 99$$

Hence,

Derivative of $f(x) = 99x$ at $x = 100$ is 99

4. Find the derivative of f(x) = x at x = 1

Solution:

Given:

$f(x) = x$

By using the derivative formula,

$$f'(a) = \lim_{h\to0}\frac{f(a+h)-f(a)}{h}$$ {Where h is a very small positive number}

Derivative of x at x = 1 is given as

$$f'(1) = \lim_{h\to0}\frac{f(1+h)-f(1)}{h}$$
$$= \lim_{h\to0}\frac{(1+h)-1}{h}$$
$$= \lim_{h\to0}\frac{1+h-1}{h} = \lim_{h\to0}\frac{h}{h}$$
$$= \lim_{h\to0}1 = 1$$

Hence,

Derivative of $f(x) = x$ at $x = 1$ is 1

5. Find the derivative of f(x) = cos x at x = 0

Solution:

Given:

f(x) = cos x

By using the derivative formula,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$ {Where h is a very small positive number}

Derivative of cos x at x = 0 is given as

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0} \frac{\cos(h)-\cos 0}{h}$$

$$= \lim_{h\to 0} \frac{\cos h - 1}{h}$$

Let us try and evaluate the limit.

We know that 1 – cos x = 2 $\sin^2(x/2)$

So,

$$= \lim_{h\to 0} \frac{-(1-\cos h)}{h} = -\lim_{h\to 0} \frac{2\sin^2\frac{h}{2}}{h}$$

Divide the numerator and denominator by 2 to get the form (sin x)/x to apply sandwich theorem.

$$= -\lim_{h\to 0} \frac{\frac{2\sin^2\frac{h}{2}}{2}}{\frac{h^2}{2}} \times h$$

By using algebra of limits we get

$$= -\lim_{h\to 0}\left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) \times \lim_{h\to 0} h$$

[By using the formula: $\lim_{x\to 0}\frac{\sin x}{x} = 1$]

f' (0) = – 1×0 = 0

∴ Derivative of f(x) = cos x at x = 0 is 0

6. Find the derivative of f(x) = tan x at x = 0

Solution:

Given:

f(x) = tan x

By using the derivative formula,

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ \{Where h is a small positive number\}}$$

Derivative of cos x at x = 0 is given as

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to 0}\frac{\tan(h)-\tan 0}{h}$$

$$=\lim_{h\to 0}\frac{\tan h}{h} \text{ [Since it is of indeterminate form]}$$

By using the formula: $\lim_{x\to 0}\frac{\tan x}{x}=1$ {i.e., sandwich theorem}

$f'(0)=1$

∴ Derivative of f(x) = tan x at x = 0 is 1

7. Find the derivatives of the following functions at the indicated points:

(i) sin x at x = π/2

(ii) x at x = 1

(iii) 2 cos x at x = π/2

(iv) sin 2xat x = π/2

Solution:

(i) sin x at x = π/2

Given:

f (x) = sin x

By using the derivative formula,

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \text{ \{Where h is a small positive number\}}$$

Derivative of sin x at x = π/2 is given as

$$f'\left(\frac{\pi}{2}\right)=\lim_{h\to 0}\frac{f(\frac{\pi}{2}+h)-f(\frac{\pi}{2})}{h}$$

$$=\lim_{h\to 0}\frac{\sin\left(\frac{\pi}{2}+h\right)-\sin\frac{\pi}{2}}{h}$$

$$=\lim_{h\to 0}\frac{\cos h-1}{h} \quad \{\because \sin(\pi/2+x)=\cos x\}$$

[Since it is of indeterminate form. Let us try to evaluate the limit.]

We know that 1 – cos x = 2 $\sin^2(x/2)$

$$= \lim_{h \to 0} \frac{-(1-\cos h)}{h} = -\lim_{h \to 0} \frac{2\sin^2\frac{h}{2}}{h}$$

Divide the numerator and denominator by 2 to get the form (sin x)/x to apply sandwich theorem.

$$= -\lim_{h \to 0} \frac{\frac{2\sin^2\frac{h}{2}}{2}}{\frac{h^2}{2}} \times h$$

Using algebra of limits we get

$$= -\lim_{h \to 0} \left(\frac{\sin\frac{h}{2}}{\frac{h}{2}}\right) \times \lim_{h \to 0} h$$

[By using the formula: $\lim_{x \to 0} \frac{\sin x}{x} = 1$]

f' (π/2) = – 1×0 = 0

∴ Derivative of f(x) = sin x at x = π/2 is 0

(ii) x at x = 1

Given:

f (x) = x

By using the derivative formula,

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ {Where h is a very small positive number}

Derivative of x at x = 1 is given as

$$f'(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{(1+h) - 1}{h}$$

$$= \lim_{h \to 0} \frac{1+h-1}{h} = \lim_{h \to 0} \frac{h}{h}$$

$$= \lim_{h \to 0} 1 = 1$$

Hence,

Derivative of f(x) = x at x = 1 is 1

(iii) 2 cos x at x = π/2

Given:

f (x) = 2 cos x

By using the derivative formula,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ {Where h is a small positive number}

Derivative of 2cos x at x = π/2 is given as

$$f'\left(\frac{\pi}{2}\right) = \lim_{h \to 0} \frac{f(\frac{\pi}{2}+h) - f(\frac{\pi}{2})}{h}$$

$$= \lim_{h \to 0} \frac{2\cos\left(\frac{\pi}{2} + h\right) - 2\cos\frac{\pi}{2}}{h}$$

$$= \lim_{h \to 0} \frac{-2\sin h}{h}$$ {∵ cos (π/2 + x) = – sin x}

[Since it is of indeterminate form]

$$= -2\lim_{h \to 0} \frac{\sin h}{h}$$

By using the formula: $\lim_{x \to 0} \frac{\sin x}{x} = 1$

f' (π/2) = – 2×1 = – 2

∴ Derivative of f(x) = 2cos x at x = π/2 is – 2

(iv) sin 2xat x = π/2

Solution:

Given:

f (x) = sin 2x

By using the derivative formula,

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$ {Where h is a small positive number}

Derivative of sin 2x at x = π/2 is given as

$$f'\left(\frac{\pi}{2}\right) = \lim_{h \to 0} \frac{f(\frac{\pi}{2}+h) - f(\frac{\pi}{2})}{h}$$

$$= \lim_{h \to 0} \frac{\sin\left\{2 \times \left(\frac{\pi}{2} + h\right)\right\} - \sin 2 \times \frac{\pi}{2}}{h}$$

$$= \lim_{h \to 0} \frac{\sin(\pi + 2h) - \sin\pi}{h}$$ {∵ sin (π + x) = – sin x & sin π = 0}

$$= \lim_{h \to 0} \frac{-\sin 2h - 0}{h}$$

$$= -\lim_{h \to 0} \frac{\sin 2h}{h}$$

[Since it is of indeterminate form. We shall apply sandwich theorem to evaluate the limit.]

Now, multiply numerator and denominator by 2, we get

$$= -\lim_{h\to 0} \frac{\sin 2h}{2h} \times 2 \; = \; -2 \lim_{h\to 0} \frac{\sin 2h}{2h}$$

By using the formula: $\lim_{x\to 0} \frac{\sin x}{x} \; = \; 1$

f' (π/2) = – 2×1 = – 2

∴ Derivative of f(x) = sin 2x at x = π/2 is – 2

EXERCISE 30.2 PAGE NO: 30.25

1. Differentiate each of the following from first principles:
(i) 2/x
(ii) 1/√x
(iii) $1/x^3$
(iv) $[x^2 + 1]/ x$
(v) $[x^2 - 1] / x$
Solution:
(i) 2/x
Given:
f (x) = 2/x
By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h}$$

$$= \lim_{h\to 0} \frac{2x - 2x - 2h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-2h}{hx(x+h)}$$

$$= \lim_{h\to 0} \frac{-2}{x(x+h)}$$

When h= 0, we get

$$= \frac{-2}{x^2}$$

$= -2x^{-2}$

∴ Derivative of f(x) = 2/x is $-2x^{-2}$

(ii) 1/√x
Given:
f (x) = 1/√x
By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

By using algebra of limits, we get

$$= \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}} \times \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}$$

$$= \lim_{h \to 0} \frac{x - x - h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \lim_{h \to 0} \frac{-h}{h\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

$$= \lim_{h \to 0} \frac{-1}{\sqrt{x}\sqrt{x+h}\left(\sqrt{x} + \sqrt{x+h}\right)}$$

When h = 0, we get

$$= \frac{-1}{\sqrt{x}\sqrt{x}\left(\sqrt{x} + \sqrt{x}\right)}$$

$$= \frac{-1}{x \times 2\sqrt{x}}$$

$$= \frac{-1}{2x^{\frac{3}{2}}}$$

$$= -\frac{1}{2}x^{\frac{-3}{2}}$$

∴ Derivative of f(x) = 1/√x is -1/2 $x^{-3/2}$

(iii) 1/x^3

Given:

f (x) = 1/x^3

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0} \frac{\frac{1}{(x+h)^3} - \frac{1}{x^3}}{h}$$

$$= \lim_{h\to 0} \frac{x^3 - (x+h)^3}{h(x+h)^3x^3}$$

By using the formula [$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$]

$$= \lim_{h\to 0} \frac{x^3 - x^3 - 3x^2h - 3xh^2 - h^3}{h(x+h)^3x^3}$$

$$= \lim_{h\to 0} \frac{-3x^2h - 3xh^2 - h^3}{h(x+h)^3x^3}$$

$$= \lim_{h\to 0} \frac{h\left(-3x^2 - 3xh - h^2\right)}{h(x+h)^3x^3}$$

$$= \lim_{h\to 0} \frac{\left(-3x^2 - 3xh - h^2\right)}{(x+h)^3x^3}$$

When h = 0, we get

$$= \frac{-3x^2}{x^6}$$

$$= \frac{-3}{x^4}$$

$$= -3x^{-4}$$

∴ Derivative of $f(x) = 1/x^3$ is $-3x^{-4}$

(iv) $[x^2 + 1]/ x$

Given:

$f(x) = [x^2 + 1]/ x$

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h \to 0} \frac{\frac{(x+h)^2+1}{x+h} - \frac{x^2+1}{x}}{h}$$

Upon expansion,

$$= \lim_{h \to 0} \frac{\frac{x^2+2xh+h^2+1}{x+h} - \frac{x^2+1}{x}}{h}$$

By using algebra of limits, we get

$$= \lim_{h \to 0} \frac{x^3 + 2x^2h + h^2x + x - x^3 - x^2h - x - h}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{x^2h + h^2x - h}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{h(x^2 + hx - 1)}{xh(x+h)}$$

$$= \lim_{h \to 0} \frac{x^2 + hx - 1}{x(x+h)}$$

When h = 0, we get

$$= \frac{x^2 - 1}{x^2}$$

$= 1 – 1/x^2$

∴ Derivative of f(x) = $1 – 1/x^2$

(v) $[x^2 - 1] / x$

Given:

f (x) = $[x^2 - 1]/ x$

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h \to 0} \frac{\frac{(x+h)^2-1}{x+h} - \frac{x^2-1}{x}}{h}$$

Upon expansion,

$$= \lim_{h\to 0} \frac{\frac{x^2+2xh+h^2-1}{x+h} - \frac{x^2-1}{x}}{h}$$

By using algebra of limits, we get

$$= \lim_{h\to 0} \frac{x^3+2x^2h+h^2x-x-x^3-x^2h+x+h}{xh(x+h)}$$

$$= \lim_{h\to 0} \frac{x^2h+h^2x+h}{xh(x+h)}$$

$$= \lim_{h\to 0} \frac{h(x^2+hx+1)}{xh(x+h)}$$

$$= \lim_{h\to 0} \frac{x^2+hx+1}{x(x+h)}$$

When h = 0, we get

$$= \frac{x^2+1}{x^2}$$

$= 1 + 1/x^2$

∴ Derivative of f(x) = $1 + 1/x^2$

2. Differentiate each of the following from first principles:

(i) e^{-x}

(ii) e^{3x}

(iii) e^{ax+b}

Solution:

(i) e^{-x}

Given:

f (x) = e^{-x}

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

By substituting the values we get,

$$\frac{d}{dx}(e^x) = \lim_{h\to 0} \frac{e^{-(x+h)}-e^{-x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{-x}e^{-h} - e^{-x}}{h}$$

Taking e^{-x} common, we have

$$= \lim_{h \to 0} \frac{e^{-x}\left(e^{-h} - 1\right)}{h}$$

$$= \lim_{h \to 0} e^{-x} \times \lim_{h \to 0} \frac{e^{-h}-1}{-h} \times (-1)$$

We know that, $\lim_{x \to 0} \frac{e^{x}-1}{x} = \log_e e = 1$

$$= -e^{-x} \lim_{h \to 0} \frac{e^{-h} - 1}{-h}$$

So,

$$= -e^{-x}(1)$$

$$= -e^{-x}$$

∴ Derivative of f (x) = $-e^{-x}$

(ii) e^{3x}

Given:

f (x) = e^{3x}

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$\frac{d}{dx}\left(e^{3x}\right) = \lim_{h \to 0} \frac{e^{3(x+h)} - e^{3x}}{h}$$

$$= \lim_{h \to 0} \frac{e^{3x}e^{3h} - e^{3x}}{h}$$

Taking e^{-x} common, we have

$$= \lim_{h \to 0} \frac{e^{3x}\left(e^{3h} - 1\right)}{3h}$$

By using algebra of limits,

$$= \lim_{h \to 0} e^{3x} \times \lim_{h \to 0} \frac{e^{3h}-1}{h}$$

Since we cannot substitute the value of h directly, we take

$$= \lim_{h \to 0} e^{3x} \times \lim_{h \to 0} \frac{e^{3h}-1}{3h} \times 3$$

We know that, $\lim_{x \to 0} \frac{e^{x}-1}{x} = \log_e e = 1$

$$= 3e^{3x} \lim_{h \to 0} \frac{e^{3h}-1}{3h}$$
$$= 3e^{3x}(1)$$
$$= 3e^{3x}$$

∴ Derivative of f (x) = $3e^{3x}$

(iii) e^{ax+b}

Given:

f (x) = e^{ax+b}

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$$

By substituting the values we get,

$$\frac{d}{dx}\left(e^{ax+b}\right) = \lim_{h \to 0} \frac{e^{a(x+h)+b}-e^{ax+b}}{h}$$
$$= \lim_{h \to 0} \frac{e^{ax+b}e^{ah}-e^{ax+b}}{h}$$

Taking e^{ax+b} common, we have

$$= \lim_{h \to 0} \frac{e^{ax+b}\left(e^{ah}-1\right)}{h}$$

By using algebra of limits,

$$= \lim_{h \to 0} e^{ax+b} \times \lim_{h \to 0} \frac{e^{ah}-1}{h}$$

Since we cannot substitute the value of h directly, we take

$$= \lim_{h \to 0} e^{ax+b} \times \lim_{h \to 0} \frac{e^{ah}-1}{ah} \times a$$

We know that, $\lim_{x \to 0} \frac{e^{x}-1}{x} = \log_e e = 1$

$$= ae^{ax+b} \lim_{h \to 0} \frac{e^{ah} - 1}{ah}$$

$$= ae^{ax+b} (1)$$

$$= ae^{ax+b}$$

∴ Derivative of f (x) = ae^{ax+b}

3. Differentiate each of the following from first principles:

(i) √(sin 2x)

(ii) sin x/x

Solution:

(i) √(sin 2x)

Given:

f (x) = √(sin 2x)

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h \to 0} \frac{\sqrt{\sin(2x+2h)} - \sqrt{\sin 2x}}{h}$$

Multiply numerator and denominator by √(sin 2(x + h)) + √(sin 2x), we have

$$= \lim_{h \to 0} \frac{\sqrt{\sin(2x+2h)} - \sqrt{\sin 2x}}{h} \times \frac{\sqrt{\sin(2x+2h)} + \sqrt{\sin 2x}}{\sqrt{\sin(2x+2h)} + \sqrt{\sin 2x}}$$

By using $a^2 - b^2 = (a + b)(a - b)$, we get

$$= \lim_{h \to 0} \frac{\sin(2x+2h) - \sin 2x}{h\left(\sqrt{\sin(2x+2h)} + \sqrt{\sin 2x}\right)}$$

By using the formula,

$$sinC - sinD = 2cos\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h\to 0} \frac{2\cos\left(\frac{2x+2h+2x}{2}\right)\sin\left(\frac{2x+2h-2x}{2}\right)}{h\left(\sqrt{\sin(2x+2h)}+\sqrt{\sin 2x}\right)}$$

$$= \lim_{h\to 0} \frac{2\cos(2x+h)\sin h}{h\left(\sqrt{\sin(2x+2h)}+\sqrt{\sin 2x}\right)}$$

By applying limits to each term, we get

$$= \lim_{h\to 0} 2\cos(2x+h) \lim_{h\to 0} \frac{\sin h}{h} \lim_{h\to 0} \frac{1}{\left(\sqrt{\sin(2x+2h)}+\sqrt{\sin 2x}\right)}$$

$$= 2\cos 2x\,(1)\,\frac{1}{\sqrt{\sin 2x}+\sqrt{\sin 2x}}$$

$$= \frac{2\cos 2x}{2\sqrt{\sin 2x}}$$

$$= \frac{\cos 2x}{\sqrt{\sin 2x}}$$

∴ Derivative of f (x) = cos 2x / √(sin 2x)

(ii) sin x/x

Given:

f (x) = sin x/x

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0} \frac{\frac{\sin(x+h)}{x+h} - \frac{\sin x}{x}}{h}$$

$$= \lim_{h\to 0} \frac{x\sin(x+h)-(x+h)\sin x}{hx\,(x+h)}$$

By using algebra of limits,

$$= \lim_{h \to 0} \frac{x(\sin x \cos h + \cos x \sin h) - x \sin x - h \sin x}{hx(x+h)}$$
$$= \lim_{h \to 0} \frac{x \sin x \cos h + x \cos x \sin h - x \sin x - h \sin x}{hx(x+h)}$$
$$= \lim_{h \to 0} \frac{x \sin x \cos h - x \sin x + x \cos x \sin h - h \sin x}{hx(x+h)}$$

By applying limits to each term, we get

$$= x \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \frac{x \cos x}{x} \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{1}{x+h} - \frac{\sin x}{x} \lim_{h \to 0} \frac{1}{x+h}$$
$$= x \sin x \lim_{h \to 0} \frac{-2\sin^2 \frac{h}{2}}{h} + \frac{x \cos x}{x} \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{1}{x+h} - \frac{\sin x}{x} \lim_{h \to 0} \frac{1}{x+h}$$
$$= x \sin x \lim_{h \to 0} \frac{-2\sin^2 \frac{h}{2}}{\frac{h^2}{4}} \times \frac{h}{4} + \frac{x \cos x}{x} \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{1}{x+h} - \frac{\sin x}{x} \lim_{h \to 0} \frac{1}{x+h}$$
$$= -x \sin x \times \lim_{h \to 0} \frac{h}{2} + \frac{x \cos x}{x} \lim_{h \to 0} \frac{\sin h}{h} \lim_{h \to 0} \frac{1}{x+h} - \frac{\sin x}{x} \lim_{h \to 0} \frac{1}{x+h}$$

When h = 0, we get

$$= -x \sin x \left(\frac{1}{2}\right)(0) + \frac{cosx}{x} - \frac{sinx}{x^2}$$
$$= \frac{\cos x}{x} - \frac{\sin x}{x^2}$$

By taking LCM, we get

$$= \frac{x \cos x - \sin x}{x^2}$$

∴ Derivative of f (x) = [x cos x – sin x]/x^2

4. Differentiate the following from first principles:

(i) $\tan^2 x$

(ii) tan (2x + 1)

Solution:

(i) $\tan^2 x$

Given:

f (x) = $\tan^2 x$

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0}\frac{\tan^2(x+h)-\tan^2 x}{h}$$

By using (a + b) (a – b) = $a^2 - b^2$, we have

$$= \lim_{h\to 0}\frac{[\tan(x+h)+\tan x]\,[\tan(x+h)-\tan x]}{h}$$

Replacing tan with sin/cos,

$$= \lim_{h\to 0}\frac{\left[\frac{\sin(x+h)}{\cos(x+h)}+\frac{\sin x}{\cos x}\right]\left[\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}\right]}{h}$$

By taking LCM,

$$= \lim_{h\to 0}\frac{[\sin(x+h)\cos x+\cos(x+h)\sin x]\,[\sin(x+h)\cos x-\cos(x+h)\sin x]}{h\cos^2 x\cos^2(x+h)}$$

$$= \lim_{h\to 0}\frac{[\sin(2x+h)]\,[\sin h]}{h\cos^2 x\cos^2(x+h)}$$

By applying limits to each term, we get

$$= \frac{1}{\cos^2 x}\lim_{h\to 0}\sin(2x+h)\lim_{h\to 0}\frac{\sin h}{h}\lim_{h\to 0}\frac{1}{\cos^2(x+h)}$$

When h = 0, we get

$$= \frac{1}{\cos^2 x}\sin(2x)\,(1)\,\frac{1}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x}2\sin x\cos x\frac{1}{\cos^2 x}$$

$$= 2\times\frac{\sin x}{\cos x}\times\frac{1}{\cos^2 x}$$

$$= 2\tan x\sec^2 x$$

∴ Derivative of f (x) = 2 tan x $\sec^2$ x

(ii) tan (2x + 1)

Given:

f (x) = tan (2x + 1)
By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0}\frac{\tan(2x+2h+1)-\tan(2x+1)}{h}$$

Replacing tan with sin/cos,

$$= \lim_{h\to 0}\frac{\frac{sin(2x+2h+1)}{\cos(2x+2h+1)}-\frac{\sin(2x+1)}{\cos(2x+1)}}{h}$$

By taking LCM,

$$= \lim_{h\to 0}\frac{sin\,(2x+2h+1)\cos(2x+1)-\cos(2x+2h+1)\sin(2x+1)}{h\cos(2x+2h+1)\cos(2x+1)}$$

$$= \lim_{h\to 0}\frac{\sin(2x+2h+1-2x-1)}{h\cos(2x+2h+1)\cos(2x+1)}$$

By applying limits to each term, we get

$$= \frac{1}{\cos(2x+1)}\lim_{h\to 0}\frac{\sin(2h)}{2h}\times 2\lim_{h\to 0}\frac{1}{\cos(2x+2h+1)}$$

When h = 0, we get

$$= \frac{1}{\cos(2x+1)}\times 2\times\frac{1}{\cos(2x+1)}$$

$$= \frac{2}{\cos^2(2x+1)}$$

$$= 2\sec^2(2x+1)$$

∴ Derivative of f (x) = 2 $\sec^2$ (2x + 1)

5. Differentiate the following from first principles:
(i) sin √2x
(ii) cos √x
Solution:
(i) sin √2x
Given:
f (x) = sin √2x

f (x + h) = sin √2(x+h)

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0}\frac{\sin\sqrt{2x+2h}-\sin\sqrt{2x}}{h}$$

By using the formula,

$$sinC - sinD = 2sin\left(\frac{C-D}{2}\right)\cos\left(\frac{C+D}{2}\right)$$

$$= \lim_{h\to 0}\frac{2\sin\left(\sqrt{2x+2h}-\sqrt{2x}\right)\cos\left(\sqrt{2x+2h}-\sqrt{2x}\right)}{h}$$

By using algebra of limits,

$$= \lim_{h\to 0}\frac{2\times 2\sin\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)\cos\left(\frac{\sqrt{2x+2h}+\sqrt{2x}}{2}\right)}{2h+2x-2x}$$

To use the sandwich theorem to evaluate the limit, we need $\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}$ in denominator.

$$= \lim_{h\to 0}\frac{2\times 2\sin\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)\cos\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}{\left(\sqrt{2x+2h}-\sqrt{2x}\right)\sqrt{2x+2h}+\sqrt{2x}}$$

$$= \lim_{h\to 0}\frac{2\times 2\sin\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)\cos\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}{2\times\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)\left(\sqrt{2x+2h}+\sqrt{2x}\right)}$$

By applying limits to each term, we get

$$= \lim_{h\to 0}\frac{\sin\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}{\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}\lim_{h\to 0}\frac{2\cos\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}{\sqrt{2x+2h}+\sqrt{2x}}$$

When h = 0, we get

$$= 1\times\frac{2\cos\sqrt{2x}}{2\sqrt{2x}}\left[\because \lim_{h\to 0}\frac{\sin\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}{\left(\frac{\sqrt{2x+2h}-\sqrt{2x}}{2}\right)}=1\right]$$

$$= \frac{\cos\sqrt{2x}}{\sqrt{2x}}$$

∴ Derivative of f (x) = cos √2x / √2x

(ii) cos √x

Given:

f (x) = cos √x

f (x + h) = cos √(x+h)

By using the formula,

$$\frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

By substituting the values we get,

$$= \lim_{h\to 0} \frac{\cos\sqrt{x+h} - \cos\sqrt{x}}{h}$$

By using the formula,

$$\cos C - \cos D = -2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{C-D}{2}\right)$$

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{h}$$

By using algebra of limits, we get

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{x+h-x}$$

To use the sandwich theorem to evaluate the limit, we need $\frac{\sqrt{x+h}-\sqrt{x}}{2}$ in denominator.

$$= \lim_{h\to 0} \frac{-2\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{2\times\left(\sqrt{x+h}+\sqrt{x}\right)\frac{(\sqrt{x+h}-\sqrt{x})}{2}}$$

By applying limits to each term, we get

$$= \lim_{h\to 0} \frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+h}-\sqrt{x}}{2}} \lim_{h\to 0} \frac{-\sin\left(\frac{\sqrt{x+h}+\sqrt{x}}{2}\right)}{\sqrt{x+h}+\sqrt{x}}$$

When h = 0, we get

$$= 1 \times \frac{-\sin\sqrt{x}}{2\sqrt{x}}\left[\because \lim_{h \to 0} \frac{\sin\left(\frac{\sqrt{x+h}-\sqrt{x}}{2}\right)}{\frac{\sqrt{x+h}-\sqrt{x}}{2}} = 1\right]$$

$$= \frac{-\sin\sqrt{x}}{2\sqrt{x}}$$

∴ Derivative of f (x) = - sin √x / 2√x

EXERCISE 30.3 PAGE NO: 30.33

Differentiate the following with respect to x:

1. x^4 – 2sin x + 3 cos x
Solution:
Given:
f (x) = x^4 – 2sin x + 3 cos x
Differentiate on both the sides with respect to x, we get

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(x^4 - 2\sin x + 3\cos x)$$

By using algebra of derivatives,

$$f' = \frac{d}{dx}(x^4) - 2\frac{d}{dx}(\sin x) + 3\frac{d}{dx}(\cos x)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$$\frac{d}{dx}(\cos x) = -\sin x$$

So,

$$= 4x^{4-1} - 2\cos x + 3(-\sin x)$$

$$= 4x^3 - 2\cos x - 3\sin x$$

∴ Derivative of f (x) is $4x^3 - 2\cos x - 3\sin x$

2. $3^x + x^3 + 3^3$
Solution:
Given:
f (x) = $3^x + x^3 + 3^3$
Differentiate on both the sides with respect to x, we get

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(3^x + x^3 + 3^3)$$

By using algebra of derivatives,

$$f' = \frac{d}{dx}(3^x) + \frac{d}{dx}(x^3) + \frac{d}{dx}(3^3)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$\frac{d}{dx}(a^x) = a^x \log a$

$\frac{d}{dx}(\text{constant}) = 0$

$f' = 3^x \log_e 3 + 3x^{3-1} + 0$

$= 3^x \log_e 3 + 3x^2$

$\therefore$ Derivative of f (x) is $3^x \log_e 3 + 3x^2$

3. $\frac{x^3}{3} - 2\sqrt{x} + \frac{5}{x^2}$

Solution:

Given:

$f(x) = \frac{x^3}{3} - 2\sqrt{x} + \frac{5}{x^2}$

Differentiate on both the sides with respect to x, we get

$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}\left(\frac{x^3}{3} - 2\sqrt{x} + \frac{5}{x^2}\right)$

By using algebra of derivatives,

$f' = \frac{d}{dx}\left(\frac{x^3}{3}\right) - 2\frac{d}{dx}(\sqrt{x}) + 5\frac{d}{dx}\left(\frac{1}{x^2}\right)$

$= \frac{1}{3}\frac{d}{dx}(x^3) - 2\frac{d}{dx}(x^{\frac{1}{2}}) + 5\frac{d}{dx}(x^{-2})$

We know that,

$\frac{d}{dx}(x^n) = nx^{n-1}$

$f' = \frac{1}{3}(3x^{3-1}) - 2 \times \frac{1}{2} x^{\frac{1}{2}-1} + 5(-2)x^{-2-1}$

$= 3 \times \frac{1}{3} x^2 - x^{-\frac{1}{2}} - 10x^{-3}$

$= x^2 - x^{(-1/2)} - 10x^{-3}$

$\therefore$ Derivative of f (x) is $x^2 - x^{(-1/2)} - 10x^{-3}$

4. $e^{x \log a} + e^{a \log x} + e^{a \log a}$

Solution:

Given:

$f(x) = e^{x \log a} + e^{a \log x} + e^{a \log a}$

We know that,

$e^{\log f(x)} = f(x)$

So,

$f(x) = a^x + x^a + a^a$

Differentiate on both the sides with respect to x, we get

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(a^x + x^a + a^a)$$

By using algebra of derivatives,

$$f' = \frac{d}{dx}(a^x) + \frac{d}{dx}(x^a) + \frac{d}{dx}(a^a)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(a^x) = a^x \log a$$

$$\frac{d}{dx}(\text{constant}) = 0$$

$$f' = a^x \log_e a - ax^{a-1} + 0$$

$$= a^x \log a - ax^{a-1}$$

∴ Derivative of f (x) is $a^x \log a - ax^{a-1}$

5. $(2x^2 + 1)(3x + 2)$

Solution:

Given:

f (x) = $(2x^2 + 1)(3x + 2)$

$= 6x^3 + 4x^2 + 3x + 2$

Differentiate on both the sides with respect to x, we get

$$\frac{d}{dx}\{f(x)\} = \frac{d}{dx}(6x^3 + 4x^2 + 3x + 2)$$

By using algebra of derivatives,

$$f' = 6\frac{d}{dx}(x^3) + 4\frac{d}{dx}(x^2) + 3\frac{d}{dx}(x) + \frac{d}{dx}(2)$$

We know that,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(\text{constant}) = 0$$

$$f' = 6(3x^{3-1}) + 4(2x^{2-1}) + 3(x^{1-1}) + 0$$

$$= 18x^2 + 8x + 3 + 0$$

$$= 18x^2 + 8x + 3$$

∴ Derivative of f (x) is $18x^2 + 8x + 3$

EXERCISE 30.4 PAGE NO: 30.39

Differentiate the following functions with respect to x:

1. x^3 sin x

Solution:

Let us consider y = x^3 sin x

We need to find dy/dx

We know that y is a product of two functions say u and v where,

u = x^3 and v = sin x

∴ y = uv

Now let us apply product rule of differentiation.

By using product rule, we get

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ … Equation (1)

As, $u = x^3$

$\frac{du}{dx} = 3x^{3-1} = 3x^2$ … Equation (2) {Since, $\frac{d}{dx}(x^n) = nx^{n-1}$}

As, v = sin x

$\frac{dv}{dx} = \frac{d}{dx}(\sin x) = \cos x$ … Equation (3) {Since, $\frac{d}{dx}(\sin x) = \cos x$}

From equation (1), we can find dy/dx

$\frac{dy}{dx} = x^3\frac{dv}{dx} + \sin x\frac{du}{dx}$

$\frac{dy}{dx} = x^3\cos x + 3x^2\sin x$ {Using equation 2 & 3}

$\therefore \frac{dy}{dx} = x^3\cos x + 3x^2\sin x$

2. x^3 e^x

Solution:

Let us consider y = x^3 e^x

We need to find dy/dx

We know that y is a product of two functions say u and v where,

u = x^3 and v = e^x

∴ y = uv

Now let us apply product rule of differentiation.

By using product rule, we get

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ … Equation (1)

As, $u = x^3$

$\frac{du}{dx} = 3x^{3-1} = 3x^2$ … Equation (2) $\{\frac{d}{dx}(x^n) = nx^{n-1}\}$

As, $v = e^x$

$\frac{dv}{dx} = \frac{d}{dx}(e^x) = e^x$ … Equation (3) {Since, $\frac{d}{dx}(e^x) = e^x\}$

Now from equation (1), we can find dy/dx

$\frac{dy}{dx} = x^3\frac{dv}{dx} + e^x\frac{du}{dx}$

$\frac{dy}{dx} = x^3 e^x + 3x^2e^x$ {Using equation 2 & 3}

$\therefore \frac{dy}{dx} = x^2e^x(3 + x)$

3. x^2 e^x log x

Solution:

Let us consider y = x^2 e^x log x

We need to find dy/dx

We know that y is a product of two functions say u and v where,

u = x^2 and v = e^x

∴ y = uv

Now let us apply product rule of differentiation.

By using product rule, we get

$\frac{dy}{dx} = uw\frac{dv}{dx} + vw\frac{du}{dx} + uv\frac{dw}{dx}$ …equation 1

As, $u = x^2$

$\frac{du}{dx} = 2x^{2-1} = 2x$ … Equation (2) {Since, $\frac{d}{dx}(x^n) = nx^{n-1}\}$

As, $v = e^x$

$\frac{dv}{dx} = \frac{d}{dx}(e^x) = e^x$ … Equation (3) {Since, $\frac{d}{dx}(e^x) = e^x\}$

As, $w = \log x$

$\frac{dw}{dx} = \frac{d}{dx}(\log x) = \frac{1}{x}$ … Equation (4) {Since, $\frac{d}{dx}(\log_e x) = \frac{1}{x}\}$

Now, from equation 1, we can find dy/dx

$\frac{dy}{dx} = x^2\log x\frac{dv}{dx} + e^x\log x\frac{du}{dx} + x^2e^x\frac{dw}{dx}$

$\frac{dy}{dx} = x^2 e^x \log x + 2xe^x \log x + x^2 e^x \frac{1}{x}$ {Using equation 2, 3 & 4}

$\therefore \frac{dy}{dx} = xe^x(1 + x \log x + 2 \log x)$

4. x^n tan x

Solution:

Let us consider y = x^n tan x

We need to find dy/dx

We know that y is a product of two functions say u and v where,

u = x^n and v = tan x

∴ y = uv

Now let us apply product rule of differentiation.

By using product rule, we get

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ … Equation 1

As, $u = x^n$

$\frac{du}{dx} = nx^{n-1}$ … Equation 2 {Since, $\frac{d}{dx}(x^n) = nx^{n-1}$}

As, v = tan x

$\frac{dv}{dx} = \frac{d}{dx}(\tan x) = \sec^2 x$ … Equation 3 {Since, $\frac{d}{dx}(\tan x) = \sec^2 x$}

Now, from equation 1, we can find dy/dx

$\frac{dy}{dx} = x^n\frac{dv}{dx} + \tan x\,\frac{du}{dx}$

$\frac{dy}{dx} = x^n \sec^2 x + nx^{n-1} \tan x$ {Using equation 2 & 3}

$\therefore \frac{dy}{dx} = x^{n-1}(n \tan x + x \sec^2 x)$

5. $x^n \log_a x$

Solution:

Let us consider y = $x^n \log_a x$

We need to find dy/dx

We know that y is a product of two functions say u and v where,

u = x^n and v = $\log_a x$

∴ y = uv

Now let us apply product rule of differentiation.

By using product rule, we get

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ … Equation (1)

As, $u = x^n$

$\frac{du}{dx} = nx^{n-1}$ … Equation (2) {Since, $\frac{d}{dx}(x^n) = nx^{n-1}$}

As, $v = \log_a x$

$\frac{dv}{dx} = \frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$ … Equation (3) {Since, $\frac{d}{dx}(\log_a x) = \frac{1}{x\log_e a}$}

Now, from equation 1, we can find dy/dx

$\frac{dy}{dx} = x^n\frac{dv}{dx} + \log_a x\frac{du}{dx}$

$\frac{dy}{dx} = x^n\frac{1}{x\log_e a} + nx^{n-1}\log_a x$ {Using equation 2 & 3}

$\therefore \frac{dy}{dx} = x^{n-1}\left(n\log_a x + \frac{1}{\log a}\right)$

EXERCISE 30.5 PAGE NO: 30.44

Differentiate the following functions with respect to x:

1. $\frac{x^2+1}{x+1}$

Solution:

Let us consider

$y = \frac{x^2+1}{x+1}$

We need to find dy/dx

We know that y is a fraction of two functions say u and v where,

$u = x^2 + 1$ and $v = x + 1$

∴ y = u/v

Now let us apply quotient rule of differentiation.

By using quotient rule, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ … Equation (1)

As, $u = x^2 + 1$

$\frac{du}{dx} = 2x^{2-1} + 0 = 2x$ … Equation (2) {Since, $\frac{d}{dx}(x^n) = nx^{n-1}$}

As, $v = x + 1$

$\frac{dv}{dx} = \frac{d}{dx}(x + 1) = 1$ … Equation (3) {Since, $\frac{d}{dx}(x^n) = nx^{n-1}$}

Now, from equation 1, we can find dy/dx

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x+1)(2x)-(x^2+1)(1)}{(x+1)^2}$$ {Using equation 2 and 3}

$$= \frac{2x^2+2x-x^2-1}{(x+1)^2}$$

$$= \frac{x^2+2x-1}{(x+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^2+2x-1}{(x+1)^2}$$

2. $\frac{2x-1}{x^2+1}$

Solution:

Let us consider

y = $\frac{2x-1}{x^2+1}$

We need to find dy/dx

We know that y is a fraction of two functions say u and v where,

u = 2x - 1 and v = x^2 + 1

∴ y = u/v

Now let us apply quotient rule of differentiation.

By using quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \ldots \text{ Equation (1)}$$

As, $u = 2x - 1$

$$\frac{du}{dx} = 2x^{1-1} - 0 = 2 \quad \ldots \text{ Equation (2) } \{\text{Since, } \frac{d}{dx}(x^n) = nx^{n-1}\}$$

As, $v = x^2 + 1$

$$\frac{dv}{dx} = \frac{d}{dx}(x^2 + 1) = 2x \quad \ldots \text{ Equation (3) } \{\text{Since, } \frac{d}{dx}(x^n) = nx^{n-1}\}$$

Now, from equation 1, we can find dy/dx

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(x^2+1)(2)-(2x-1)(2x)}{(x^2+1)^2} \quad \{\text{Using equation 2 and 3}\}$$

$$= \frac{2x^2+2-4x^2+2x}{(x^2+1)^2}$$

$$= \frac{-2x^2+2x+2}{(x^2+1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{2(1+x-x^2)}{(x^2+1)^2}$$

3. $\boldsymbol{\frac{x+e^x}{1+\log x}}$

Solution:

Let us consider

y = $\frac{x+e^x}{1+\log x}$

We need to find dy/dx

We know that y is a fraction of two functions say u and v where,

u = x + e^x and v = 1 + log x

∴ y = u/v

Now let us apply quotient rule of differentiation.
By using quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \text{ … Equation 1}$$

As, $u = x + e^x$

$$\frac{du}{dx} = \frac{d}{dx}(x + e^x) \left\{\text{Since, } \frac{d}{dx}(x^n) = nx^{n-1} \;\&\; \frac{d}{dx}(e^x) = e^x\right\}$$

$$\frac{du}{dx} = \frac{d}{dx}(x) + \frac{d}{dx}(e^x) = 1 + e^x \text{ … Equation 2}$$

As, $v = 1 + \log x$

$$\frac{dv}{dx} = \frac{d}{dx}(\log x + 1)$$

$$= \frac{d}{dx}(1) + \frac{d}{dx}(\log x)$$

$$\frac{dv}{dx} = 0 + \frac{1}{x} = \frac{1}{x} \text{ … Equation 3 } \left\{\text{Since, } \frac{d}{dx}(\log x) = \frac{1}{x}\right\}$$

Now, from equation 1, we can find dy/dx

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$\frac{dy}{dx} = \frac{(1 + \log x)(1 + e^x) - (x + e^x)\left(\frac{1}{x}\right)}{(\log x + 1)^2} \text{ {Using equation 2 and 3}}$$

$$= \frac{1 + e^x + \log x + e^x \log x - 1 - \frac{e^x}{x}}{(\log x + 1)^2}$$

$$= \frac{x\log x(1 + e^x) + e^x(x-1)}{x(\log x + 1)^2}$$

$$\therefore \frac{dy}{dx} = \frac{x \log x(1 + e^x) - e^x(1 - x)}{x(1 + \log x)^2}$$

4. $\dfrac{e^x - \tan x}{\cot x - x^n}$

Solution:

Let us consider

$$y = \frac{e^x - \tan x}{\cot x - x^n}$$

We need to find dy/dx

We know that y is a fraction of two functions say u and v where,

$u = e^x - \tan x$ and $v = \cot x - x^n$

$\therefore y = u/v$

Now let us apply quotient rule of differentiation.
By using quotient rule, we get

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \ldots \text{Equation (1)}$$

As, $u = e^x - \tan x$

$$\frac{du}{dx} = \frac{d}{dx}(e^x - \tan x) \quad \left\{\text{Since, } \frac{d}{dx}(\tan x) = \sec^2 x \ \& \ \frac{d}{dx}(e^x) = e^x\right\}$$

$$\frac{du}{dx} = -\frac{d}{dx}(\tan x) + \frac{d}{dx}(e^x) = \sec^2 x + e^x \quad \ldots \text{Equation (2)}$$

As, $v = \cot x - x^n$

$$\frac{dv}{dx} = \frac{d}{dx}(\cot x - x^n)$$

$$= \frac{d}{dx}(\cot x) - \frac{d}{dx}(x^n) \quad \left\{\text{Since, } \frac{d}{dx}(\cot x) = -\text{cosec}^2 x \ \& \ \frac{d}{dx}(x^n) = nx^{n-1}\right\}$$

$$\frac{dv}{dx} = -\text{cosec}^2 x - nx^{n-1} \quad \ldots \text{Equation (3)}$$

Now, from equation 1, we can find dy/dx

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2} \quad \{\text{Using equation 2 and 3, we get}\}$$

$$\frac{dy}{dx} = \frac{(\cot x - x^n)(\sec^2 x + e^x) - (e^x - \tan x)(-\text{cosec}^2 x - nx^{n-1})}{(\cot x - x^n)^2}$$

$$\therefore \frac{dy}{dx} = \frac{(\cot x - x^n)(e^x - \sec^2 x) + (e^x - \tan x)(\text{cosec}^2 x + nx^{n-1})}{(\cot x - x^n)^2}$$

5. $\dfrac{ax^2 + bx + c}{px^2 + qx + r}$

Solution:

Let us consider

$$y = \frac{ax^2 + bx + c}{px^2 + qx + r}$$

We need to find dy/dx

We know that y is a fraction of two functions say u and v where,

$u = ax^2 + bx + c$ and $v = px^2 + qx + r$

$\therefore y = u/v$

Now let us apply quotient rule of differentiation.

By using quotient rule, we get

$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$ … Equation (1)

As, $u = ax^2 + bx + c$

$\frac{du}{dx} = 2ax + b$ … Equation (2) {Since, $\frac{d}{dx}(x^n) = nx^{n-1}$}

As, $v = px^2 + qx + r$

$\frac{dv}{dx} = \frac{d}{dx}(px^2 + qx + r) = 2px + q$ … Equation (3)

Now, from equation 1, we can find dy/dx

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(px^2 + qx + r)(2ax + b) - (ax^2 + bx + c)(2px + q)}{(px^2 + qx + r)^2}$$ {Using equation 2 and 3}

$$= \frac{2apx^3 + bpx^2 + 2aqx^2 + bqx + 2arx + br - 2apx^3 - aqx^2 - 2bpx^2 - bqx - 2pcx - qc}{(px^2 + qx + r)^2}$$

$$= \frac{aqx^2 - bpx^2 + 2arx + br - 2pcx - qc}{(px^2 + qx + r)^2}$$

$$= \frac{x^2(aq - bp) + 2x(ar - pc) + br - qc}{(px^2 + qx + r)^2}$$

$$\therefore \frac{dy}{dx} = \frac{x^2(aq - bp) + 2x(ar - pc) + br - qc}{(px^2 + qx + r)^2}$$