Exercise Solutions

Question 1: Question 1: Show that the dimensions of the displacement current Єo [dϕ_E/dt] are that of an electric current.

Solution:

The displacement current $I_d = \epsilon_o \, d\phi_E/dt$ ...(1)

Where ϕ_E is the electric flux produced by the time varying electric field.

According to Gauss's law, the electric flux ϕ_E through a surface:

$\phi_E = q_{enclosed}/\epsilon_o$

(1)=>

$$I_d = \epsilon_0 \times \frac{d(\frac{q_{enclosed}}{\epsilon_0})}{dt}$$

$$= \epsilon_0 \times \frac{1}{\epsilon_0} \times \frac{dq_{enclosed}}{dt}$$

$$I_d = \frac{d(q_{enclosed})}{dt}$$

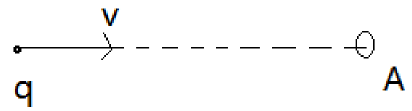
Now,
Dimension of I_d = [TA]/[T] = [A]

As [A] is the dimension of electric current which are

Since dimension of displacement current has same dimension as that of electric current.

So, dimension of displacement current = [A]

Question 2: A point charge is moving along a straight line with a constant velocity v. Consider a small area A perpendicular to the direction of motion of the charge figure. Calculate the displacement current through the area when its distance from the charge is x. The value of x is not large so that the electric field at any instant is essentially given by Coulomb's law.



Solution:

The electric field produced by the charge when it is at a distance x from the surface is given by Coulomb's law is E i.e.

$E = q/(4\pi\epsilon_o x^2)$

And E produces an electric flux through the area whose magnitude is given by Gauss's law

$$\phi_E = \oint \vec{E} \cdot \vec{ds} = \oint E\, dS\, cos\theta = \oint E\, dS$$

As, electric field lines are directed along the normal to the area vector of the surface.

So, $\phi_E = EA = qA/(4\pi\epsilon_o x^2)$

The displacement current through the area:

$$I_d = \epsilon_0 \left|\frac{d(\phi_E)}{dt}\right| = \epsilon_0 \left|\frac{d(\frac{qA}{4\pi\epsilon_0 x^2})}{dt}\right|$$

$$= \epsilon_0 \times \frac{qA}{4\pi\epsilon_0} \times \left|\frac{d(x^{-2})}{dt}\right|$$

$$= \frac{qA}{2\pi x^3} \times \frac{dx}{dt}$$

$$I_d = \frac{qAv}{2\pi x^3}$$

Where dx/dt = v, as "x" is the distance of the charge from the area at different intervals of time.

Question 3: A parallel-plate capacitor having plate-area A and plate separation d is joined to a battery of emf ϵ and internal resistance R at t = 0.
Consider a plane surface of area A/2, parallel to the plates and
situated symmetrically between them. Find the displacement current
through this surface as a function of time.

Solution: The displacement current Id is generated due to the fact that the charge on capacitor plates is changing with time.

$I_d = \epsilon_0 \,|d/dt\,(\phi_E)|$

If the charge on the capacitor plate is Q and the area of the plate is "A", then

$\oint E.\,ds = Q/\epsilon_0$

If dS is a small area element on the plate.

$\oint E.ds = \oint Eds \cos 0^o = EA$

$\Rightarrow EA = Q/\epsilon_0$

or $E = Q/A\epsilon_0$, which is the electric field between the plates.

This electric field produces and electric flux through the plane surface.

$$\oint EdS = E\oint dS = \frac{EA}{2} = \frac{Q}{A\epsilon_0}\times\frac{A}{2} = \frac{Q}{2\epsilon_0} = \phi_E$$

If the capacitance of the capacitor is C, then the charge Q at time t is

$Q = \varepsilon C(1 - e^{-t/RC})$

Now,

The displacement current as a function of time, I_d:

$$I_d = \epsilon_0\left|\frac{d(\phi_E)}{dt}\right| = \epsilon_0\left|\frac{d(\frac{Q}{2\epsilon_0})}{dt}\right|$$

$$= \frac{1}{2}\left|\frac{d(\epsilon C(1 - e^{\frac{-t}{RC}}))}{dt}\right|$$

$$= \frac{1}{2}\left|\epsilon C \times \left(-\frac{1}{RC}\right) \times \left(-e^{\frac{-t}{RC}}\right)\right|$$

$$= \frac{\epsilon}{2R}\, e^{\frac{-t}{RC}}$$

Question 4: Consider the situation of the previous problem. Define displacement resistance $R_d = V/i_d$ of the space between the plates where V is the potential difference between the plates and id is the displacement current. Show that R_d varies with time as $R_d = R\,(e^{t/\tau} - 1)$

Solution:

Displacement current = $I_d = \epsilon_o\, |d/dt\,(\phi_E)|$

Electric flux, Using Gauss's law:

$\phi_E = Q/\epsilon_o$

Where, Q = charge enclosed

And $Q = CV = CV_o\, e^{(-t/RC)}$

[As the capacitor is charging, the charge will be a function of time]

Now, the flux is:

$\phi_E = [CV_o\, e^{(-t/RC)}]/\, \epsilon_o$

$E = q/(4\pi\epsilon_o x^2)$

Now, let us find the displacement current:

$$I_d = \epsilon_0 \left|\frac{d(\phi_E)}{dt}\right|$$

$$= \epsilon_0 \left|\frac{d\left(\frac{CV_0 e^{\frac{-t}{RC}}}{\epsilon_0}\right)}{dt}\right|$$

$$= CV_0 \left|\left(\frac{-1}{RC}\right) \times \left(e^{-t/RC}\right)\right|$$

$$=> \; I_d = \frac{V_0}{R} e^{\frac{-t}{RC}}$$

We know, $R_d = \frac{V_0}{I_d} - R$

$$R_d = \frac{V_0}{\frac{V_0 e^{\frac{-t}{RC}}}{R}} - R = Re^{\frac{t}{RC}} - R = R\left(e^{\frac{t}{\tau}} - 1\right)$$

Here τ is the time constant, thus R_d varies with time.

Question 5: Using $B = \mu_0 H$ find the ratio E_0/H_0 for a plane electromagnetic wave propagating through vacuum. Show that it has the dimensions of electric resistance. This ratio is a universal constant called the impedance of free space.

Solution:
For free space, relation between magnetic field and electric field,
$B_0 = E_0/C$ ....(1)

Also, for free space given relation can be written as, $B_0 = \mu_0 H_o$ ...(2)

From (1) and (2), we have

$\mu_0 H_o = E_0 /C$

$\Rightarrow E_o/H_o = \mu_0 C$ ...(3)

Dimension of $C = [L T^{-1}]$

Find the dimension of μ_0:

According to scalar form of Biot Savart law:

$B = \mu_0/4\pi \times iL/R^2$

Or $\mu_0 = [4\pi R^2 B] / iL$

Dimensions of $\mu_0 = [MT^{-2}A^{-1}][L^2]/[L][A] = [MLT^{-2}A^{-2}]$

(3)=> Dimensions of $E_o/H_o = [MLT^{-2}A^{-2}] [L T^{-1}] = [ML^2T^{-3}A^{-2}]$

Therefore, Dimensions of E_o/H_o has the same dimensions as that of electrical resistance i.e. [R].

Question 6: The sunlight reaching the earth has maximum electric field of 810 V m–1. What is the maximum magnetic field in this light?

Solution:
We know, maximum magnetic field = $B_0 = E_0/C$

Where, C is the speed of light in vacuum and B_0 is the amplitude of maximum magnetic field.

$B_0 = E_0/C = 810/[3\times10^8] = 2.7 \times 10\text{-}6$ T = 2.7 µ T

Question 7: The magnetic field in a plane electromagnetic wave is given by
B = (200 μT) sin [$(4.0 \times 10^{15}\ s^{-1})$ (t–x/c)].
Find the maximum electric field and the average energy density corresponding to the electric field.

Solution:

Given: B = (200 μT) sin [$(4.0 \times 10^{15}\ s^{-1})$ (t–x/c)]

We know, relation between magnetic field and electric field : $B_0 = E_0/C$

The electric field intensity: $E_0 = B_0 C$

=> $E_0 = (3x10^8) \times 200 = 6 \times 10^4$ N/C

The energy density associated with an electric field:

$U_d = (1/2)\ \epsilon_o E_o^2$

$U_d = (1/2) \times (8.85 \times 10^{-12}) \times (6x10^4)^2$

$= 159.3 \times 10^{-4}\ Jm^{-3}$

=> $U_d = 1.593 \times 10^{-2}\ Jm^{-3}$

Question 8: A laser beam has intensity $2.5 \times 10^{14}\ W\ m^{-2}$. Find the amplitudes of electric and magnetic fields in the beam.

Solution:
The amplitude of the electric field is related to the intensity of the wave by the relation:

$I = (1/2)\ \epsilon_o E_o^2 C$

or $E_o^2 = 2I/\ \epsilon_o C$

$= [2x2.5x10^{14}]/[3x10^8x8.85x10^{-12}]$

Or $E_o = 4.339 \times 10^8\ NC^{-1}$

=>Electric field amplitude = $E_o = 4.339 \times 10^8\ NC^{-1}$

The relation between amplitudes of electric and magnetic fields is given as:

$B_o = E_o/C$

So, magnetic field amplitude = B_o = $[4.339 \times 10^8]/[3 \times 10^8]$ = 1.43 T

Question 9: The intensity of the sunlight reaching Earth is 1380 W m−2. Assume this light to be a plane, monochromatic wave. Find the amplitudes of electric and magnetic fields in this wave.

Solution:
The relation between amplitude of the electric field and the intensity of the wave is

$I = (1/2)\, \epsilon_o E_o^2 C$

or $E_o^2 = 2I/ \epsilon_o C$

$= [2 \times 1380]/[3 \times 10^8 \times 8.85 \times 10^{-12}]$

Or $E_o = 1.02 \times 10^3\ NC^{-1}$

=>Electric field amplitude = $E_o = 1.02 \times 10^3\ NC^{-1}$

The relation between amplitudes of electric and magnetic fields is given as:

$B_o = E_o/C$

Value of Magnetic field amplitude = $B_o = [1.02 \times 10^3]/[3 \times 10^8] = 3.4 \times 10^{-6}$ T = 3.4 μT.