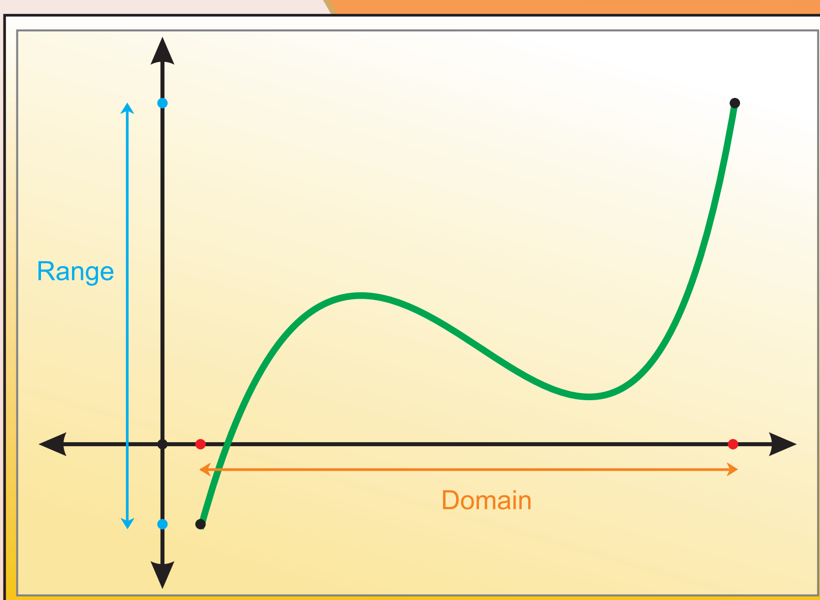
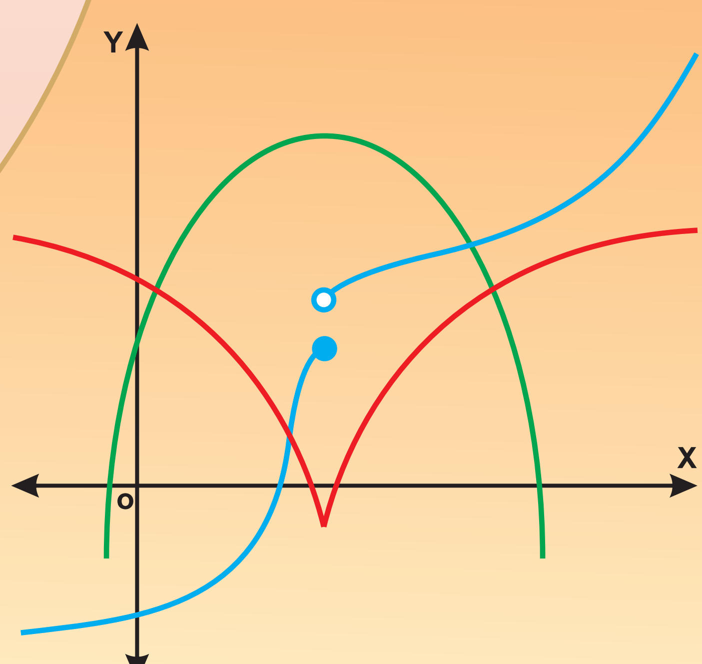
Standard XI

Mathematics & Statistics

(Arts & Science) Part 2





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Mathematics and Statistics

(Arts and Science)

Part - 2

STANDARD - XI

महाराष्ट्र राज्य पाठ्यपुस्तक निर्मिती व अभ्यासक्रम संशोधन मंडळ

Maharashtra State Bureau of Textbook Production and Curriculum Research, Pune - 411 004

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First Edition : 2019

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The Constitution of India

Preamble

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the unity and integrity of the Nation;

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949, do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

NATIONAL ANTHEM

Jana-gana-mana-adhināyaka jaya hē
Bhārata-bhāgya-vidhātā,

Panjāba-Sindhu-Gujarāta-Marāthā
Drāvida-Utkala-Banga

Vindhya-Himāchala-Yamunā-Gangā
uchchala-jaladhi-taranga

Tava subha nāmē jāgē, tava subha āsisa māgē,
gāhē tava jaya-gāthā,

Jana-gana-mangala-dāyaka jaya hē
Bhārata-bhāgya-vidhātā,

Jaya hē, Jaya hē, Jaya hē,
Jaya jaya jaya, jaya hē.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders respect, and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone lies my happiness.

PREFACE

Dear Students,

Welcome to the eleventh standard!

You have successfully completed your secondary education and have entered the higher secondary level. You will now need to learn certain mathematical concepts and acquire some statistical skills to add more applicability to your work. Maharashtra State Bureau of Textbook Production and Curriculum Research has modified and restructured the curriculum in Mathematics in accordance with changing needs.

The curriculum of Mathematics is divided in two parts. Part 1 covers topics in Trignometry, Algebra, Co-ordinate Geometry and Statistics. Part 2 covers Complex Numbers, Sets and Relations, Calculus and Combinatorics. There is a special emphasis on applications. Activities are added in chapters for creative thinking. Some material will be made available on E-balbharati website (ebalbharati.in). It contains a list of specimen practical problems on each chapter. Students should complete the practical exercises under the guidance of their teachers. Maintain a journal and take teacher's signature on every completed practical.

You are encouraged to use modern technology in your studies. Explore the Internet for more recent information on topics in the curriculum. Get more examples and practice problems from the Internet. You will enjoy learning if you study the textbook thoroughly and manage to solve problems.

On the title page Q.R. code is given. It will help you to get more knowledge and clarity about the contents.

This textbook is prepared by Mathematics Subject Commitee and members of study group. This book has also been reviewed by senior teachers and subject experts. The Bureau is grateful to all of them and would like to thank for their contribution in the form of creative writing, constructive criticism, and valuable suggestions in making this book useful to you and helpful to your teachers.

The Bureau hopes that the textbook will be received well by all stakeholders in the right spirit.

You are now ready to study. Best wishes for a happy learning experience.

Pune

Date : 20 June 2019

Indian Solar Date : 30 Jyeshtha 1941

(Dr. Sunil Magar)
Director
Maharashtra State Bureau of Textbook
Production and Curriculum Research, Pune.

Mathematics and Statistics Std XI (Part II)
Arts and Science

Sr. No	Area	Topic	Competency Statement
1	Complex Numbers	Complex Numbers	The students will be able to - • understand set of complex numbers and different ways of expressing complex numbers. • perform algebraic operations on complex numbers. • simplify algebraic expressions involving complex numbers.
2	Sequences and Series	Sequences and Series	• Revise AP, learn GP and HP. • Find the general term and the sum of the first n terms of these sequences.
3	Permutations and combinations	Permutations, Combinations	• count the number of arrangements of given objects satisfying specific conditions. • count the number of possible selections of objects with certain conditions.
4	Method of Induction and Binomial theorem	Method of Induction	• understand the method of induction and apply it to verify mathematical statements.
		Binomial Theorem	• expand binomial expressions and find its general term. • simplify the binomial expression for negative index or fractional power.
5	Sets and relations	Sets	• work with sets and operations on sets. • construct sets from given conditions. • solve problems on applications of set theory.
		Relations	• identify the types of relations. • study equivalence relations.

6	Functions	Functions	• work with function defined on different domains. • identify different types of functions. • carry out algebraic operations on functions.
7	Limits	Limits	• understand the concept of limit of a function. • determine the limits of functions if they exist.
8	Continuity	Continuity	• Define and study the continuity of a function at a point and in an interval.
9	Differentiation	Differentiation	• understand and study the differentiability of a function. • understand and study differentiation of various functions.

INDEX

1 COMPLEX NUMBERS

Let's Study

- A complex number (C.N.)
- Algebra of C.N.
- Geometrical Representation of C.N.
- Polar & Exponential form of C.N.
- De Moivre's Theorem.

Let's Recall

- Algebra of real numbers.
- Solution of linear and quadratic equations
- Representation of a real number on the number line
- Representation of point in a plane
- Trigonometric ratios

Introduction:

Consider, the equation $x^2 + 1 = 0$. This equation has no solution in the set of real numbers because there is no real number whose square is -1. We need to extend the set of real numbers to a larger set, which would include solutions of such equations.

We introduce a symbol i (greek letter iota) such that $i = \sqrt{-1}$ and $i^2 = -1$. i is called as an **imaginary unit** or an **imaginary number.**

Swiss mathematician Leonard Euler (1707-1783) was the first mathematician to introduce the symbol i with $i = \sqrt{-1}$ and $i^2 = -1$.

1.1 A Complex number :

1.1(a) Imaginary Number :

A number of the form bi, where b$\in$ R, b $\neq$ 0, $i = \sqrt{-1}$ is called an imaginary number.

Ex : $\sqrt{-25} = 5i$, $2i$, $\frac{2}{7}i$, $-11i$ etc.

Note:

The number i satisfies following properties,

i) $i \times 0 = 0$

ii) If a$\in$R, then $\sqrt{-a^2} = \sqrt{i^2 a^2} = \pm ia$

iii) If a, b$\in$R, and ai = bi then a = b

1.1 (b) Complex Number :

Definition : A number of the form a+ib, where a, b$\in$ R and $i = \sqrt{-1}$ with $i^2 = -1$ is called a complex number and is usually denoted by z.

That is z = a+ib, a, b$\in$R and $i = \sqrt{-1}$

Here 'a' is called the real part of z and is denoted by **Re(z) or R(z).** 'b' is called imaginary part of z and is denoted by **Im(z) or I(z)**

The set of complex numbers is denoted by C

$\therefore$ C = {a+ib / a, b$\in$ R, and $i = \sqrt{-1}$ }

Ex :

z	a+ib	Re(z)	Im(z)
$2+4i$	$2+4i$	2	4
$5i$	$0+5i$	0	5
$3-4i$	$3-4i$	3	-4
$5+\sqrt{-16}$	$5+4i$	5	4
$2+\sqrt{5}i$	$2+\sqrt{5}\,i$	2	$\sqrt{5}\,i$
$7+\sqrt{3}$	$(7+\sqrt{3})+0i$	$(7+\sqrt{3})$	0

Note :

1) A complex number whose real part is zero is called a purely imaginary number. Such a number is of the form $z = 0 + ib = ib$

2) A complex number whose imaginary part is zero is a real number.

$z = a + 0i = a$, is a real number.

3) A complex number whose both real and imaginary parts are zero is the zero complex number. $0 = 0 + 0i$

4) The set R of real numbers is a subset of the set C of complex numbers.

5) The real part and imaginary part cannot be combined to form single term. e.g. $2 + 3i \neq 5i$

1.2 Algebra of Complex Numbers :

1.2.1 Equality of two Complex Numbers :

Definition : Two complex numbers $z_1 = a+ib$ and $z_2 = c + id$ are said to be equal if their corresponding real and imaginary parts are equal.

i.e. $a + ib = c + id$ if $a = c$ and $b = d$

Ex. : i) If $x + iy = 4 + 3i$ then $x = 4$ and $y = 3$

Ex. : ii) If $7a + i(3a - b) = 21 - 3i$ then find a and b.

Solution : $7a + (3a - b)i = 21 - 3i$

By equality of complex numbers

$7a = 21 \quad \therefore a = 3$

and $3a - b = -3 \quad \therefore 3(3) + 3 = b$

$\therefore 12 = b$

Note : The order relation (inequality) of complex number can not be defined. Hence, there does not exist a smaller or greater complex number than given complex number. We cannot say $i < 4$.

1.2.2 Conjugate of a Complex Number:

Definition : The conjugate of a complex number $z = a + ib$ is defined as $a - ib$ and is denoted by $\bar{z}$

Ex : 1)

z	$\bar{z}$
$3 + 4i$	$3 - 4i$
$7i - 2$	$-7i - 2$
3	3
$5i$	$-5i$
$2 + \sqrt{3}$	$2 + \sqrt{3}$
$7 + \sqrt{5}\,i$	$7 - \sqrt{5}\,i$

2) Properties of $\bar{z}$

1) $\overline{(\bar{z})} = z$

2) If $z = \bar{z}$, then z is purely real.

3) If $z = -\bar{z}$, then z is purely imaginary.

Now we define the four fundamental operations of addition, subtraction, multiplication and division of complex numbers.

1.2.3 Addition of complex numbers :

Let $z_1 = a+ib$ and $z_2 = c+id$

then $z_1 + z_2 = (a+ib) + (c+id)$

$= (a+c) + (b+d)\, i$

In other words, $\text{Re}(z_1 + z_2) = \text{Re}(z_1) + \text{Re}(z_2)$

and $\text{Im}(z_1 + z_2) = \text{Im}(z_1) + \text{Im}(z_2)$

Ex. 1) $(2 + 3i) + (4 + 3i) = (2+4) + (3+3)i$

$= 6 + 6i$

2) $(-2 + 5i) + (7 + 3i) + (6 - 4i)$

$= [(-2) + 7 + 6] + [5 + 3 + (-4)]i$

$= 11 + 4i$

Properties of addition : If z_1, z_2, z_3 are complex numbers then

i) $z_1 + z_2 = z_2 + z_1$ (commutative)

ii) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$ (associative)

iii) $z_1 + 0 = 0 + z_1 = z_1$ (identity)

iv) $z + \overline{z} = 2\text{Re}(z)$ (Verify)

v) $\overline{(z_1+z_2)} = \overline{z_1} + \overline{z_2}$

1.2.4 Scalar Multiplication :

If $z = a+ib$ is any complex number, then for every real number k, define $kz = ka + i\,(kb)$

Ex. 1) If $z = 7 + 3i$ then

$5z = 5(7 + 3i) = 35 + 15i$

2) $z_1 = 3 - 4i$ and $z_2 = 10 - 9i$ then

$2z_1 + 5z_2 = 2(3 - 4i) + 5(10 - 9i)$

$= 6 - 8i + 50 - 45i$

$= 56 - 53i$

Note: 1) $0.z = 0(a + ib) = 0 + 0i = 0$

1.2.5 Subtraction of complex numbers :

Let $z_1 = a+ib$, $z_2 = c+id$ then define

$z_1 - z_2 = z_1 + (-1)z_2 = (a+ib) + (-1)(c+id)$

$= (a+ib) + (-c - id)$

$= (a-c) + i\ (b-d)$

Hence, $\text{Re}(z_1 - z_2) = \text{Re}(z_1) - \text{Re}(z_2)$

$\text{Im}(z_1 - z_2) = \text{Im}(z_1) - \text{Im}(z_2)$

Ex. 1) $z_1 = 4+3i$, $z_2 = 2+i$

$\therefore z_1 - z_2 = (4+3i) - (2+i)$

$= (4-2) + (3-1)\,i$

$= 2 + 2i$

2) $z_1 = 7+i$, $z_2 = 4i$, $z_3 = -3+2i$

then $2z_1 - (5z_2 + 2z_3)$

$= 2(7+i) - [5(4i) + 2(-3+2i)]$

$= 14 + 2i - [20i - 6 + 4i]$

$= 14 + 2i - [-6 + 24i]$

$= 14 + 2i + 6 - 24i$

$= 20 - 22i$

Properties of Subtraction :

1) $z - \overline{z} = 2\,\text{Im}(z)$ (Verify)

1) $\overline{(z_1 - z_2)} = \overline{z_1} - \overline{z_2}$

1.2.6 Multiplication of complex numbers:

Let $z_1 = a+ib$ and $z_2 = c+id$. We denote multiplication of z_1 and z_2 as $z_1.z_2$ and is given by

$z_1.z_2 = (a+ib)(c+id) = a(c+id)+ib(c+id)$

$= ac + adi + bci + i^2bd$

$= ac + (ad+bc)i - bd \qquad (\because i^2 = -1)$

$z_1.z_2 = (ac-bd) + (ad+bc)i$

Ex. 1) $z_1 = 2+3i$, $z_2 = 3-2i$

$\therefore z_1.z_2 = (2+3i)(3-2i) = 2(3-2i) + 3i(3-2i)$

$= 6 - 4i + 9i - 6i^2$

$= 6 - 4i + 9i + 6 \qquad (\because i^2 = -1)$

$= 12 + 5i$

Ex. 2) $z_1 = 2-7i$, $z_2 = 4-3i$, $z_3 = 1+i$ then

$(2\,z_1) . (z_2) . (z_3) = 2(2-7i) . (4-3i) . (1+i)$

$= (4-14i) . [4+4i-3i-3i^2]$

$= (4-14i) . [7+i]$

$= 28 + 4i - 98i - 14i^2$

$= 42 - 94i$

Properties of Multiplication :

i) $z_1.z_2 = z_2.z_1$ (commutative)

ii) $(z_1.z_2).z_3 = z_1.(z_2.z_3)$ (associative)

iii) $(z_1.1) = 1.z_1 = z_1$ (identity)

iv) $\overline{(z_1.z_2)} = \overline{z_1} . \overline{z_2}$ (Verify)

v) If $z = a+ib$ $z.\overline{z} = a^2 + b^2$

1.2.7. Powers of i : We have $\sqrt{-1} = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$. Let us consider i^n, where n is a positive integer and $n > 4$.

We divide n by 4 and obtain the remainder 'r'. Let m be the quotient.

Then, $n = 4m + r$, where $0 \le r < 4$

$\therefore \quad i^n = i^{4m+r} = i^{4m}.\, i^r = (i^4)^m.\, i^r = 1.i^r = i^r$

Similarly,

$$i^{4m} = 1 = \frac{1}{i^{4m}} = i^{-4m}$$

$$i^2 = -1,\ i^3 = i \times i^2 = -i$$

$$i^{-1} = \frac{1}{i} = \frac{i}{i^2} = \frac{i}{-1} = -i$$

The above equations help us to find i^k for any integer k.

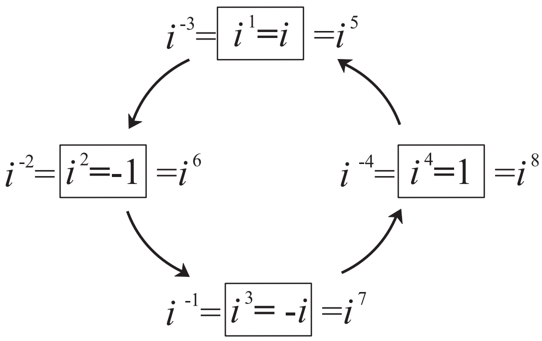


Fig. 1.1

e.g. (i) $\quad i^{50} = (i^4)^{12}.\ i^2 = i^2 = -1$

(ii) $\quad i^{318} = (i^4)^{79}.\ i^2 = -1$

(iii) $\quad i^{999} = (i^4)^{249+3} = (i^4)^{249}.\ i^3 = -i$

Remark : In general,

$i^{4n} = 1, \qquad i^{4n+1} = i,$

$i^{4n+2} = -1, \qquad i^{4n+3} = -i$ where $n \in Z$

1.2.8 Division of complex number :

Let $z_1 = a+ib$ and $z_2 = c+id$ be any two complex numbers such that $z_2 \neq 0$

Now,

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id} \quad \text{where } z_2 \neq 0 \text{ i.e. } c+id \neq 0$$

Multiply and divide by conjugate of z_2.

$$\therefore \frac{z_1}{z_2} = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$= \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{ac-bd}{c^2+d^2}\right)i$$

Where $\left(\frac{ac+bd}{c^2+d^2}\right) \in R$ and $\left(\frac{ac-bd}{c^2+d^2}\right) \in R$

Illustration : If $z_1 = 3+2i$, & $z_2 = 1+i$,

then $\dfrac{z_1}{z_2} = \dfrac{3+2i}{1+i}$

By multiplying numerator and denominator with $\overline{z}_2 = 1 - i$

$$\frac{z_1}{z_2} = \frac{3+2i}{1+i} \times \frac{1-i}{1-i}$$

$$= \frac{3-3i+2i-2i^2}{1+1}$$

$$= \frac{5-i}{2}$$

$$\therefore \frac{z_1}{z_2} = \frac{5}{2} - \frac{1}{2}i$$

Properties of Division :

1) $\dfrac{1}{i} = \dfrac{1}{i} \times \dfrac{i}{i} = \dfrac{i}{-1} = -i$

2) $\dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$

SOLVED EXAMPLES

Ex. 1 : Write $(1+2i)(1+3i)(2+i)^{-1}$ in the form a+ib

Solution :

$$(1+2i)(1+3i)(2+i)^{-1} = \frac{(1+2i)(1+3i)}{2+i}$$

$$= \frac{1+3i+2i+6i^2}{2+i} = \frac{-5+5i}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{-10+5i+10i-5i^2}{4-i^2} = \frac{-5+15i}{4+1} \qquad (\because i^2=-1)$$

$$= \frac{-5+15i}{5} = -1+3i$$

Ex. 2 : (Activity) Express $\frac{1}{i}+\frac{2}{i^2}+\frac{3}{i^3}+\frac{5}{i^4}$ in the form of $(a+ib)$.

Solution : $i^2=-1$, $i^3=-i$, $i^4=1$

$$\frac{1}{i}=\boxed{},\ \frac{1}{i^2}=\boxed{},\ \frac{1}{i^3}=\boxed{},\ \frac{1}{i^4}=\boxed{}$$

$$\therefore \frac{1}{i}+\frac{2}{i^2}+\frac{3}{i^3}+\frac{5}{i^4}$$

$$= 1(\boxed{})+2(\boxed{})+3(\boxed{})+5(\boxed{})$$

$$= \boxed{}+i\boxed{}$$

$$\therefore\ a=\boxed{},\ b=\boxed{}$$

Ex. 3 : If a and b are real and $(i^4+3i)a+(i-1)b+5i^3=0$, find a and b.

Solution : $(i^4+3i)a+(i-1)b+5i^3=0+0i$

i.e. $(1+3i)a+(i-1)b-5i=0+0i$

$\therefore$ a + 3ai + bi − b − 5i = 0+0i

i.e. $(a-b)+(3a+b-5)i=0+0i$

By equality of complex numbers, we get

$a-b=0$ and $3a+b-5=0$

$\therefore$ a=b and 3a+b = 5

$\therefore$ 3a+a = 5

$\therefore$ 4a = 5

$\therefore a=\frac{5}{4}$

$\therefore a=b=\frac{5}{4}$

Ex. 4 : If $x+2i+15i^6y = 7x+i^3(y+4)$ find $x+y$, given that $x, y \in$ R.

Solution :

$x+2i+15i^6y = 7x+i^3(y+4)$

$\therefore x+2i-15y = 7x-(y+4)i$

$(\because i^6=-1, i^3=-i)$

$\therefore x-15y+2i = 7x-(y+4)i$

Equating real and imaginary parts, we get

$x-15y = 7x$ and $2=-(y+4)$

$\therefore -6x-15y = 0$ .... (i) $\quad y+6=0$ ....(ii)

$\therefore y=-6,\ x=15$ [Solving (i) and (ii)]

$\therefore x+y=15-6=9$

Ex. 5 : Show that $\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^3 = i$.

Solution :

$$\text{L.H.S.} = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^3 = \left(\frac{\sqrt{3}+i}{2}\right)^3$$

$$= \frac{\left(\sqrt{3}\right)^3+3\left(\sqrt{3}\right)^2 i+3\sqrt{3}i^2+(i)^3}{(2)^3}$$

$$= \frac{3\sqrt{3}+9i-3\sqrt{3}-i}{8}$$

$$= \frac{8i}{8}$$

$$= i$$

= R.H.S.

EXERCISE 1.1

1) Simplify :

i) $\sqrt{-16}+3\sqrt{-25}+\sqrt{-36}-\sqrt{-625}$

ii) $4\sqrt{-4}+5\sqrt{-9}-3\sqrt{-16}$

2) Write the conjugates of the following complex numbers

i) $3+i$ ii) $3-$ iii) $-\sqrt{5}-\sqrt{7}i$

iv) $-\sqrt{-5}$ v) $5i$ vi) $\sqrt{5}-i$

vii) $\sqrt{2}+\sqrt{3}\,i$ viii) $\cos\theta + i\sin\theta$

3) Find a and b if

i) $a + 2b + 2ai = 4 + 6i$

ii) $(a-b) + (a+b)i = a + 5i$

iii) $(a+b)(2 + i) = b + 1 + (10 + 2a)i$

iv) $abi = 3a - b + 12i$

v) $\dfrac{1}{a+ib} = 3 - 2i$

vi) $(a + ib)(1 + i) = 2 + i$

4) Express the following in the form of $a+ib$, $a, b \in R$ $i = \sqrt{-1}$. State the values of a and b.

i) $(1+2i)(-2+i)$ ii) $(1+i)(1-i)^{-1}$

iii) $\dfrac{i(4+3i)}{(1-i)}$ iv) $\dfrac{(2+i)}{(3-i)(1+2i)}$

v) $\left(\dfrac{1+i}{1-i}\right)^2$ vi) $\dfrac{3+2i}{2-5i}+\dfrac{3-2i}{2+5i}$

vii) $(1+i)^{-3}$ viii) $\dfrac{2+\sqrt{-3}}{4+\sqrt{-3}}$

ix) $(-\sqrt{5}+2\sqrt{-4})+(1-\sqrt{-9})+(2+3i)(2-3i)$

x) $(2+3i)(2-3i)$ xi) $\dfrac{4i^8-3i^9+3}{3i^{11}-4i^{10}-2}$

5) Show that $(-1+\sqrt{3}i)^3$ is a real number.

6) Find the value of $\left(3+\dfrac{2}{i}\right)(i^6-i^7)(1+i^{11})$

7) Evaluate the following :

i) i^{35} ii) i^{888} iii) i^{93} iv) i^{116}

v) i^{403} vi) $\dfrac{1}{i^{58}}$ vii) i^{-888}

viii) $i^{30}+i^{40}+i^{50}+i^{60}$

8) Show that $1 + i^{10} + i^{20} + i^{30}$ is a real number.

9) Find the value of

i) $i^{49}+i^{68}+i^{89}+i^{110}$

ii) $i+i^2+i^3+i^4$

10) Simplify : $\dfrac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$

11) Find the value of $1 + i^2 + i^4 + i^6 + i^8 + \ldots + i^{20}$

12) Show that $1 + i^{10} + i^{100} - i^{1000} = 0$.

13) Is $(1 + i^{14} + i^{18} + i^{22})$ a real number? Justify your answer.

14) Evaluate : $\left(i^{37}+\dfrac{1}{i^{67}}\right)$

15) Prove that $(1+i)^4 \times \left(1+\dfrac{1}{i}\right)^4 = 16$.

16) Find the value of $\dfrac{i^6+i^7+i^8+i^9}{i^2+i^3}$

17) If $a = \dfrac{-1+\sqrt{3}i}{2}$, $b = \dfrac{-1-\sqrt{3}i}{2}$ then show that $a^2 = b$ and $b^2 = a$.

18) If $x+iy = (a+ib)^3$, show that $\dfrac{x}{a}+\dfrac{y}{b} = 4(a^2-b^2)$

19) If $\dfrac{a+3i}{2+ib} = 1-i$, show that $(5a-7b) = 0$.

20) If $x+iy = \sqrt{\dfrac{a+ib}{c+id}}$,

prove that $(x^2+y^2)^2 = \dfrac{a^2+b^2}{c^2+d^2}$

21) If $(a+ib) = \dfrac{1+i}{1-i}$, then prove that $(a^2+b^2) = 1$.

22) Show that $\left(\dfrac{\sqrt{7}+i\sqrt{3}}{\sqrt{7}-i\sqrt{3}}+\dfrac{\sqrt{7}-i\sqrt{3}}{\sqrt{7}+i\sqrt{3}}\right)$ is real.

23) If $(x+iy)^3 = y+vi$ the show that

$\left(\dfrac{y}{x}+\dfrac{v}{y}\right) = 4(x^2-y^2)$

24) Find the value of x and y which satisfy the following equations $(x, y \in R)$

i) $(x+2y) + (2x-3y)\,i + 4i = 5$

ii) $\dfrac{x+1}{1+i} + \dfrac{y-1}{1-i} = i$

iii) $\dfrac{(x+iy)}{2+3i} + \dfrac{2+i}{2+3i} = \dfrac{9}{13}(1+i)$

iv) If $x(1+3i) + y(2-i) - 5 + i^3 = 0$, find $x+y$

v) If $x+2i+15i^6y = 7x+i^3(y+4)$, find $x+y$

1.3 Square root of a complex number :

Consider $z = x+iy$ be any complex number

Let $\sqrt{x+iy} = a+ib$, a, b$\in$R

On squaring both the sides, we get

$x+iy = (a+ib)^2$

$x+iy = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we get

$x = (a^2-b^2)$ and $y = 2ab$

Solving these equations simultaneously, we can get the values of a and b.

Solved Examples:

Ex.1 : Find the square root of $6+8i$.

Solution :

Let, $\sqrt{6+8i} = a+ib$ (a, b$\in$R)

On squaring both the sides, we get

$6+8i = (a+ib)^2$

$\therefore 6+8i = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we have

$6 = a^2-b^2$ .... (1)

$8 = 2ab$ .... (2)

$\therefore a = \dfrac{4}{b}$

$\therefore$ (1) becomes $6 = \left(\dfrac{4}{b}\right)^2 - b^2$

i.e. $6 = \dfrac{16}{b^2} - b^2$

$\therefore b^4+6b^2-16 = 0$ i.e. $(b^2)^2 + 6b^2 - 16 = 0$

put $b^2 = m$

$\therefore m^2+6m-16 = 0$

$\therefore (m+8)(m-2) = 0$

$\therefore m = -8$ or $m = 2$

i.e. $b^2 = -8$ or $b^2 = 2$

but b is a real number $\therefore b^2 \neq -8$

So, $b^2 = 2$ $\therefore b = \pm\sqrt{2}$

For, $b = \sqrt{2}$, $a = 2\sqrt{2}$

$\therefore \sqrt{6+8i} = 2\sqrt{2} + \sqrt{2}\,i = \sqrt{2}\,(2+i)$

For, $b = -\sqrt{2}$, $a = -2\sqrt{2}$

$\therefore \sqrt{6+8i} = -2\sqrt{2} - \sqrt{2}\,i = -\sqrt{2}\,(2+i)$

$\therefore \sqrt{6+8i} = \pm\sqrt{2}\,(2+i)$

Ex. 2 : Find the square root of $3 - 4i$

Solution :

Let $\sqrt{3-4i} = a+ib$ a, b$\in$R

On squaring both the sides, we have

$3 - 4i = (a+ib)^2$

$\therefore 3 - 4i = (a^2-b^2) + (2ab)\,i$

Equating real and imaginary parts, we have

$a^2-b^2 = 3$, $2ab = -4$

As $(a^2+b^2)^2 = (a^2-b^2)^2 + (2ab)^2$

$(a^2+b^2)^2 = 3^2 + (-4)^2 = 9 + 16 = 25$

$(a^2+b^2)^2 = 5^2$

$\therefore a^2+b^2 = 5$

Solving $a^2+b^2 = 5$ and $a^2-b^2 = 3$ we get

$2a^2 = 8$

$a^2 = 4$

$\therefore$ $a = \pm 2$

For $a = 2$;

$b = \dfrac{-4}{2a} = \dfrac{-4}{2(2)} = -1$

For, $a = -2$, $b = \dfrac{-4}{2(-2)} = 1$

$\therefore \sqrt{3-4i} = 2 - i$ or $-2 + i$

1.4 Fundamental Theorem of Algebra :

'A polynomial equation with real coefficients has at least one root' in C.

or 'A polynomial equation with complex coefficients and of degree n has n complex roots'.

1.4.1 Solution of a Quadratic Equation in complex number system :

Let the given equation be $ax^2 + bx + c = 0$ where $a, b, c \in$ R and $a \neq 0$

$\therefore$ The solution of this quadratic equation is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Hence, the roots of the equation $ax^2 + bx + c = 0$

are $\dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $\dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$

The expression $(b^2 - 4ac)$ = D is called the discriminant.

If D < 0 then the roots of the given quadratic equation are complex.

Note : If $p + iq$ is the root of equation $ax^2 + bx + c = 0$ where $a, b, c \in$ R and $a \neq 0$ then $p - iq$ is also a solution of the given equation. Thus, complex roots occur in conjugate pairs.

Solved Examples :

Ex. 1 : Solve $x^2 + x + 1 = 0$

Solution : Given equation is $x^2 + x + 1 = 0$

Comparing with $ax^2 + bx + c = 0$ we get

$a = 1, \quad b = 1, c = 1$

These roots are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1 \pm \sqrt{-3}}{2}$$

$$= \frac{-1 \pm \sqrt{3}\, i}{2}$$

$\therefore$ Roots are $\dfrac{-1 + \sqrt{3}\, i}{2}$ and $\dfrac{-1 - \sqrt{3}\, i}{2}$

Ex. 2 : Solve $x^2 - (2\sqrt{3} + 3i)\, x + 6\sqrt{3}\, i = 0$

Solution : Given equation is

$x^2 - (2\sqrt{3} + 3i)\, x + 6\sqrt{3}\, i = 0$

The method of finding the roots of

$ax^2 + bx + c = 0$, is applicable even if a, b, c

are complex numbers. where $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Here, $a = 1$, $b = -(2\sqrt{3} + 3i)$, $c = 6\sqrt{3}\, i$

$b^2 - 4ac = [-(2\sqrt{3} + 3i)]^2 - 4 \times 1 \times 6\sqrt{3}\, i$

$= 12 - 9 + 12\sqrt{3}\, i - 24\sqrt{3}\, i$

$= 3 - 12\sqrt{3}\, i$

$= 3\,(1 - 4\sqrt{3}\, i)$

So, the given equation has complex roots. These roots are given by

$$x = \frac{\left(2\sqrt{3} + 3i\right) \pm \sqrt{3(1 - 4\sqrt{3}\, i)}}{2}$$

Now, we shall find $\sqrt{1 - 4\sqrt{3}\, i}$

Let $a + ib = \sqrt{1 - 4\sqrt{3}\, i}$

$\therefore\ a^2 - b^2 + 2iab = 1 - 4\sqrt{3}\, i$

$\therefore\ a^2 - b^2 = 1$ and $2ab = -4\sqrt{3}$

$a^2 - b^2 = 1$ and $ab = -2\sqrt{3}$

Consider $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4\, a^2b^2$

$= 1 + 4\,(12) = 49$

$\therefore\ a^2 + b^2 = 7$ ...(1)

and also $a^2 - b^2 = 1$ ...(2)

Solving (1) and (2)

$2a^2 = 8 \qquad \therefore\ a^2 = 4$

$\therefore\ a = \pm 2$ and $b = \pm\sqrt{3}$

$\therefore$ We have four choices

$a = 2, \quad b = \sqrt{3}$ ... (3)

$a = 2, \quad b = -\sqrt{3}$ ... (4)

$a = -2, \quad b = \sqrt{3} \quad \ldots (5)$

$a = -2, \quad b = -\sqrt{3} \quad \ldots (6)$

With this combination, the condition $ab = -2\sqrt{3}$ should also be satisfied.

We can verify the values of a and b given by (4) and (5) satisfy the condition and that from (3) and (6) do not satisfy the condition.

Hence, $a = 2, \quad b = -\sqrt{3}$

and $a = -2, \quad b = \sqrt{3}$

$\therefore \sqrt{1-4\sqrt{3}\,i} = \pm(2-\sqrt{3}\,i)$

$\therefore$ The roots are given by

$$x = \frac{\left(2\sqrt{3}+3i\right) \pm \sqrt{3}\left(2-\sqrt{3}\,i\right)}{2}$$

So, the roots are $2\sqrt{3}$ and $3i$.

Ex. 3 : Find the value of $x^3 - x^2 + 2x + 10$ when $x = 1 + \sqrt{3}\,i$.

Solution : Since $x = 1 + \sqrt{3}\,i$

$\therefore x-1 = \sqrt{3}\,i$

squaring both sides, we get

$(x-1)^2 = (\sqrt{3}\,i)^2$

$\therefore x^2 - 2x + 1 = 3i^2$

i.e. $x^2 - 2x + 1 = -3$

$\therefore x^2 - 2x = -4 \quad \ldots\ldots\ldots (I)$

$x^3 - x^2 + 2x + 10 = x^3 - (x^2 - 2x) + 10 \quad$ (By I)

$= x^3 - (-4) + 10 = x^3 + 14$

$= (1 + \sqrt{3}\,i)^3 + 14$

$= 1 + 3\sqrt{3}\,i - 9 - 3\sqrt{3}\,i + 14$

$= 6$

Ex. 4 : If $x = -5 + 2\sqrt{-4}$, find the value of $x^4 + 9x^3 + 35x^2 - x + 64$.

Solution : $x = -5 + 2\sqrt{-4} = -5 + 2(2i) = -5 + 4i$

Let $P(x) = x^4 + 9x^3 + 35x^2 - x + 64$

Form the quadratic expression $q(x)$ with real coefficients and roots $\alpha = -5+4i$ and $\overline{\alpha} = -5-4i$.

$\alpha\overline{\alpha} = (-5+4i)(-5-4i) = 25 + 16 = 41$

$\alpha + \overline{\alpha} = (-5+4i) + (-5-4i) = -10$

Therefore $q(x) = x^2 + 10x + 41$

Now $q(\alpha) = \alpha^2 + 10\alpha + 41 = 0 \quad \ldots(1)$

We divide the given polynomial $p(x)$ by $q(x)$.

$$\begin{array}{r|l}
 & x^2 - x + 4 \\
\hline
x^2+10x+41 & x^4 + 9x^3 + 35x^2 - x + 64 \\
 & x^4 + 10x^3 + 41x^2 \\
\hline
 & \quad -x^3 + 6x^2 - x \\
 & \quad -x^3 - 10x^2 - 41x \\
\hline
 & \quad\quad 4x^2 + 40x + 64 \\
 & \quad\quad 4x^2 + 40x + 164 \\
\hline
 & \quad\quad\quad -100
\end{array}$$

$\therefore \quad p(x) = (x^2 + 10x + 41)(x^2 - x + 4) - 100$

$\therefore \quad p(\alpha) = q(\alpha)(\alpha^2 - \alpha + 4) - 100$

$\therefore \quad p(\alpha) = 0 - 100 = -100 \quad$ (By I)

EXERCISE 1.2

1) Find the square root of the following complex numbers

i) $-8-6i$ ii) $7+24i$ iii) $1+4\sqrt{3}\,i$

iv) $3+2\sqrt{10}\,i$ v) $2(1-\sqrt{3}\,i)$

2) Solve the following quadratic equations.

i) $8x^2 + 2x + 1 = 0$

ii) $2x^2 - \sqrt{3}\,x + 1 = 0$

iii) $3x^2 - 7x + 5 = 0$

iv) $x^2 - 4x + 13 = 0$

3) Solve the following quadratic equations.

i) $x^2 + 3ix + 10 = 0$

ii) $2x^2 + 3ix + 2 = 0$

iii) $x^2 + 4ix - 4 = 0$

iv) $ix^2 - 4x - 4i = 0$

4) Solve the following quadratic equations.

i) $x^2 - (2+i)\,x - (1-7i) = 0$

ii) $x^2 - (3\sqrt{2} + 2i)\,x + 6\sqrt{2}\,i = 0$

iii) $x^2 - (5-i)\,x + (18+i) = 0$

iv) $(2 + i)x^2 - (5-i)\,x + 2\,(1-i) = 0$

5) Find the value of

iii) $x^3 - x^2 + x + 46$, if $x = 2+3i$.

iv) $2x^3 - 11x^2 + 44x + 27$, if $x = \dfrac{25}{3-4i}$.

v) $x^3 + x^2 - x + 22$, if $x = \dfrac{5}{1-2i}$.

vi) $x^4 + 9x^3 + 35x^2 - x + 4$, if $x = -5 + \sqrt{-4}$.

vii) $2x^4 + 5x^3 + 7x^2 - x + 41$, if $x = -2 - \sqrt{3}\,i$.

1.5 Argand Diagram or Complex Plane :

A complex number $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$ is expressed as a point in the plane whose co-ordinates are ordered pair (x, y). Jean Robert Argand used the one to one correspondence between a complex number and the points in a the plane.

Let $z = x + iy$ be a complex number.

Then the point $P(x, y)$ represents the complex number $z = x + iy$ (fig.1.2) i.e. $x + iy \equiv (x, y)$, $x = \text{Re}(z)$ is represented on the X-axis. So, X-axis is called the real axis. Similarly, $y = \text{Im}(z)$ is represented on the Y-axis, so the Y-axis is called the **imaginary axis.**

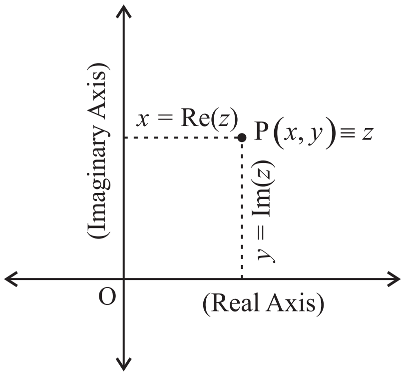


Fig. 1.2

e.g. (1) $(1, 2) \equiv 1 + 2i$ (2) $-4+3i \equiv (-4, 3)$
(3) $(0, 0) \equiv 0 + 0i$ (4) $5+0i \equiv (5, 0)$
(5) $(0, -1) \equiv 0 - i$ (6) $-2-2i \equiv (-2,-2)$

A diagram which represents complex numbers by points in a plane with reference to the real and imaginary axes is called **Argand's diagram on complex plane.**

1.5.1 Modulus of z :

If $z = a+ib$ is a complex number then the modulus of z, denoted by $|z|$ or r, is defined as $|z| = \sqrt{a^2 + b^2}$. (From fig. 1.3), point $P(a, b)$ represents the complex number $z = a+ib$.

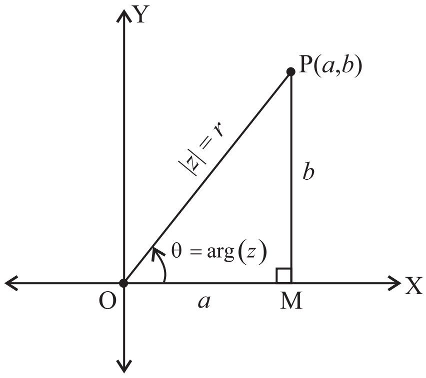
$$\therefore\ r = |z| = \sqrt{a^2 + b^2} = OP$$

Hence, modulus of z is the distance of point P from the origin where the point P represents the complex number z in the plane. e.g. For, $z = 4 + 3i$,

Modulus of z $= |z| = \sqrt{16+9} = \sqrt{25} = 5$

1.5.2 Argument of z :

OP makes an angle θ with positive direction of X-axis. θ is called the **argument** or **amplitude** of the complex number $z = a+ib$, denoted by arg(z).



Modulus and Argument of z

Fig. 1.3

$$\therefore \sin\theta = \frac{b}{r}, \quad \cos\theta = \frac{a}{r}, \quad r \neq 0$$

$$\therefore b = r\sin\theta, \quad a = r\cos\theta$$

and $\tan\theta = \dfrac{b}{a}$, if $a \neq 0$

$$\therefore \theta = \tan^{-1} = \left(\frac{b}{a}\right) = \arg(z),$$

e.g. If $z = 2+2i$ then

$$\arg(z) = \theta = \tan^{-1}\left(\frac{2}{2}\right)$$

$$\therefore \tan^{-1}(1) = \frac{\pi}{4}$$

Note : If $\tan x = y$ then its inverse funtion is given by $x = \tan^{-1} y$ or $x = \text{arc}\tan y$

eg:

1) As $\tan \dfrac{\pi}{6} = \dfrac{1}{\sqrt{3}}$ then $\tan^{-1} = \left(\dfrac{1}{\sqrt{3}}\right) = \dfrac{\pi}{6}$

2) As $\tan\left(-\dfrac{\pi}{4}\right) = -\tan\dfrac{\pi}{4} = -1$ then

$$\tan^{-1}(-1) = -\frac{\pi}{4}$$

1.5.3 Argument of z in different quadrants/axes :

$z = a + ib$	**Example**	**Quadrant/Axis**	$\theta = \arg z$ $= \tan^{-1}\left(\frac{b}{a}\right)$, $(0 \leq \theta < 2\pi)$	**from Example**
$a > 0,\ b = 0$	$z = 3$	On positive real (X) axis	$\theta = 0$	$\theta = 0$
$a > 0,\ b > 0$	$z = 1 + i$	In quadrant I	$\theta = \tan^{-1}\left(\frac{b}{a}\right)$, $(0 < \theta < \frac{\pi}{2})$	$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$
$a = 0,\ b < 0$	$z = 5i$	On Positive imaginary (Y) axis	$\theta = \frac{\pi}{2}$	$\theta = \frac{\pi}{2}$
$a < 0,\ b > 0$	$z = -\sqrt{3} + i$	In quadrant II	$\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$ $(\frac{\pi}{2} < \theta < \pi)$	$\theta = \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi$ $= \frac{-\pi}{6} + \pi = \frac{5\pi}{6})$
$a < 0,\ b = 0$	$z = -6$	On negative real (X) axis	$\theta = \pi$	$\theta = \pi$

$a < 0, b < 0$	$z = -1 - \sqrt{3i}$	In quadrant III	$\theta = \tan^{-1}\left(\frac{b}{a}\right) + \pi$ $(\pi < \theta < \frac{3\pi}{2})$	$\theta = \tan^{-1}\left(\frac{-\sqrt{3}}{-1}\right) + \pi$ $= \frac{\pi}{3} + \pi = \frac{4\pi}{3})$
$a = 0, b < 0$	$z = -2i$	On negative imaginary (Y) axis	$\theta = \frac{3\pi}{2}$	$\theta = \frac{3\pi}{2}$
$a > 0, b < 0$	$z = 1 - i$	In quadrant IV	$\theta = \tan^{-1}\left(\frac{b}{a}\right) + 2\pi$ $(\frac{3\pi}{2} < \theta < 2\pi)$	$\theta = \tan^{-1}\left(\frac{-1}{1}\right) + 2\pi$ $= \frac{-\pi}{4} + 2\pi = \frac{3\pi}{4})$

SOLVED EXAMPLES

Ex. 1 : If $z = 1+3i$, find the modulus and amplitude of z.

Solution : $z = 1+3i$ here a=1, b=3 and a, b > 0

$$\therefore \; |z| = \sqrt{1^2+3^2} = \sqrt{10}$$

$$\text{amp } z = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{3}{1}\right) = \tan^{-1}(3)$$

Ex. 2 : Find the modulus, argument of the complex number $-7 + 24i$.

Solution : let $z = -7+24i$ a = −7, b = 24

$$\therefore |z| = \sqrt{(7)^2+(24)^2} = \sqrt{625} = 25$$

Here,

$$\arg z = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{24}{-7}\right)$$

As a < 0, b > 0, θ lies in 2nd quadrant.

Properties of modulus of complex numbers :

If z_1, z_2, z_3 are complex numbers, then

i) $|z| = 0 \Leftrightarrow z = 0$ i.e. $\text{Re}(z) = \text{I}m(z) = 0$

ii) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

iii) $-|z| \le \text{Re}(z) \le |z|; -|z| \le \text{I}m(z) \le |z|$

iv) $z\bar{z} = |z|^2$

v) $|z_1 z_2| = |z_1|\,|z_2|$

vi) $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}, \; z_2 \ne 0$

vii) $|z_1 + z_2| \le |z_1| + |z_2|$

viii) $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z_2})$

ix) $|z_1-z_2|^2 = |z_1|^2 + |z_2|^2 - 2\text{Re}(z_1\bar{z_2})$

x) $|z_1+z_2|^2 + |z_1-z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

xi) $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$

$= (a^2+b^2)\;(|z_1|^2 + |z_2|^2)$ where $a, b \in R$

Properties of arguments :

1) $\arg(z_1 . z_2) = \arg z_1 + \arg z_2$

2) $\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$

3) $\arg z + \arg \bar{z} = 0, \; \arg(z\bar{z}) = 0$

4) $\arg \bar{z} = -\arg z = \arg \frac{1}{z}$

5) If $\arg z = 0$ then z is real

1.5.4 Polar form of a complex number :

Let the complex number z = a+ib be represented by the point P(a, b) (see fig 1.4)

Let $m\angle XOP = \theta = \tan^{-1}\left(\frac{b}{a}\right)$ and $l(OP) = r = \sqrt{a^2+b^2} > 0$, then P($r$, θ) are called the Polar Co-ordinates of P.

We call the origin as pole. (figure 1.4)

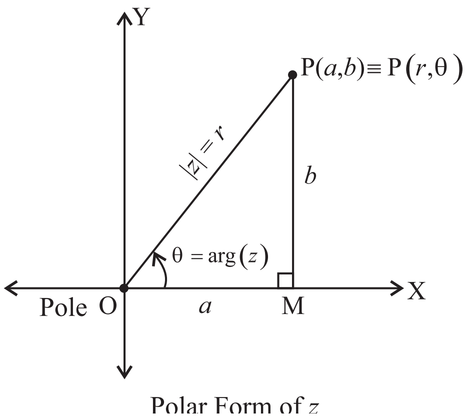


Fig.1.4

As $a = r\cos\theta$, $b = r\sin\theta$

$\therefore$ $z = a + ib$ becomes

$z = r\cos\theta + ir\sin\theta$

$\therefore$ $z = r(\cos\theta + i\sin\theta)$

This is called polar form of complex number $z = a + ib$

1.5.5 Exponential form :

It is known and can be proved using special series that $e^{i\theta} = \cos\theta + i\sin\theta$

$\therefore$ $z = a + ib = r(\cos\theta + i\sin\theta) = r\, e^{i\theta}$

where $r = |z|$ and $\theta = \arg z$ is called an exponential form of complex number.

Solved Example:

Ex. 1 : Represent the complex numbers $z = 1+i$, $\bar{z} = 1-i$, $-\bar{z} = -1+i$, $-z = -1-i$ in Argand's diagram and hence find their arguments from the figure.

Solution :

arg z is the angle made by the segment OA with the positive direction of the X-axis. (Fig.1.5)

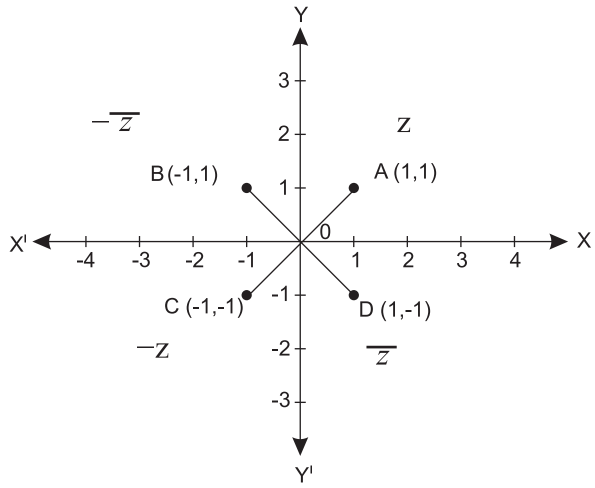


Fig. 1.5

Thus, θ together with r gives the position of the point A in the Argand's diagram.

Hence, from the figure 1.5,

$$\arg z = 45° = \frac{\pi}{4}, \quad \arg(-\bar{z}) = 135° = \frac{3\pi}{4}$$

$$\arg(-z) = 225° = \frac{5\pi}{4}, \quad \arg \bar{z} = 315° = \frac{7\pi}{4}$$

Ex. 2 : Represent the following complex numbers in the polar form and in the exponential form

i) $4+4\sqrt{3}\,i$ ii) -2 iii) $3i$ iv) $-\sqrt{3}+i$

Solution :

i) Let, $z = 4 + 4\sqrt{3}\,i$

$a = 4$, $b = 4\sqrt{3}$

$$r = \sqrt{4^2 + (4\sqrt{3})^2} = \sqrt{16+48} = \sqrt{64} = 8$$

As θ lies in quadrant I

$$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \tan^{-1}\left(\frac{4\sqrt{3}}{4}\right)$$

$$= \tan^{-1}(\sqrt{3}) = \frac{\pi}{3} \text{ or } 60°$$

$\therefore$ The polar form of $z = 4 + 4\sqrt{3}\,i$ is

$z = r(\cos\theta + i\sin\theta)$

$z = 8(\cos 60° + i\sin 60°)$

$= 8\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$\therefore$ The exponential form of $z = 8\,e^{i\left(\frac{\pi}{3}\right)}$

ii) Let $z = -2$

$\therefore$ $a = -2,\ b = 0$

Hence, $r = \sqrt{(-2)^2 + 0^2} = \sqrt{4} = 2$

As point $(-2, 0)$ lies on negative real axis $\theta = \pi^c$ or $180°$

$\therefore$ The polar form of $z = 2(\cos 180° + i\sin 180°)$
$= 2(\cos\pi + i\sin\pi)$

$\therefore$ The exponential form of $z = 2e^{i\pi}$

iii) Let $z = 3i$

$a=0,\ b=3$

Hence, $r = \sqrt{0^2 + 3^2} = 3$

As point $(0, 3)$ lies on positive imaginary axis

$\theta = \frac{\pi^c}{2}$ or $90°$

$\therefore$ The polar form of $z = 3(\cos 90° + i\sin 90°)$

$= 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$\therefore$ The exponential form of $z = 3e^{i\frac{\pi}{2}}$

iv) Let, $z = -\sqrt{3} + i$

$\therefore a = -\sqrt{3},\quad b = 1$

$r = \sqrt{(-\sqrt{3})^2 + 1^2} = \sqrt{3+1} = \sqrt{4} = 2$

As $(-\sqrt{3}, 1)$ lies in quadrant II

$\theta = \tan^{-1}\left(\frac{b}{a}\right) = \pi$

$= \tan^{-1}\left(\frac{1}{-\sqrt{3}}\right) + \pi = -\frac{\pi}{6} + \pi$

$= \frac{5\pi}{6}$

$\therefore$ The polar form of

$z = r(\cos\theta + i\sin\theta)$

$z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$

$\therefore$ The exponential form of

$z = re^{i\theta} = 2\,e^{i\left(\frac{5\pi}{6}\right)}$

Ex. 3 : Express $z = \sqrt{2}.e^{\frac{3\pi}{4}i}$ in the $a + ib$ form.

Solution: $z = \sqrt{2}.e^{\frac{3\pi}{4}i} = r\,e^{i\theta}$

$\therefore$ $r = \sqrt{2},\ \theta = \frac{3\pi}{4}$

As the polar form of z is

$z = r(\cos\theta + i\sin\theta)$

$= \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

By using allied angles results in trigonometry, we get

$\cos\frac{3\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{1}{\sqrt{2}}$

$\sin\frac{3\pi}{4} = \sin\left(\pi - \frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\therefore$ $z = \sqrt{2}\left[\frac{-1}{\sqrt{2}} + i\left(\frac{1}{\sqrt{2}}\right)\right]$

$= -1 + i$

Ex. 4 : Express (i) $3.e^{\frac{5\pi}{12}i} \times 4.e^{\frac{\pi}{12}i}$

ii) $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$ in $a + ib$ form

Solution: (i) $3.e^{\frac{5\pi}{12}i} \times 4.e^{\frac{\pi}{12}i}$

$$= (3 \times 4)\ e^{\left(\frac{5\pi}{12}+\frac{\pi}{12}\right)i}$$

$$= 12\ e^{\frac{6\pi}{12}i} = 12\ e^{\frac{\pi}{2}i}$$

$$= 12\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$$

$$= 12\ (0 + i) = 12i$$

(ii) $$\frac{\sqrt{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)} = \frac{\sqrt{2}\,e^{\frac{\pi}{12}i}}{2e^{\frac{5\pi}{6}i}}$$

$$= \left(\frac{\sqrt{2}}{2}\right)e^{\left(\frac{\pi}{12}-\frac{5\pi}{6}\right)i} = \left(\frac{\sqrt{2}}{2}\right)e^{\left(-\frac{3\pi}{4}\right)i}$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left[\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right]$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left[-\frac{1}{\sqrt{2}}+i\left(-\frac{1}{\sqrt{2}}\right)\right]$$

$$= -\frac{1}{2}-\frac{i}{2}$$

EXERCISE 1.3

1) Find the modulus and amplitude for each of the following complex numbers.

i) $7 - 5i$ ii) $\sqrt{3}+\sqrt{2}i$ iii) $-8 + 15i$

iv) $-3(1-i)$ v) $-4-4i$ vi) $\sqrt{3}-i$

vii) 3 viii) $1 + i$ iv) $1 + i\sqrt{3}$

x) $(1+2i)^2\ (1-i)$

2) Find real values of θ for which $\left(\frac{4+3i\sin\theta}{1-2i\sin\theta}\right)$ is purely real.

3) If $z = 3 + 5i$ then represent the $z,\ \bar{z}, -z, -\bar{z}$ in Argand's diagram.

4) Express the following complex numbers in polar form and exponential form.

i) $-1+\sqrt{3}\,i$ ii) $-i$ iii) -1 iv) $\frac{1}{1+i}$

v) $\frac{1+2i}{1-3i}$ vi) $\frac{1+7i}{(2-i)^2}$

5) Express the following numbers in the form $x+iy$

i) $\sqrt{3}\,(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

ii) $\sqrt{2}\,(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4})$

iii) $7\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$

iv) $e^{\frac{\pi}{3}i}$ v) $e^{\frac{-4\pi}{3}i}$ vi) $e^{\frac{5\pi}{6}i}$

6) Find the modulus and argument of the complex number $\frac{1+2i}{1-3i}$.

7) Convert the complex number

$$z = \frac{i-1}{\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}}$$ in the polar form.

8) For $z = 2+3i$ verify the following :

i) $\overline{(\bar{z})} = z$ ii) $z\bar{z} = |z|^2$

iii) $(z+\bar{z})$ is real iv) $z - \bar{z} = 6i$

9) $z_1 = 1 + i,\ z_2 = 2 - 3i$. Verify the following :

i) $\overline{z_1+z_2} = \overline{z_1} + \overline{z_2}$

ii) $\overline{z_1-z_2} = \overline{z_1} - \overline{z_2}$

iii) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

iv) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$

1.6 De Moivres Theorem:

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$

Then $z_1 . z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2})$

$= r_1 . r_2 \, e^{i(\theta_1 + \theta_2)}$

That is if two complex numbers are multiplied then their modulii get multiplied and arguments get added.

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) . e^{i(\theta_1 - \theta_2)}$$

That is, if one complex numbers is divided by the other, then their modulii get divided and arguments get substracted.

In 1730, De Moivre propsoed the following theorem for finding the power of a complex number $z = r(\cos\theta + i \sin\theta)$, as $[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)]$

for any $n \in z$.

The proof of this theorem can be given using the Method of Induction (Chapter 4).

For example:

i) $(\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta)$

ii) $(\cos\theta + i\sin\theta)^{-1} = \cos(-\theta) + i\sin(-\theta)$

iii) $(\cos\theta + i\sin\theta)^{\frac{2}{3}} = \cos\left(\frac{2}{3}\theta\right) + i\sin\left(\frac{2}{3}\theta\right)$

Solved Examples:

Ex. 1 Use De Moivres Theorom and simplify.

i) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

ii) $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

iii) $\dfrac{(\cos 5\theta + i\sin 5\theta)^2}{(\cos 4\theta - i\sin 4\theta)^3}$

Solution: (i) $\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^8$

$= \cos\left(8\times\frac{\pi}{3}\right) + i\sin\left(8\times\frac{\pi}{3}\right)$

$= \cos\left(\frac{2\pi}{3}\right) - i\sin\left(\frac{2\pi}{3}\right)$

$\left(\because \frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}\right)$

$= \cos\left(\pi - \frac{\pi}{3}\right) + i\sin\left(\pi - \frac{\pi}{3}\right)$

$= -\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2}$

(ii) $\left(\cos\frac{\pi}{10} - i\sin\frac{\pi}{10}\right)^{15}$

$= \cos\left(\frac{15\pi}{10}\right) - i\sin\left(\frac{15\pi}{10}\right)$

$= \cos\left(\frac{3\pi}{2}\right) - i\sin\left(\frac{3\pi}{2}\right)$

$= \cos\left(\pi + \frac{\pi}{2}\right) - i\sin\left(\pi + \frac{\pi}{2}\right)$

$= -\cos\frac{\pi}{2} - i\left(-\sin\frac{\pi}{2}\right)$

$= -0 - i(-1) = i$

(iii) $\dfrac{(\cos 5\theta + i\sin 5\theta)^2}{(\cos 4\theta - i\sin 4\theta)^3}$

$= \dfrac{\cos 10\theta + i\sin 10\theta}{\cos 12\theta - i\sin 12\theta}$

$= \dfrac{\cos 10\theta + i\sin 10\theta}{\cos(-12\theta) + i\sin(-12\theta)}$

$= \cos[10\theta - (-12\theta)] + i\sin[10\theta - (-12\theta)]$

$= \cos 22\theta + i\sin 22\theta$

Ex. 2 Express $(1 + i)^4$ in $a + ib$ form.

Solution : Let $z = 1 + i$

$\therefore\ x = 1,\ y = 1$

$r = \sqrt{1+1} = \sqrt{2}$,

$\theta = \tan^{-1} \dfrac{x}{y}$

$= \tan^{-1} 1 = \dfrac{\pi}{4}$

$\therefore z = r(\cos\theta + i\sin\theta) = \sqrt{2}\left[\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right]$

$\therefore z^4 = (1+i)^4 = \sqrt{2}\left[\cos\dfrac{\pi}{4} + i\sin\dfrac{\pi}{4}\right]^4$

$= \sqrt{2}\left[\cos\dfrac{4\pi}{4} + i\sin\dfrac{4\pi}{4}\right]$

$= \sqrt{2}\left[\cos\pi + i\sin\pi\right]$

$= \sqrt{2}\left[-1+i(0)\right]$

$= \sqrt{2}\left[-1\right] = -\sqrt{2} = -\sqrt{2} + 0i$

1.7 Cube roots of unity :

Number 1 is often called unity. Let x be the cube root of unity i.e. 1

$\therefore \quad x^3 = 1$

$\therefore \quad x^3 - 1 = 0$

$\therefore \quad (x-1)(x^2+x+1) = 0$

$\therefore \quad x-1 = 0$ or $x^2+x+1 = 0$

$\therefore \quad x = 1$ or $x = \dfrac{-1 \pm \sqrt{(1)^2 - 4\times1\times1}}{2\times1}$

$\therefore \quad x = 1$ or $x = \dfrac{-1 \pm \sqrt{-3}}{2}$

$\therefore \quad x = 1$ or $x = \dfrac{-1 \pm i\sqrt{3}}{2}$

$\therefore$ Cube roots of unity are

$1, \dfrac{-1+i\sqrt{3}}{2}, \dfrac{-1-i\sqrt{3}}{2}$

Among the three cube roots of unity, one is real and other two roots are complex conjugates of each other.

$= \left(\dfrac{-1+i\sqrt{3}}{2}\right)^2$

$= \dfrac{1}{4}[(-1)^2+2\times(-1)\times i\sqrt{3} + (i\sqrt{3})^2]$

$= \dfrac{1}{4}(1-2i\sqrt{3}-3)$

$= \dfrac{1}{4}(-2-2i\sqrt{3})$

$= \dfrac{-1-i\sqrt{3}}{2}$

Similarly it can be verified that $\left(\dfrac{-1-i\sqrt{3}}{2}\right)^2$

$= \dfrac{-1+i\sqrt{3}}{2}$

Thus cube roots of unity are 1,

$\dfrac{-1+i\sqrt{3}}{2}, \left(\dfrac{-1+i\sqrt{3}}{2}\right)^2$

Let $\dfrac{-1+i\sqrt{3}}{2} = w$, then $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^2 = w^2$

Hence, cube roots of unity are $1, w, w^2$

where $w = \left(\dfrac{-1+i\sqrt{3}}{2}\right)$ and $w^2 = \left(\dfrac{-1+i\sqrt{3}}{2}\right)^2$

Also note that $1 = e^{2\pi i}$, $w = e^{\frac{2\pi}{3}i}$, $w^2 = e^{\frac{4\pi}{3}i}$

Properties of $1, w, w^2$

i) w is complex cube root of 1.

$\therefore w^3 = 1$

ii) $w^3 - 1 = 0$

i.e. $(w-1)(w^2+w+1) = 0$

$\therefore w = 1$ or $w^2+w+1 = 0$

but $w \neq 1$

$\therefore w^2+w+1 = 0$

iii) $w^2 = \dfrac{1}{w}$ and $\dfrac{1}{w^2} = w$

iv) $w^3 = 1$ so $w^{3n} = 1$

v) $w^4 = w^3 . w = w$ so $w^{3n+1} = w$

vi) $w^5 = w^2 . w^3 = w^2 . 1 = w^2$ So $w^{3n+2} = w^2$

vii) $\overline{w} = w^2$

viii) $\overline{w}^2 = w$

1.8 Set of points in complex plane

If $z = x + iy$ represents the variable point $P(x,y)$ and $z_1 = x_1 + iy_1$, represents the fixed point A (x_1,y_1) then (i) $|z-z_1|$ represents the length of AP

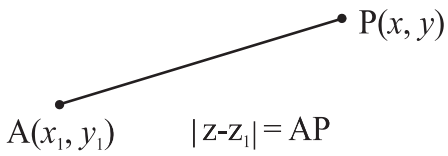


Fig. 1.6

(2) $|z-z_1| = a$ represents the circle with centre $A(x_1,y_1)$ and radius a.

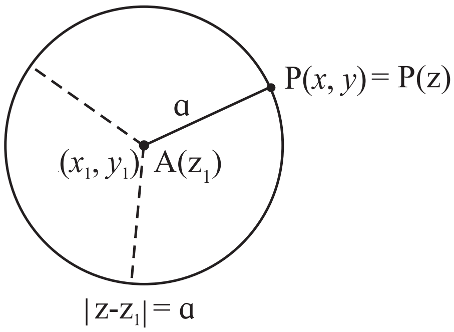


Fig. 1.7

(3) $|z-z_1| = |z-z_2|$ represents the perpendicular bisector of the line joining the points A and B.

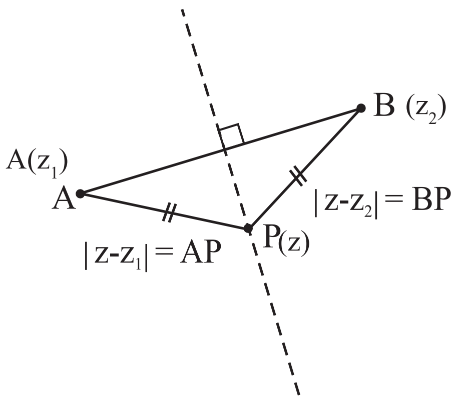


Fig. 1.8

Illustration:

For $z_1 = 2+3i$, $z_2 = 1+i$ and $z = x+iy$

(i) $|z-z_1| = |(x+iy)-(2+3i)| = |x+iy-2-3i|$

$= |(x-2)+i(y-3)| = \sqrt{(x-2)^2+(y-3)^2}$

represents the distance between (x,y) and $(2,3)$.

(ii) If $|z-z_1| = 5$, then

$|(x+iy)-(-1+i)| = |(x+1)+i(y-1)| = 5$

$\therefore \sqrt{(x+1)^2+(y-1)^2} = 5$

$\therefore (x+1)^2+(y-1)^2 = 5^2$

represents the circle with centre $(-1,1)$ and radius 5.

(iii) If $|z-z_1| = |z-z_2|$ then

$|(x+iy)-(2+3i)| = |(x+iy)-(-1+i)|$

$\therefore\ |(x-2)+i(y-3)| = |(x+1)+i(y-1)|$

$\therefore (x-2)^2+(y-3)^2 = (x+1)^2+(y-1)^2$

$\therefore\ x^2-4x+4+y^2-6y+9$

$= x^2+2x+1+y^2-2y+1$

$\therefore\ -6x+4y+11=0 \quad i.e. \quad 6x+4y-11=0$

represents the perpendicular bisector of line joining points $(2,3)$ and $(-1,1)$.

SOLVED EXAMPLES

Ex. 1 : If w is a complex cube root of unity, then prove that

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = -1$

ii) $(1+w^2)^3 = -1$

iii) $(1-w+w^2)^3 = -8$

Solution : Given, w is a complex cube root of unity.

$\therefore\ w^3 = 1$ Also $w^2+w+1 = 0$

$\therefore\ w^2+1 = -w$ and $w+1 = -w^2$

i) $\dfrac{1}{w} + \dfrac{1}{w^2} = \dfrac{w+1}{w^2} = \dfrac{-w^2}{w^2} = -1$

ii) $(1+w^2)^3 = (-w)^3 = -w^3 = -1$

iii) $(1-w+w^2)^3 = (1+w^2-w)^3$

$= (-w-w)^3 \quad (\because 1+w^2 = -w)$

$= (-2w)^3 = -8w^3 = -8 \times 1 = -8$

Ex. 2 : If w is a complex cube root of unity, then show that

i) $(1-w+w^2)^5 + (1+w-w^2)^5 = 32$

ii) $(1-w)(1-w^2)(1-w^4)(1-w^5) = 9$

Solution :

i) Since $w^3 = 1$

and $w \neq 1\ \therefore\ w^2+w+1 = 0$

Also $w^2+1 = -w,\ w^2+w = -1$

and $w+1 = -w^2$

Now, $(1-w+w^2)^5$

$= (-w-w)^5 \quad (\because 1+w^2 = -w)$

$= (-2w)^5$

$= -32w^5$

$(1+w-w^2)^5 = (-w^2-w^2)^5$

$= (-2w^2)^5$

$= -32w^{10}$

$\therefore\ (1-w+w^2)^5 + (1+w-w^2)^5 = -32w^5 - 32w^{10}$

$= -32w^5(1+w^5)$

$= -32w^5 \times (-w) = 32w^6 = 32(w^3)^2$

$= 32 \times (1)^2 = 32$

ii) $(1-w)(1-w^2)(1-w^4)(1-w^5)$

$= (1-w)(1-w^2)(1-w^3.w)(1-w^3.w^2)$

$= (1-w)(1-w^2)(1-w)(1-w^2)$

$= (1-w)^2(1-w^2)^2$

$= [(1-w)(1-w^2)]^2$

$= (1-w^2-w+w^3)^2$

$= [1-(w^2+w)+1]^2$

$= [1-(-1)+1]^2 = (1+1+1)^2 = (3)^2 = 9$

Ex. 3 : If w is a complex cube root of unity such that x=a+b, y=aw+bw^2 and z=aw^2+bw, a, b$\in$R prove that

i) $x+y+z=0$ ii) $x^3+y^3+z^3=3(a^3+b^3)$

Solution : Since w is a complex cube root of unity

$\therefore\ w^3 = 1$ and $w^2+w+1 = 0$ but $w \neq 1$ given

$\therefore\ w^2+1 = -w,\ w+1 = -w^2,\ w^3 = w^6 = 1$

i) $x+y+z = a+b+aw+bw^2+aw^2+bw$

$= a(1+w+w^2)+b(1+w+w^2)$

$= a.0 + b.0$

$= 0$

ii) $x^3 = a^3+3a^2b+3ab^2+b^3$

$y^3 = a^3w^3+3a^2bw^4+3ab^2w^5+b^3w^6$

$z^3 = a^3w^6+3a^2bw^5+3ab^2w^4+b^3w^3$

adding the above three equations

$x^3+y^3+z^3 = a^3(1+w^3+w^6)+3a^2b(1+w+w^2) + 3ab^2(1+w^2+w)+b^3(1+w^6+w^3)$

$= 3a^3 + 3a^2b(0) + 3ab^2(0) + 3b^3$

$= 3(a^3 + b^3)$

Ex. 4 : Prove that

$1+w^n+w^{2n} = 3$, if n is multiple of 3

$1+w^n+w^{2n} = 0$, if n is not multiple of 3, n$\in$N

Solution : for n, k$\in$N, if n is multiple of 3 then n=3k and if n is not multiple of 3 then n=3k+1 or 3k+2

$\therefore$ if n is multiple of 3

i.e. n=3k

then $1+w^n+w^{2n} = 1+w^{3k}+w^{2\times 3k}$

$= 1+(w^3)^k+(w^3)^{2k}$

$= 1+(1)^k+(1)^{2k}$

$=1+1+1$

$= 3$

If $n = 3k + r$, $r = 1,2$.

As w is complex root of Unity.

w^r, $r = 1,2$ is also complex root of Unity.

$\therefore\ 1 + w^r + w^{2r} = 0$

we have, $1+w^n+w^{2n}=0$, if n is not a multiple of 3.

EXERCISE 1.4

1) Find the value of

i) w^{18} ii) w^{21} iii) w^{-30} iv) w^{-105}

2) If w is a complex cube root of unity, show that

i) $(2-w)(2-w^2) =7$

ii) $(1+w-w^2)^6 = 64$

iii) $(1+w)^3-(1+w^2)^3 = 0$

iv) $(2+w+w^2)^3 -(1-3w+w^2)^3 = 65$

v) $(3+3w+5w^2)^6 -(2+6w+2w^2)^3 = 0$

vi) $\dfrac{a+bw+cw^2}{c+aw+bw^2} = w^2$

vii) $(a + b) + (aw + bw^2) + (aw^2 + bw) = 0$

viii) $(a - b)(a - bw)(a - bw^2) = a^3 - b^3$

ix) $(a + b)^2 + (aw + bw^2)^2 + (aw^2 + bw)^2 = 6ab$

3) If w is a complex cube root of unity, find the value of

i) $w+\dfrac{1}{w}$ ii) $w^2+w^3+w^4$ iii) $(1+w^2)^3$

iv) $(1-w-w^2)^3 + (1-w+w^2)^3$

v) $(1+w)(1+w^2)(1+w^4)(1+w^8)$

4) If α and β are the complex cube root of unity, show that

(a) $\alpha^2+\beta^2+\alpha\beta = 0$

(b) $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1} = 0$

5) If $x = a+b$, $y = \alpha a+\beta b$ and $z = a\beta+b\alpha$ where α and β are the complex cube-roots of unity, show that $xyz = a^3+b^3$

6) Find the equation in cartesian coordinates of the locus of z if

(i) $|z| = 10$ (ii) $|z - 3| = 2$

(iii) $|z - 5 + 6i| = 5$ (iv) $|z + 8| = |z - 4|$

(v) $|z - 2 - 2i| = |z + 2 + 2i|$

(vi) $\dfrac{|z+3i|}{|z-6i|} = 1$

7) Use De Moivres theorem and simplify the following

i) $\dfrac{(\cos 2\theta + i\sin 2\theta)^7}{(\cos 4\theta + i\sin 4\theta)^3}$

ii) $\dfrac{\cos 5\theta + i\sin 5\theta}{(\cos 3\theta - i\sin 3\theta)^2}$

iii) $\dfrac{\left(\cos\dfrac{7\pi}{13} + i\sin\dfrac{7\pi}{13}\right)^4}{\left(\cos\dfrac{4\pi}{13} - i\sin\dfrac{4\pi}{13}\right)^6}$

8) Express the following in the form $a + ib$, $a, b \in R$, using De Moivre's theorem.

i) $(1 - i)^5$ ii) $(1 + i)^6$ iii) $(1 - \sqrt{3}\,i)^4$

iv) $(-2\sqrt{3} - 2i)^5$

Let's Remember

- A number $a+ib$, where $a, b \in R$ and $i = \sqrt{-1}$, is a complex number.
- Let $z_1 = a+ib$ and $z_2 = c+id$. Then

  $z_1 + z_2 = (a+c) + (b+d)i$

$z_1 - z_2 = (a-c) + (b-d)i$

$z_1 . z_2 = (ac-bd) + (ad+bc)i$

$$\frac{z_1}{z_2} = \left(\frac{ac+bd}{c^2+d^2}\right) + \left(\frac{bc-ad}{c^2+d^2}\right)i$$

- For any non-zero complex number $z = a+ib$ $\frac{1}{z} = \frac{a}{a^2+b^2} + i\,\frac{-b}{a^2+b^2}$
- For any positive integer k, $i^{4k} = 1,\ i^{4k+1} = i,\ i^{4k+2} = -1,\ i^{4k+3} = -i$
- The conjugate of $z = a+ib$ is $\overline{z}$, is given by $\overline{z} = a-ib$
- The polar form of the complex number $z = x+iy$ is $r(\cos\theta + i\sin\theta) = r\,e^{i\theta}$ where $r = \sqrt{x^2+y^2}$ is called modulus and $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ (θ is called argument of z) $\arg(z) = \tan^{-1}\left(\frac{y}{x}\right)$.
- w is complex cute root of unity then $1 + w + w^2 = 0$, $w^3 = 1$.

MISCELLANEOUS EXERCISE - 1

I) Select the correct answer from the given alternatives.

9) If n is an odd positive integer then the value of $1 + (i)^{2n} + (i)^{4n} + (i)^{6n}$ is :

A) $-4i$ B) 0 C) $4i$ D) 4

10) The value of is $\frac{i^{592}+i^{590}+i^{588}+i^{586}+i^{584}}{i^{582}+i^{580}+i^{578}+i^{576}+i^{574}}$ is equal to :

A) -2 B) 1 C) 0 D) -1

3) $\sqrt{-3}\,\sqrt{-6}$ is equal to

A) $-3\sqrt{2}$ B) $3\sqrt{2}$ C) $3\sqrt{2}\,i$ D) $-3\sqrt{2}\,i$

4) If w is a complex cube root of unity, then the value of $w^{99}+w^{100}+w^{101}$ is :

A) -1 B) 1 C) 0 D) 3

5) If $z=r(\cos\theta+i\sin\theta)$, then the value of $\frac{z}{\overline{z}}+\frac{\overline{z}}{z}$

A) $\cos 2\theta$ B) $2\cos 2\theta$ C) $2\cos\theta$ D) $2\sin\theta$

6) If $w(\neq 1)$ is a cube root of unity and $(1+w)^7 = A + Bw$, then A and B are respectively the numbers

A) 0, 1 B) 1,1 C) 1, 0 D) -1,1

7) The modulus and argument of $(1+i\sqrt{3})^8$ are respectively

A) 2 and $\frac{2\pi}{3}$ B) 256 and $\frac{8\pi}{3}$

C) 256 and $\frac{2\pi}{3}$ D) 64 and $\frac{4\pi}{3}$

8) If $\arg(z) = \theta$, then $\arg\overline{(z)}$ =

A) $-\theta$ B) θ C) $\pi-\theta$ D) $\pi+\theta$

9) If $-1+\sqrt{3}\,i = re^{i\theta}$, then θ = ...

A) $-\frac{2\pi}{3}$ B) $\frac{\pi}{3}$ C) $-\frac{\pi}{3}$ D) $\frac{2\pi}{3}$

10) If $z = x+iy$ and $|z-zi| = 1$ then

A) z lies on x-asis B) z lies on y-asis

D) z lies on a rectangle C) z lies on a circle

II) Answer the following.

1) Simplify the following and express in the form a+ib.

i) $3+\sqrt{-64}$ ii) $(2i^3)^2$ iii) $(2+3i)(1-4i)$

iv) $\frac{5}{2}i(-4-3i)$ v) $(1+3i)^2(3+i)$ vi) $\frac{4+3i}{1-i}$

vii) $(1+\frac{2}{i})(3+\frac{4}{i})(5+i)^{-1}$ viii) $\frac{\sqrt{5}+\sqrt{3}i}{\sqrt{5}-\sqrt{3}i}$

ix) $\frac{3i^5+2i^7+i^9}{i^6+2i^8+3i^{18}}$ x) $\frac{5+7i}{4+3i} + \frac{5+7i}{4-3i}$

2) Solve the following equations for $x, y \in R$

i) $(4-5i)x + (2+3i)y = 10-7i$

ii) $\dfrac{x+iy}{2+3i} = 7-i$

iii) $(x+iy)\ (5+6i) = 2+3i$

iv) $2x+i^9y\ (2+i) = xi^7+10i^{16}$

11) Evaluate i) $(1-i+i^2)^{-15}$ ii) $(i^{131}+i^{49})$

12) Find the value of

i) $x^3+2x^2-3x+21$, if $x = 1+2i$.

ii) $x^4+9x^3+35x^2-x+164$, if $x = -5+4i$.

13) Find the square roots of

i) $-16+30i$ ii) $15-8i$ iii) $2+2\sqrt{3}\,i$

iv) $18i$ v) $3-4i$ vi) $6+8i$

14) Find the modulus and argument of each complex number and express it in the polar form.

i) $8+15i$ ii) $6-i$ iii) $\dfrac{1+\sqrt{3}i}{2}$ iv) $\dfrac{-1-i}{\sqrt{2}}$

v) $2i$ vi) $-3i$ vii) $\dfrac{-1}{\sqrt{2}} + \dfrac{-1}{\sqrt{2}}i$

15) Represent $1+2i$, $2-i$, $-3-2i$, $-2+3i$ by points in Argand's diagram.

16) Show that $z = \dfrac{5}{(1-i)(2-i)(3-i)}$ is purely imaginary number.

17) Find the real numbers x and y such that

$$\frac{x}{1+2i} + \frac{y}{3+2i} = \frac{5+6i}{-1+8i}$$

18) Show that $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^{10} + \left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right)^{10} = 0$

19) Show that $\left(\dfrac{1+i}{\sqrt{2}}\right)^8 - \left(\dfrac{1-i}{\sqrt{2}}\right)^8 = 2.$

20) Convert the complex numbers in polar form and also in exponential form.

i) $z = \dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

ii) $z = -6+\sqrt{2}\,i$

iii) $\dfrac{-3}{2} + \dfrac{3\sqrt{3}\,i}{2}$

21) If $x+iy = \dfrac{a+ib}{a-ib}$, prove that $x^2+y^2 = 1$.

22) Show that $z = \left(\dfrac{-1+\sqrt{-3}}{2}\right)^3$ is a rational number.

23) Show that $\dfrac{1-2i}{3-4i} + \dfrac{1+2i}{3+4i}$ is real.

24) Simplify i) $\dfrac{i^{29}+i^{39}+i^{49}}{i^{30}+i^{40}+i^{50}}$ ii) $\left(i^{65} + \dfrac{1}{i^{145}}\right)$

iii) $\dfrac{i^{238}+i^{236}+i^{234}+i^{232}+i^{230}}{i^{228}+i^{226}+i^{224}+i^{222}+i^{220}}$

25) Simplify $\left[\dfrac{1}{1-2i}+\dfrac{3}{1+i}\right]\left[\dfrac{3+4i}{2-4i}\right]$

26) If α and β are complex cube roots of unity, prove that $(1-\alpha)(1-\beta)\ (1-\alpha^2)(1-\beta^2) = 9$

27) If w is a complex cube root of unity, prove that $(1-w+w^2)^6+(1+w-w^2)^6 = 128$

28) If w is the cube root of unity then find the value of

$$\left(\frac{-1+i\sqrt{3}}{2}\right)^{18} + \left(\frac{-1-i\sqrt{3}}{2}\right)^{18}$$

2 SEQUENCES AND SERIES

Let's Study

- A.P. and G.P.
- Sum of n terms of a G.P.
- Sum of infinite terms of a G.P.
- H.P. and A.G.P.
- A.M., G.M, H.M.

Let's Recall

2.1 Sequence :

A set of numbers where the numbers are arranged in a definite order,like the natural numbers, is called a sequence.

Examples : Natural numbers, Even integers between 10 and 100, Squares of integers.

In general, a sequence is written as $t_1, t_2, t_3, t_4 ..., t_n ...$ where t_1 - first term, t_4 - fourth term, ..., t_n - n^{th} term.

Finite sequence – A sequence containing finite number of terms is called a finite sequence.

It is written as $\{t_1, t_2, t_3 ....., t_n\}$ for some positive integer n.

Infinite sequence – A sequence is said to be infinite if it is not a finite sequence.

It is written as $\{t_1, t_2, t_3 .....\}$ or $\{t_n\}$ $n \geq 1$

Sequences that follow specific patterns are called **progressions.**

In the previous class, we have studied Arithmetic Progression (A.P.).

2.2 Arithmetic Progression– (A.P.)

In a sequence if the difference between any term and its preceding term $(t_{n+1} - t_n)$ is constant,then the sequence is called an Arithmetic Progression (A.P.)

Consider the following sequences

1) 2, 5, 8, 11, 14, …

2) 4, 10, 16, 22, 28, …

3) 4, 16, 64, 256, …

4) $\frac{1}{5}, \frac{1}{25}, \frac{1}{125}, \ldots$

5) −3, 2, 7, 12, 17, …

The sequences 1), 2) and 5) are A. P. but the terms in sequences 3) and 4) are not in A. P. as the difference between their consecutive terms is not constant.

If $t_1, t_2, t_3 ....., t_n$ are in A.P. then $t_{n+1} - t_n = d$, is constant for all n.

Hence the sequence can also be written as a, a+d, a+2d, ... Its n^{th} term is $t_n = a + (n-1)d$, $t_1 = a$ and sum of n terms i.e.

$$S_n = t_1 + t_2 + ... + t_n = \frac{n}{2}[2a + (n-1)d].$$

If $t_1, t_2, t_3 ....., t_n$ are in A.P. then

i) $t_1 + k, \ t_2 + k, t_3 + k, ... t_n + k$ are also in A.P.

ii) $k\,t_1, kt_2, kt_3, ... , kt_n$ are also in A.P. $(k \neq 0)$

Let's Learn

2.3 Geometric progression :

A sequence $t_1, t_2, t_3 ....., t_n$ ..... is G.P. (Geometric progression) if the common ratio $\frac{t_{n+1}}{t_n} = r$ is constant for all n.

Hence a G.P. can also be written as $a, ar^2, ar^3, ...$ where a is first term and r is common ratio.

Examples :

i) 2, 4, 8, 16, ... [here $a = 2$, r = 2]

ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, ...$ [$a = 1$, $r = \frac{1}{3}$]

iii) 1, −1, 1, −1, 1, −1, ... [$a = 1$, r = −1]

2.3.1 The General term or the nth term of a G.P.

For G.P., $t_1 = a$, $t_2 = ar$, $t_3 = ar^2$, $t_4 = ar^3$, ...

If a and r are the first term and common ratio of a G.P. respectively. Then its nth term is given by $t_n = ar^{n-1}$. (Verify)

Ex. Find nth term of the following G.P.

i) 3, −6, 12, −24, …

Here a = 3 , r = −2

Since $t_n = ar^{n-1} = 3\,(-2)^{n-1}$

ii) $5, 1, \frac{1}{5}, \frac{1}{25}, \ldots$

Here a = 5 , $r = \frac{1}{5}$

Since $t_n = ar^{n-1} = 5\left(\frac{1}{5}\right)^{n-1} = \left(\frac{1}{5}\right)^{n-2}$.

Properties of Geometric Progression.

If $t_1, t_2, t_3, ...t_n$ are in G.P. then

(i) $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}, ... \frac{1}{t_n}$ are also in G.P.

ii) ($k \neq 0$) $k\,t_1, kt_2, kt_3, ... kt_n$ are also in G.P.

iii) $t_1^n, t_2^n, t_3^n, ...$ are also in G.P.

SOLVED EXAMPLES

Ex. 1) Verify whether $1, \frac{-3}{2}, \frac{9}{4}, \frac{-27}{4}, \ldots$ is a G.P., if it is a G.P. Find its ninth term.

Solution : Here $t_1 = 1$, $t_2 = \frac{-3}{2}$, $t_3 = \frac{9}{4}$,

Consider $= \frac{t_2}{t_1} = \frac{\frac{-3}{2}}{1} = \frac{-3}{2}$

and $\frac{t_3}{t_2} = \left(\frac{9}{4}\right)\left(-\frac{2}{3}\right) = \frac{-3}{2}$

Here the ratio of any two consecutive terms is constant hence the given sequence is a G.P.

Now $t_9 = ar^{9-1} = a\,r^{8} = 1\left(\frac{-3}{2}\right)^8$

Ex. 2) Which term of the sequence

$\sqrt{3}$, 3, $3\sqrt{3}$, ... is 243?

Solution : Here $a = \sqrt{3}$, $r = \sqrt{3}$, $t_n = 243$

and $t_n = a\,r^{n-1}$

$\therefore\ a\,r^{n-1} = 243$

$\sqrt{3}.(\sqrt{3})^{n-1} = 243 = 3^5$

$\therefore\ (\sqrt{3})^{n} = 3^{\frac{n}{2}} = 3^5$

$\therefore\ \frac{n}{2} = 5$

$\therefore$ n = 10

$\therefore$ Tenth term of the G.P. is 243.

Ex. 3) For a G.P. If a = 3 and $t_7 = 192$ find r and t_{11}.

Solution : Given $a = \square$, $t_7 = ar^{\square} = 192$

$\therefore\ \square\,(r)^{\square} = 192$, $r^{\square} = \frac{192}{3} = 64$

$\therefore\ r^{\square} = 2^{\square}$,

$\therefore$ r= 2 .

also $t_{11} = a\,r^{\square} = 3\,(2)^{\square} = \square$.

Ex 4) In a G.P. ,if the third term is $\frac{1}{5}$ and sixth term is $\frac{1}{625}$, find its nth term .

Solution : Here $t_3 = \frac{1}{5}$, $t_6 = \frac{1}{625}$

$t_3 = ar^2 = \frac{1}{5}$ ...(1),

$t_6 = ar^5 = \frac{1}{625}$ ...(2)

Divide equation (2) by equation (1), we get

$$\frac{ar^5}{ar^2} = \frac{(1/625)}{(1/5)}$$

$$r^3 = \frac{1}{125} = \frac{1}{5^3}$$

$$\therefore\ r = \frac{1}{5}.$$

Substitute in equation (2), we get

$$a\left(\frac{1}{5}\right)^2 = \frac{1}{5}$$

$$\therefore\ a = 5.$$

$$t_n = ar^{n-1} = 5\left(\frac{1}{5}\right)^{n-1} = 5 \times (5)^{1-n} = (5)^{2-n}.$$

Ex 5) If for a sequence $\{t_n\}$, $t_n = \frac{5^{n-2}}{4^{n-3}}$ show that the sequence is a G.P. Find its first term and the common ratio.

Solution: $t_n = \frac{5^{n-2}}{4^{n-3}}$

$$\therefore\ t_{n+1} = \frac{5^{n-1}}{4^{n-2}}$$

Consider $\frac{t_n+1}{t_n} = \frac{\frac{5^{n-1}}{4^{n-2}}}{\frac{5^{n-2}}{4^{n-3}}} = \frac{5^{n-1}}{4^{n-2}} \times \frac{4^{n-3}}{5^{n-2}}$

$$\frac{5^{n-1-n+2}}{4^{n-2-n+3}} = \frac{5}{4} = \text{constant},\ \forall\ n \in N.$$

The given sequence is a G.P. with $r = \frac{5}{4}$.

and $t_1 = \frac{16}{5}$.

Let's Note :

i) Three numbers in G.P. can be conveniently assumed as $\frac{a}{r}$, a, ar

iii) Assume 4 numbers in a G.P. as

$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$ (Here ratio is r^2)

iv) Assume 5 numbers in a G.P. as

$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

Ex 6) Find three numbers in G.P. such that their sum is 42 and their product is 1728.

Solution : Let the three numbers in G.P. be $\frac{a}{r}, a, ar$.

As their product is 1728

$$\frac{a}{r}.a.ar = 1728$$

$$\therefore\ a^3 = 1728 = (12)^3$$

$$\therefore\ a = 12.$$

According to first condition, their sum is 42

$$\therefore\ \frac{a}{r} + a + ar = 42$$

$$\therefore\ \frac{12}{r} + 12 + 12r = 42$$

$$\therefore\ \frac{12}{r} + 12r = 30$$

Multiply by r

$$\therefore 12 + 12r^2 = 30r$$

$$\therefore 12r^2 - 30r + 12 = 0$$

Dividing by 6, we get

$$\therefore\ 2r^2 - 5r + 2 = 0$$

$$\therefore\ (2r-1)(r-2) = 0$$

$$\therefore\ 2r = 1 \text{ or } r = 2$$

$$\therefore\ r = \frac{1}{2}\quad r = 2$$

If a=12, $r = \frac{1}{2}$ then the required numbers are 24, 12, 6.

If a = 12, r =2 then the required numbers are 6, 12, 24.

$\therefore$ 24, 12, 6 or 6, 12, 24 are the three required numbers in G.P.

Ex 7) Find four numbers in G. P. such that their product is 64 and sum of the second and third number is 6.

Solution : Let the four numbers be $\frac{a}{r^3}, \frac{a}{r}$, ar , ar^3 (common ratio is r^2)

According to the first condition

$$\frac{a}{r^3} \times \frac{a}{r} \times ar \times ar^3 = 64$$

$\therefore$ $a^4 = 64$

$\therefore$ $a = (2^6)^{1/4} = 2^{3/2}$

$\therefore$ $a = 2\sqrt{2}$.

Now using second condition $\frac{a}{r} + ar = 6$

$$\frac{2\sqrt{2}}{r} + 2\sqrt{2}r = 6.$$

Multiplying by r,

$$2\sqrt{2} + 2\sqrt{2}\ r^2 = 6r$$

Dividing by 2

$$\sqrt{2} + \sqrt{2}\ r^2 = 3r$$

$$\sqrt{2}\ r^2 - 3r + \sqrt{2} = 0,$$

$$\sqrt{2}\ r^2 - 2r - r + \sqrt{2} = 0,$$

$$\sqrt{2}\ r(r - \sqrt{2}) - 1(r - \sqrt{2}) = 0.$$

$$(r - \sqrt{2})(\sqrt{2}\ r - 1) = 0.$$

$$r = \sqrt{2} \text{ or } r = \frac{1}{\sqrt{2}} \quad .$$

If $a = 2\sqrt{2}$, $r = \sqrt{2}$ then 1, 2, 4, 8 are the four required numbers in G.P.

If $a = 2\sqrt{2}$, $r = \frac{1}{\sqrt{2}}$ then 8, 4, 2, 1 are the four required numbers in G.P.

Ex 8) If p, q, r, s are in G.P. then show that

$$(q-r)^2 + (r-p)^2 + (s-q)^2 = (p-s)^2$$

Solution : Since, p, q, r, s are in G.P. $\frac{q}{p} = \frac{r}{q} = \frac{s}{r}$

$\therefore$ $q^2 = pr$, $r^2 = qs$, $qr = ps$

consider L.H.S. $= (q - r)^2 + (r - p)^2 + (s - q)^2$

$= q^2 - 2qr + r^2 + r^2 - 2rp + p^2 + s^2 - 2sq + q^2$

$= pr - 2qr + qs + qs - 2rp + p^2 + s^2 - 2sq + pr$

$= -2qr + p^2 + s^2 = -2ps + p^2 + s^2$ ($\because$ qr = ps)

$= (p-s)^2 =$ R.H.S.

Ex 9) Shraddha deposited Rs. 8000 in a bank which pays annual interest rate of 8%.She kept it with the bank for 10 years with compound interest. Find the total amount she will receive after 10 years. [given $(1.08)^{10} = 2.1575$)]

Solution:

The Amount deposited in a bank is Rs 8000 with 8% compound interest.

Each year , the ratio of the amount to the principal to that year is constant = $\frac{108}{100}$

Hence we get a G.P. of successive amounts.

We consider the amount at the end of each year for Rs 100 , the amount is 108.

The ratio of $\frac{\text{amount}}{\text{principal}} = \frac{108}{100}$.

For P = 8000 ,

the amount after 1 year is

$$8000 + 8000 \times \frac{8}{100} = 8000\left(1+\frac{8}{100}\right)$$

$$= 8000 \times \frac{108}{100}$$

the amount after 2 years is $8000 \times \frac{108}{100} \times \frac{108}{100}$

$$= 8000 \times \left(\frac{108}{100}\right)^2$$

the amount after 3 years is $8000 \times \frac{108}{100} \times \frac{108}{100} \times \frac{108}{100}$

$$= 8000 \times \left(\frac{108}{100}\right)^3.$$

Therefore after 10 years the amount is

$$= 8000\left(\frac{108}{100}\right)^{10}$$

$$= 8000\,(1.08)^{10}$$

$$= 8000 \times 2.1575 = 17260.$$

Thus Shraddha will get Rs 17260 after 10 years.

The formula to find amount by compound interest is

$$A = P\left(1+\frac{R}{100}\right)^N$$

$$\therefore \quad \frac{A}{P} = \left(1+\frac{R}{100}\right)^N$$

Note that, ($\left(1+\frac{R}{100}\right)^N$ is a G.P.]

EXERCISE 2.1

1) Check whether the following sequences are G.P. If so, write t_n.
   i) 2, 6, 18, 54, …
   ii) 1, −5, 25, −125 …
   iii) $\sqrt{5}, \frac{1}{\sqrt{5}}, \frac{1}{5\sqrt{5}}, \frac{1}{25\sqrt{5}}, \ldots$
   iv) 3, 4, 5, 6, …
   v) 7, 14, 21, 28, …

2) For the G.P.
   i) If $r = \frac{1}{3}$, $a = 9$ find t_7
   ii) If $a = \frac{7}{243}$, $r = 3$ find t_6.
   iii) If $r = -3$ and $t_6 = 1701$, find a.
   iv) If $a = \frac{2}{3}$, $t_6 = 162$, find r .

3) Which term of the G.P. 5, 25, 125, 625, … is 5^{10}?

4) For what values of *x, the terms* $\frac{4}{3}, x, \frac{4}{27}$ are in G.P?

5) If for a sequence, $t_n = \frac{5^{n-3}}{2^{n-3}}$, show that the sequence is a G.P. Find its first term and the common ratio.

6) Find three numbers in G.P. such that their sum is 21 and sum of their squares is 189.

7) Find four numbers in G.P. such that sum of the middle two numbers is 10/3 and their product is 1.

8) Find five numbers in G. P. such that their product is 1024 and fifth term is square of the third term.

9) The fifth term of a G.P. is x , eighth term of a G.P. is y and eleventh term of a G.P. is z verify whether $y^2 = xz$.

10) If p,q,r,s are in G.P. show that p+q , q+r , r+s are also in G.P.

11) The number of bacteria in a culture doubles every hour. If there were 50 bacteria originally in the culture, how many bacteria will be there at the end of 5th hour ?

12) A ball is dropped from a height 0f 80 ft. The ball is such that it rebounds $\left(\frac{3}{4}\right)^{th}$ of the height it has fallen. How high does the ball rebound on 6th bounce? How high does the ball rebound on nth bounce?

13) The numbers 3, x and $x + 6$ form are in G.P. Find (i) x, (ii) 20th term (iii) nth term.

14) Mosquitoes are growing at a rate of 10% a year. If there were 200 mosquitoes in the begining. Write down the number of mosquitoes after (i) 3 years (ii) 10 years (iii) n years.

15) The numbers $x - 6$, $2x$ and x^2 are in G.P. Find (i) x (ii) 1st term (iii) nth term.

Let's Learn

2.3.2 Sum of the first n terms of a G.P. (S_n)

Consider the G.P. t_1, t_2, t_3, ... t_n .... . we write the sum of first n terms

$t_1 + t_2 + t_3 + \ldots\ldots + t_n$ as $\sum_{r=1}^{n} t_r = S_n$

Note : $\sum$ is the notation of summation, the sum is of all t_r $(1 \leqslant r \leqslant n)$

In $\sum_{r=1}^{n}$ the variable is r.

Theorem : If $a, ar, ar^2, \ldots, ar^{n-1}$ $(r \neq 1)$ is a G.P. then

$S_n = a + ar^2 + \ldots + ar^{n-1} = t_1 + t_2 + t_3 + \ldots\ldots + t_n$

$$= \sum_{r=i}^{n} t_r = \frac{a(1-r^n)}{(1-r)} \ldots (r \neq 1)$$

Proof : Consider $S_n = a + ar + ar^2 + \ldots + ar^{n-1}$

$S_n = a(1+r+r^2+r^3+ \ldots + r^{n-1})$ ... (1)

Multiplying both sides by r we get

$r\,S_n = a(r+r^2+r^3+ \ldots + r^n)$ ... (2)

Subtract (2) from (1) we get $S_n - S_n r = a(1-r^n)$

$\therefore$ $S_n(1-r) = a(1-r^n)$

$$\therefore \; S_n = \frac{a(1-r^n)}{1-r}, \; r \neq 1.$$

Let's note :

1) If r is positive and r < 1, it is convenient to write $S_n = \dfrac{a(1-r^n)}{1-r}$

2) If r is positive and r > 1, it is convenient to write $S_n = \dfrac{a(r^n-1)}{r-1}$

3) If r = 1 then G.P. is a, a, a ...a (n times), So, $S_n = a.\,n$

4) $S_n - S_{n-1} = t_n$

SOLVED EXAMPLES

Ex 1) If a = 1, r = 2 find S_n for the G.P.

Solution : a = 1 , r = 2

$$S_n = a\left(\frac{r^n-1}{r-1}\right) = 1 \times \left(\frac{2^n-1}{2-1}\right) = 2^n - 1 .$$

Ex 2) For a G.P. 0.02, 0.04, 0.08, 0.016, …, find S_n.

Solution : Here a = 0.02, $r = \dfrac{0.04}{0.02} = 2$

$$S_n = a\left(\frac{r^n-1}{r-1}\right) = 0.02\left(\frac{2^n-1}{2-1}\right)$$

$$= 0.02\,.\,(2^n - 1)$$

Ex 3) For the following G.P. 3, −3, 3, −3, …, find S_n.

Solution :

Case (i)

If n is even , n = 2k

$S_{2k} = (3-3) + (3-3) + (3-3) + (3-3) + \ldots$ 2k terms

= (3+3+ ... k terms) + (−3, −3, −3, ... k terms)

= 3k − 3k = 0

Case (ii)

If n is odd , n = 2k +1

$S_{2k+1} = (3 + 3 + 3 \ldots$ k terms)
+ (−3, −3, −3, ... k terms) + 3 = 3k − 3k + 3 = 3

Ex 4) For a G.P. if a=6 , r =2 , find S_{10}.

Solution: $S_n = a\left(\frac{r^n - 1}{r-1}\right)$,

$$S_{10} = 6\left(\frac{2^{10}-1}{2-1}\right) = 6\left(\frac{1023}{1}\right) = 6\,(1023) = 6138 .$$

Ex 5) How many terms of G.P.

$2, 2^2, 2^3, 2^4, \ldots$ are needed to give the sum 2046.

Solution : Here a = 2 , r = 2 , let S_n = 2046.

$$\therefore 2046 = a\left(\frac{r^n-1}{r-1}\right) = 2\left(\frac{2^n-1}{2-1}\right) = 2\,(2^n-1)$$

$1023 = 2^n - 1$, $2^n = 1024 = 2^{10}$ $\therefore$ n = 10

Ex 6) If for a G.P. r=2 , S_{10}=1023 , find a .

Solution : $S_{10} = a\left(\frac{2^{10}-1}{2-1}\right)$

$\therefore$ 1023 = a (1023)

$\therefore$ a =1.

Ex 7) For a G.P. a = 3 , r = 2 , S_n = 765 , find n .

Solution : $S_n = 765 = 3\left(\frac{2^n-1}{2-1}\right) = 3\,(2^n-1)$,

$\therefore 255 = 2^n - 1$,

$2^n = 256 = 2^8$,

n = 8.

Ex 8) For a G.P. if S_3= 16, S_6=144, find the first term and the common ratio of the G.P.

Solution : Given

$$S_3 = a\left(\frac{r^3-1}{r-1}\right) = 16 \qquad \ldots (1)$$

$$S_6 = a\left(\frac{r^6-1}{r-1}\right) = 144 \qquad \ldots (2)$$

Dividing (2) by (1) we get,

$$\frac{S_6}{S_3} = \frac{r^6-1}{r^3-1} = \frac{144}{16}$$

$$\frac{(r^3-1)(r^3+1)}{(r^3-1)} = 9,$$

$r^3 + 1 = 9$,

$r^3 = 8 = 2^3$,

r = 2

Substitute r = 2 in (1) We get

$$a\left(\frac{2^3-1}{2-1}\right) = 16,$$

$$a\left(\frac{8-1}{2-1}\right) = 16,$$

a (7) = 16 ,

a = 16 / 7

Ex 9) Find 5+55+555+5555+ ... upto n terms.

Solution:

Let S_n,

= 5+55+555+5555+... upto n terms.

= 5 (1+11+111+... upto n terms)

= $\frac{5}{9}$ (9+99+999+... upto n terms)

= $\frac{5}{9}$ [(10–1) + (100–1) + (1000–1) + ... to n brackets]

= $\frac{5}{9}$ [(10 + 100 + 1000 +... upto n terms)

– (1+1+1+ …..upto n terms)]

[(a = 10, r = 10) and (a = 1, r = 1)]

$$= \frac{5}{9}\left[10\left(\frac{10^n-1}{10-1}\right) - n\right]$$

$$= \frac{5}{9}\left[\frac{10}{9}\left(10^n-1\right) - n\right]$$

Ex 10) Find the sum to n terms 0.3+0.33+0.333+ ... n terms

Solution :

S_n = 0.3+0.33+0.333+ upto n terms

= 3 [0.1+0.11+ 0.111+ ... n terms]

Multiply and divide by 9

$= \frac{3}{9}$ [0.9+0.99+ 0.999+ ... n terms]

$= \frac{3}{9}$ [(1–0.1)+(1–0.01)+(1–0.001)+ ... to n terms]

$= \frac{3}{9}$ [(1+1+1 ...... n terms)

– (0.1+0.01+ 0.001+ ...... n terms)]

[(a = 1, r = 1) and (a = 0.1, r = 0.1)]

$$= \frac{3}{9}\left[n - 0.1\frac{(1-0.1^n)}{(1-0.1)}\right]$$

$$= \frac{3}{9}\left[n - \frac{1}{9}(1-0.1^n)\right]$$

Ex 11) Find the n^{th} term of the sequence 0.4 , 0.44, 0.444,…

Solution :

Here t_1=0.4

t_2=0.44 = 0.4 + 0.04

t_3= 0.444 = 0.4+0.04+0.004

t_n= 0.4 + 0.04 + 0.004 + 0.0004 + ... upto n terms

here t_n is the sum of first n terms of a G.P.

with a = 0.4 and r = 0.1

$$t_n = 0.4\left(\frac{1-0.1^n}{1-0.1}\right) = \frac{0.4}{0.9}[1-0.1)^n]$$

$$= \frac{4}{9}[1-(0.1)^n].$$

Ex 12) For a sequence , if Sn = $5(4^n–1)$, find the n^{th} term, hence verify that it is a G.P., Also find r.

Solution : $S_n = 5(4^n-1)$, $S_{n-1} = 5(4^{n-1}-1)$

We know that $t_n = S_n - S_{n-1}$

$= 5(4^n-1) - 5(4^{n-1}-1)$

$= 5(4^n) - 5 - 5(4^{n-1})+5$

$= 5(4^n - 4^{n-1})$

$= 5(4^{n-1})(4-1)$

$= 5(4^{n-1})(3)$

$\therefore t_n = 15(4^{n-1})$

Then, $t_{n+1} = 15(4^n)$

and $\frac{t_{n+1}}{t_n} = \frac{15(4^n)}{15(4^{n-1})}$

$= 4^{n-n+1} = 4$

= constant $\forall$ n $\in$ N.

r = 4.

The sequence is a G.P. with $t_n=15(4^{n-1})$.

Ex 13) A teacher wanted to reward a student by giving some chocolates. He gave the student two choices. He could either have 60 chocolates at once or he could get 1 chocolate on the first day, 2 on the second day, 4 on the third day and so on for 6 days.Which option should the student choose to get more chocolates?

Ans : We need to find sum of chocolates in 6 days.

According to second option teacher gives 1 chocolate on the first day , 2 on the second day, 4 on the third day, and so on.Hence it is a G. P . with a = 1, r = 2 .

By using $S_n = a\left(\frac{r^n-1}{r-1}\right)$

$$S_6 = 1\left(\frac{2^6-1}{2-1}\right)$$

$$= 64 - 1 = 63$$

Hence the student should choose the second way to get more chocolates.

Ex 14) Mr. Pritesh got a job with an annual salary package of Rs. 4,00,000 with 10% annual increment. Find his salary in the 5th year and also find his total earnings through salary in 10 years.

[Given $(1.1)^4 = 1.4641$, $(1.1)^{10} = 2.59374$]

Solution : In the year he will get a salary of Rs. 4,00,000.

He gets an increment of 10% so in the second year his salary will be

$$4,00,000 \times \left(\frac{110}{100}\right) = 4,40,000$$

In the third year his salary will be

$$4,00,000 \times \left(\frac{110}{100}\right)^2 \text{ and so on ...}$$

hence it is a G.P. with a = 4,00,000 & r = 1.1.

So, his salary in the fifth year will be

$$t_5 = ar^4 = 4,00,000 \times \left(\frac{110}{100}\right)^4 = 585640.$$

His total income through salary in 10 years

will be $S_{10} = a\left(\frac{r^{10}-1}{r-1}\right)$

[$\because (1.1)^{10} = 2.59374$]

$$= 4,00,000 \times \left(\frac{2.59374-1}{0.1}\right)$$

$$= 4,00,000 \left(\frac{1.59374}{0.1}\right)$$

= 4,00,000 [15.9374] = 63,74,960.

Mr. Pritesh will get Rs. 5,85,640 in the fifth year and his total earnings through salary in 10 years will be Rs 63,74,960.

EXERCISE 2.2

1) For the following G.P.s , find S_n
   i) 3, 6, 12, 24, ...
   ii) p, q, $\frac{q^2}{p}$, $\frac{q^3}{p}$, ...
   iii) 0.7, 0.07, 0.007, ...
   iv) $\sqrt{5}, -5, 5\sqrt{5}, -25\ldots$

2) For a G.P.
   i) a = 2, r = $-\frac{2}{3}$, find S_6
   ii) If $S_5 = 1023$, r = 4, Find a

3) For a G.P.
   i) If a =2, r = 3, $S_n = 242$ find n.
   ii) For a G.P. sum of first 3 terms is 125 and sum of next 3 terms is 27, find the value of r .

4) For a G.P.
   i) If $t_3 = 20$, $t_6 = 160$, find S_7
   ii) If $t_4 = 16$, $t_9 = 512$, find S_{10}

5) Find the sum to n terms
   i) 3 + 33 + 333 + 3333 + ...
   ii) 8 +88 + 888 + 8888 + ...

6) Find the sum to n terms
   i) 0.4 + 0.44 + 0.444 + ...
   ii) 0.7 + 0.77+ 0.777 + ...

7) Find the sum to n terms of the sequence
   i) 0.5, 0.05, 0.005, ...
   ii) 0.2, 0.02, 0.002, ...

8) For a sequence, if $S_n = 2(3^n - 1)$, find the n^{th} term, hence show that the sequence is a G.P.

9) If S,P,R are the sum , product and sum of the reciprocals of n terms of a G. P. respectively, then verify that $\left[\frac{S}{R}\right]^n = P^2$.

10) If S_n, S_{2n}, S_{3n} are the sum of n,2n,3n terms of a G.P. respectively , then verify that
$S_n(S_{3n} - S_{2n}) = (S_{2n} - S_n)^2$.

11) Find (i) $\sum_{r=1}^{10}(3\times 2^r)$ (ii) $\sum_{r=1}^{10} 5\times 3^r$

12) The value of a house appreciates 5% per year. How much is the house worth after 6 years if its current worth is Rs. 15 Lac. [Given : $(1.05)^5 = 1.28$, $(1.05)^6 = 1.34$]

13) If one invests Rs. 10,000 in a bank at a rate of interest 8% per annum, how long does it take to double the money by compound interest? $[(1.08)^5 = 1.47]$

2.4 Sum of infinite terms of G. P.

Consider a G.P. of the positive terms. The sum of first n terms is $\frac{a(r^n - 1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)}$

If $|r| > 1$, $r - 1$ is constant but r^n approaches ∞ as n approaches ∞, so the infinite terms cannot be summed up.

If $|r| < 1$, r^n approaches 0, as n approaches ∞ and the sum Sn = $\frac{a(1-r^n)}{(1-r)}$ approaches $\frac{a}{1-r}$

Here the infinite sum $\sum_{r=1}^{\infty} t_r$ is said to be $\frac{a}{1-r}$

Example : $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

Solution :

here $a = 1$, $r = \frac{1}{2}$, i.e. $|r| < 1$

$\therefore$ Sum to infinity is given by

$$\frac{a}{1-r} = \frac{1}{1-\left(\frac{1}{2}\right)} = \frac{1}{\left(\frac{1}{2}\right)} = 2$$

Visual proof : This can be visualised taking a rectangle of 2×1 containing smaller rectangles of area 1, $\frac{1}{2}$, $\frac{1}{4}$, ... square units.

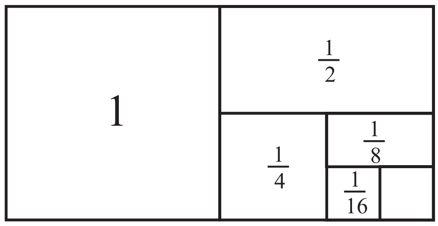


Fig. 2.1

These rectangles are seen to be completely in the big rectangle and slowly fills up the big rectangle of area two square units.

That is $1 + \frac{1}{2} + \frac{1}{4} + \dots = 2$.

SOLVED EXAMPLES

Ex. 1) Determine whether the sum of all the terms in the series is finite. In case it is finite find it.

i) $\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \dots$

ii) $\frac{3}{5}, \frac{-9}{25}, \frac{27}{125}, \frac{-81}{625} \dots$

iii) $1, -3, 9, -27, 81, \dots$

Solution:

i) Here $a = \frac{1}{3}$, $|r| = \left|\frac{1}{3}\right| = \frac{1}{3} < 1$

$\therefore$ Sum to infinite terms is finite.

$$S = \frac{a}{1-r} = \frac{\left(\frac{1}{3}\right)}{1-\left(\frac{1}{3}\right)} = \frac{\left(\frac{1}{3}\right)}{\left(\frac{2}{3}\right)} = \frac{1}{2}$$

ii) Here $a = \frac{3}{5}$, $r = -\frac{3}{5}$, $|r| = \left|\frac{-3}{5}\right| = \frac{3}{5} < 1$

$\therefore$ Sum to infinite terms is finite.

$$S = \frac{a}{1-r} = \frac{\left(\frac{3}{5}\right)}{1-(-\frac{3}{5})} = \frac{\left(\frac{3}{5}\right)}{\left(\frac{8}{5}\right)} = \frac{3}{8}$$

iii) Here a=1 , $r = -3$ $|r| = |-3| = 3 \nless 1$

$\therefore$ sum to infinity does not exist

2.4.1 Expressing recurring decimals as rational numbers :

We know that recurring decimal fraction can be written as rational numbers e.g. $0.666 \ldots 0.\overline{6} = \frac{2}{3}$. This can also be checked using G.P.

Ex i) 0.66666…

$= 0.6 + 0.06 + 0.006 + \ldots$

the terms are in G.P. with a=0.6, $|r| = |0.1| < 1$

$\therefore$ Sum to infinite terms is finite and is

$$\frac{a}{1-r} = \frac{0.6}{1-0.1} = \frac{0.6}{0.9} = \frac{6}{9} = \frac{2}{3}$$

ii) $0.\overline{46} = 0.46 + 0.0046 + 0.000046 + \ldots$ the terms are in G.P. with a = 0.46 , $|r| = |0.01| < 1$.

$\therefore$ Sum to infinite terms is finite and is

$$\frac{a}{1-r} = \frac{0.46}{1-(0.01)} = \frac{0.46}{0.99} = \frac{46}{99}\ .$$

iii) $2.\overline{5} = 2 + 0.5 + 0.05 + 0.005 + 0.0005 + \ldots$
After the first term,the terms are in G.P. with a = 0.5 , $|r| = |0.1| < 1$

$\therefore$ Sum to infinite terms is finite and is

$$\frac{a}{1-r} = \frac{0.5}{1-0.1} = \frac{0.5}{0.9} = \frac{5}{9}$$

$$2.\overline{7} = 2 + \frac{5}{9} = \frac{23}{9}$$

EXERCISE 2.3

1) Determine whether the sum to infinity of the following G.P.s exist, if exists find them

i) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

ii) $2, \frac{4}{3}, \frac{8}{9}, \frac{16}{27}, \ldots$

iii) $-3, 1, \frac{-1}{3}, \frac{1}{9}, \ldots$

iv) $\frac{1}{5}, \frac{-2}{5}, \frac{4}{5}, \frac{-8}{5}, \frac{16}{5}, \ldots$

v) 9, 8.1, 7.29, ...

2) Express the following recurring decimals as a rational number.

i) $0.\overline{7}$

ii) $2.\overline{4}$

iii) $2.3\overline{5}$

iv) $51.0\overline{2}$

3) If the common ratio of a G.P. is $\frac{2}{3}$ and sum to infinity is 12. Find the first term.

4) If the first term of the G.P. is 16 and its sum to infinity is $\frac{96}{17}$ find the common ratio.

5) The sum of an infinite G.P. is 5 and the sum of the squares of these terms is 15 find the G.P.

6) Find (i) $\sum_{r=1}^{\infty} 4(0.5)^r$ (ii) $\sum_{r=1}^{\infty}\left(-\frac{1}{3}\right)^r$

(iii) $\sum_{r=0}^{\infty}(-8)\left(-\frac{1}{2}\right)^n$ (iv) $\sum_{n=1}^{\infty} 0.4^n$

7) The mid points of the sides of a square of side 1 are joined to form a new square. This procedure is repeated indefinitely. Find the sum of (i) the areas of all the squares (ii) the perimeters of all the square.

8) A ball is dropped from a height of 10m. It bounces to a height of 6m, then 3.6m and so on. Find the total distance travelled by the ball.

2.5 Harmonic Progression (H. P.)

Definition : A sequence $t_1, t_2, t_3, t_4, \ldots, t_n, \ldots$ ($t_n \neq 0$, $n \in N$) is called a harmonic progression if $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3} \ldots, \frac{1}{t_n}, \ldots$ are in A.P.

e.g., i) $\frac{1}{7}, \frac{1}{11}, \frac{1}{15} \ldots$ are in H. P. as

$\frac{1}{\left(\frac{1}{7}\right)}, \frac{1}{\left(\frac{1}{11}\right)}, \frac{1}{\left(\frac{1}{15}\right)}$ i.e. 7, 11, 15 … are in A.P.

ii) $\frac{1}{4}, \frac{3}{14}, \frac{3}{16} \ldots$ is H.P. as $4, \frac{14}{3}, \frac{16}{3}, \ldots$ are in A.P.

SOLVED EXAMPLES

Ex. 1) Find the n^{th} term of the H.P.

$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7} \ldots$

Solution : Here $2, \frac{5}{2}, 3, \frac{7}{2} \ldots$ are in A.P. with $a = 2$ and $d = \frac{1}{2}$ hence $\frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \ldots$ are in H.P.

For A.P. $t_n = a + (n-1)d = 2 + (n-1)\frac{1}{2}$

$= 2 + \frac{1}{2}n - \frac{1}{2}$

$= \frac{3}{2} + \frac{n}{2}$

$= \frac{3+n}{2}$.

Therefore in H.P. $t_n = \frac{2}{3+n}$

Ex. 2) Find the n^{th} term of H.P. $\frac{1}{5}, 1, \frac{-1}{3}, \frac{-1}{7}, \ldots$

Solution: Since 5, 1, −3, −7, … are in A.P.

with $a = 5$ and $d = -4$

Hence $t_n = a + (n-1)d$

$= 5 + (n-1)(-4)$

$= 5 - 4n + 4 = 9 - 4n$.

For H.P. $t_n = \frac{1}{9-4n}$

2.6 Types of Means:

2.6.1 Arithmetic mean (A. M.):

If x and y are two numbers, their A.M. is given by $A = \frac{x+y}{2}$.

We observe that x, A, y form an A.P.

i.e. $A - x = y - A \quad \therefore 2A = x + y \quad \therefore A = \frac{x+y}{2}$

2.6.2 Geometric mean (G. M.) :

If x and y are two numbers having same sign (positive or negative), their G.M. is given by $G = \sqrt{xy}$.

We observe that x, G, y form a G.P.

i.e. $\frac{G}{x} = \frac{y}{G} \quad \therefore G^2 = xy$

$\therefore G = \sqrt{xy}$

2.6.3 Harmonic mean (H.M.) :

If x and y are two numbers, their H.M. is given by $H = \frac{2xy}{x+y}$.

We observe that x, H, y form an HP .

i.e. $\frac{1}{x}, \frac{1}{H}, \frac{1}{y}$ is in A.P.

$$\therefore \frac{1}{H} = \frac{\frac{1}{x}+\frac{1}{y}}{2} = \frac{x+y}{2xy} \quad \therefore H = \frac{2xy}{x+y}$$

Note 1) : These results can be extended to n numbers as follows

$$A = \frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}$$

$$G = \sqrt[n]{x_1 . x_2 . x_3 \ldots x_n}$$

$$H = \frac{n}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}+\ldots.+\frac{1}{x_n}}$$

2) If $x = y$ then A = G = H

Theorem : If A, G and H are A.M., G.M., H.M. of two positive numbers x and y respectively, then

i) $G^2 = AH$ ii) $A \geq G \geq H$

Proof : Let A, G and H be A.M., G.M and H.M. of two positive numbers x and y

Then

$$A = \frac{x+y}{2}, \; G = \sqrt{xy}, \; H = \frac{2xy}{x+y}$$

i) $$RHS = AH = \left(\frac{x+y}{2}\right)\left(\frac{2xy}{x+y}\right)$$

$$= xy = G^2 = \text{L.H.S.}$$

ii) $$\text{Consider} A - G = \frac{x+y}{2} - \sqrt{xy}$$

$$= \frac{1}{2}(x+y-2\sqrt{xy})$$

$$= \frac{1}{2}\left[\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 - 2\sqrt{x}\sqrt{y}\right]$$

$$= \frac{1}{2}\left(\sqrt{x}-\sqrt{y}\right)^2 \geqslant 0$$

Since squares are always non-negative.

$\therefore A - G \geq 0 \quad \therefore A \geq G$ (I)

$\therefore \frac{A}{G} \geq 1$ (II)

Now since $G^2 = AH$

$\frac{G}{H} = \frac{A}{G} \geq 1$ (From II)

$\therefore \frac{G}{H} \geq 1 \; \therefore G \geq H$ (III)

From (I) and (III)

$A \geq G \geq H$

SOLVED EXAMPLES

Ex. 1 : Find A.M.,G.M.,H.M. of the numbers 4 and 16

Solution : Here x = 4 and y = 16

$$A = \frac{x+y}{2} = \frac{4+16}{2} = \frac{20}{2} = 10$$

$$G = \sqrt{xy} = \sqrt{4\times16} = \sqrt{64} = 8$$

$$H = \frac{2xy}{x+y} = \frac{2\times4\times16}{4+16} = \frac{128}{20} = \frac{32}{5}$$

Ex. 2 : Insert 4 terms between 2 and 22 so that the new sequence is in AP.

Solution: Let A_1, A_2, A_3, A_4 be the four terms between 2 and 22 so that

2, A_1, A_2, A_3, A_4, 22 are in AP with

a = 2, t_6 = 22, n = 6 .

$\therefore$ 22 = 2 + (6–1)d = 2 + 5d

20 = 5d, d = 4

A_1 = a+d = 2+4 = 6,

A_2 = a+2d = 2+8 = 10,

$A_3 = a+3d = 2+3\times4 = 2+12 = 14$

$A_4 = a+4d = 2 + 4\times4 = 2+16 = 18.$

$\therefore$ the 4 terms between 2 and 22 are 6, 10, 14, 18.

Ex: 3 Insert two numbers between $\frac{2}{9}$ and $\frac{1}{12}$ so that the resulting sequence is a HP.

Solution : Let the required numbers be $\frac{1}{H_1}$ and $\frac{1}{H_2}$

$\therefore \frac{2}{9}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{12}$ are in HP.

$\therefore \frac{9}{2}, H_1, H_2, 12$ are in A.P.

$t_1 = a = \frac{9}{2}, t_4 = 12 = a+3d \therefore \frac{9}{2} + 3d = 12$

$\therefore 3d = 12 - \frac{9}{2} = \frac{24-9}{2} = \frac{15}{2}$

$\therefore d = \frac{5}{2}$

$t_2 = H_1 = a+d = \frac{9}{2} + \frac{5}{2} = \frac{14}{2} = 7.$

$t_3 = H_2 = a+2d = \frac{9}{2} + 2 \times \frac{5}{2} = \frac{19}{2}$

$\frac{1}{7}$ and $\frac{2}{19}$ are to be

inserted between $\frac{2}{9}$ and $\frac{1}{12}$

Ex: 4 Insert two numbers between 1 and 27 so that the resulting sequence is a G.P.

Solution: Let the required numbers be G_1 and G_2

$\therefore$ 1, G_1, G_2, 27 are in G.P.

$\therefore t_1 = 1, t_2 = G_1, t_3 = G_2, t_4 = 27$

$\therefore a = 1, t_4 = ar^3 = 27 \quad \therefore r^3 = 27$

$\therefore r^3 = 27 = 3^3 \therefore r = 3$

$t_2 = G_1 = ar = 1 \times 3 = 3.$

$t_3 = G_2 = ar^2 = 1(3)^2 = 9$

$\therefore$ 3 and 9 are the two required numbers between 1 and 27.

Ex: 5 The A.M. of two numbers exceeds their G.M. by 2 and their H.M. by $\frac{18}{5}$. Find the Numbers.

Solution : Given A= G + 2

$\therefore G = A - 2$ .... (I)

Also $A = H + \frac{18}{5} \quad \therefore H = A - \frac{18}{5}$ ..... (II)

We know that $G^2 = AH$

$(A-2)^2 = A\left(A - \frac{18}{5}\right)$ .... from (I) and (II)

$A^2 - 4A + 2^2 = A^2 - \frac{18}{5}A$

$\frac{18}{5}A - 4A = -4$

$-2A = -4 \times 5,$

$\therefore A = 10$

Also $G = A-2 = 10 - 2 = 8$

Now, let the two numbers be x and y.

As $A = \frac{x+y}{2} = 10,\ x+y = 20,$

$\therefore y = 20 - x$ ... (III)

Now $G = \sqrt{xy} = 8$

$\therefore xy = 64$ ... (IV)

$\therefore x(20 - x) = 64$ ... from (III) and (IV)

$20x - x^2 = 64$

$x^2 - 20x + 64 = 0$

$(x-16)(x-4) = 0$

$x = 16$ or $x = 4$

If $x = 16$ then $y = 4$

If $x = 4$ then $y = 16$.

The required numbers are 4 and 16.

EXERCISE 2.4

1) Verify whether the following sequences are H.P.

i) $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \ldots$

ii) $\frac{1}{3}, \frac{1}{6}, \frac{1}{12}, \frac{1}{24}, \ldots$

iii) $5, \frac{10}{17}, \frac{10}{32}, \frac{10}{47}, \ldots$

2) Find the n^{th} term and hence find the 8^{th} term of the following HPs

i) $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \ldots$

ii) $\frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \ldots$

iii) $\frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \ldots$

3) Find A.M. of two positive numbers whose G.M. and H. M. are 4 and $\frac{16}{5}$ respectively.

4) Find H.M. of two positive numbers A.M. and G.M. are $\frac{15}{2}$ and 6

5) Find GM of two positive numbers whose A.M. and H.M. are 75 and 48

6) Insert two numbers between $\frac{1}{4}$ and $\frac{1}{3}$ so that the resulting sequence is a HP.

7) Insert two numbers between 1 and −27 so that the resulting sequence is a G.P.

8) If the A.M. of two numbers exceeds their G.M. by 2 and their H.M. by $\frac{18}{5}$, find the numbers.

9) Find two numbers whose A.M. exceeds their GM by $\frac{1}{2}$ and their HM by $\frac{25}{26}$

Let's Learn

2.7 ARITHMETICO-GEOMETRIC PROGRESSION (A.G.P.) :

Definition : A sequence in which each term is the product of the corresponding terms of an A.P. and G.P. is called and arithmetico – geometric progression (AGP).

e.g. Consider the sequence

	(a)	$(a+d)$	$(a+2d)$	$(a+3d)$
A.P.	2	5	8	11
G.P.	1	3	9	27
	(1)	(r)	(r^2)	(r^3)

A.G.P. is a, $(a+d)r$, $(a+2d)\,r^2$, $(a+3d)\,r^3$, …

i.e. 2×1, 5×3, 8×9, 11×27,...

Here the first factors of the terms are in AP and the second factors are in G.P.

Therefore the given sequence forms an A.G.P.

n^{th} term of A.G.P.

$t_1 = a \times 1$, $t_2 = (a+d)r$, $t_3 = (a+2d)r^2$, ...

$\therefore\ t_n = [a + (n-1)d]\ r^{n-1}$

2.7.1 Sum of n terms of A.G.P.:

$$\text{Let } S_n = a + (a+d)r + (a+2d)r^2 + \ldots + [a+(n-1)d]r^{n-1}$$

$$\therefore\ S_n = a + ar + dr + ar^2 + 2dr^2 + \ldots + ar^{n-1} + (n-1)dr^{n-1} \quad \ldots(i)$$

$$\therefore\ rS_n = ar + ar^2 + dr^2 + ar^3 + 2dr^3 + \ldots + ar^n + (n-1)dr^n \quad \ldots(ii)$$

Subtracting (ii) from (i) we get

$S_n - rS_n = a + dr + dr^2 + ... dr^{n-1} - ar^n - (n-1)dr^n$

$\therefore\ S_n(1-r) = a + dr + dr^2 + ...$

$+ dr^{n-1} - [a + n(n-1)d]r^n$

$$S_n(1-r) = a + dr\left(\frac{1-r^{n-1}}{(1-r)}\right) - [a+(n-1)d]r^n$$

$$S_n = \frac{a}{1-r} + \frac{dr(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}$$

$(r \neq 1)$

Note that, sum to infinity of A.G.P. is

$$\frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

SOLVED EXAMPLE

Ex 1) Find t_n and the sum of n terms of

1, 4, 12, 32, 80, 192, ...

Solution: Given sequence can be written as

$1\times1,\ 2\times2,\ 3\times4,\ 4\times8,\ 5\times16, \ldots$

We observe that first factors in each term 1, 2, 3, 4, 5, ... are in A.P. with a = 1,d = 1

and the second factors in each term 1, 2, 4, 8, 16, … are in G.P with r = 2

$\therefore\ t_n = [a + (n-1)d]r^{n-1}$

$= [1 + (n-1)]2^{n-1} = n.2^{n-1}$

We know that S_n of A.G.P. is given by

$$S_n = \frac{a}{1-r} + dr\frac{(1-r^{n-1})}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r} \quad (r \neq 1)$$

Substituting the values of a,d,r we get ,

$$S_n = \frac{1}{1-2} + 1\times 2\frac{(1-2^{n-1})}{(1-2)^2} - \frac{[1+(n-1)1]2^n}{1-2}$$

$$S_n = \frac{1}{-1} + 2(1-2^{n-1}) + n.\ 2^n$$

$S_n = -1 + 2 - 2^n + n\ .\ 2^n$

$S_n = 1 - 2^n + n\ 2^n$

$S_n = 1 - 2^n(1-n) = 1 + (n-1)\ 2^n$.

EXERCISE 2.5

1) Find S_n of the following arithmetico – geometric sequences

i) $2, 4x, 6x^2, 8x^3, 10x^4, \ldots$

ii) $1, 4x, 7x^2, 10x^3, 13x^4, \ldots$

iii) $1, 2\times3, 3\times9, 4\times27, 5\times81, \ldots$

iv)) 3, 12, 36, 96, 240, …

2) Find the sum to infinity of the following arithmetico – geometric sequence

i) $1, \frac{2}{4}, \frac{3}{16}, \frac{4}{64}, \ldots$

ii) $3, \frac{6}{5}, \frac{9}{25}, \frac{12}{125}, \frac{15}{625}, \ldots$

iii) $1, \frac{-4}{3}, \frac{7}{9}, \frac{-10}{27} \ldots$

Properties of Summation

i) $\sum_{r=1}^{n} kt_r = k\sum_{r=1}^{n} t_r$, ... $k \neq 0$

where k is a nonzero constant.

ii) $\sum_{r=1}^{n} (a_r + b_r) = \sum_{r=1}^{n} a_r + \sum_{r=1}^{n} b_r$

iii) $\sum_{1}^{n} 1 = n$

iv) $\sum_{1}^{n} k = k \sum_{1}^{n} 1 = kn$, ... $k \neq 0$

Result: 1) The Sum of the first n natural numbers

$$= \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$$

Result 2) The Sum of squares of first n natural numbers

$$= \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

Result 3) The sum of the cubes of the first n natural numbers $= \sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

The above results can be proved using Mathematical Induction. (Chapter No. 4)

SOLVED EXAMPLES

Ex 1) Evaluate $\sum_{r=1}^{n} (8r-7)$

Solution : $\sum_{r=1}^{n} (8r-7) = \sum_{r=1}^{n} 8r - \sum_{r=1}^{n} 7$

$= 8\sum_{r=1}^{n} r - 7\sum_{r=1}^{n} 1 = 8\left(\frac{n(n+1)}{2}\right) - 7n$

$= 4(n^2+n) - 7n = 4n^2 + 4n - 7n = 4n^2 - 3n.$

Ex 2) Find $3^2 + 4^2 + 5^2 + \ldots + 29^2$.

Solution: $3^2 + 4^2 + 5^2 + \ldots + 29^2$

$= (1^2 + 2^2 + 3^2 + \ldots + 29^2) - (1^2 + 2^2)$

$= \sum_{r=1}^{29} r^2 - \sum_{r=1}^{2} r^2$

$= \frac{29(29+1)(58+1)}{6} - \frac{2(2+1)(4+1)}{6}$

$= (29 \times 30 \times 59)/6 - (2 \times 3 \times 5)/6$

$= (29 \times 5 \times 59) - 5$

$= 5(29 \times 59 - 1) = 5(1711-1)$

$= 5(1710) = 8550$

Ex 3) Find $100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^2$

Solution : $100^2 - 99^2 + 98^2 - 97^2 + \ldots + 2^2 - 1^2$

$= (100^2 + 98^2 + 96^2 + \ldots + 2^2) - (99^2 + 97^2 + 95^2 + \ldots + 1^2)$

$= \sum_{r=1}^{50} (2r)^2 - \sum_{r=1}^{50} (2r-1)^2$

$= \sum_{r=1}^{50} (4r^2 - 4r^2 + 4r - 1)$

$= \sum_{r=1}^{50} (4r-1)$

$= \sum_{r=1}^{50} 4r - \sum_{r=1}^{50} 1$

$= 4\sum_{r=1}^{50} r - \sum_{r=1}^{50} 1 = 4 \times \frac{50(50+1)}{2} - 50$

$= 2 \times 50 \times 51 - 50$

$= 50(2 \times 51 - 1)$

$= 50(101)$

$= 5050.$

Ex 4) Find $1 \times 5 + 3 \times 7 + 5 \times 9 + 7 \times 11$ ... upto n terms.

Solution : Note that first factors of each term 1, 3, 5, 7, ... are in A.P. with a =1, d=2.

$t_r = a + (r-1)d = 1 + (r-1)2 = 2r-1.$

Also the second factors 5,7,9,11, ... are also in A.P. with a= 5 , d = 2 .

$t_r = 5 + (r-1)2 = 5 + 2r - 2 = 2r+3$

$\therefore S_n = \sum_{r=1}^{n} (2r-1)(2r+3)$

$= \sum_{r=1}^{n} (4r^2 + 4r - 3)$

$= 4\sum_{r=1}^{n} r^2 + 4\sum_{r=1}^{n} r - \sum_{r=1}^{n} 3$

$= 4\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} - 3n$

$= n\left[\frac{2(n+1)(2n+1)}{3}\right] + 2n(n+1) - 3n$

$= \frac{n}{3}[2(2n^2 + n + 2n + 1) + 6(n+1) - 9]$

$= \frac{n}{3}[(4n^2 + 6n + 2) + 6n - 3]$

$= \frac{n}{3}(4n^2 + 12n - 1)$

2.8 Power Series

Some functions can be expressed as infinite sums of powers of x. They are called powers series.

Some examlpes of Power Series are,

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$

2) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

3) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

4) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} \ldots$

5) If $|x| < 1$ then

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$$

The proofs of the equations given above are obtained at more advanced stage in mathematics.

EXERCISE 2.6

1) Find the sum $\sum_{r=1}^{n} (r+1)(2r-1)$

2) Find $\sum_{r=1}^{n} (3r^2 - 2r + 1)$

3) Find $\sum_{r=1}^{n} \left(\frac{1+2+3.....+r}{r}\right)$

4) Find $\sum_{r=1}^{n} \left(\frac{1^3+2^3+.....+r^3}{r(r+1)}\right)$

5) Find the sum $5 \times 7 + 9 \times 11 + 11 \times 3 + \ldots$ upto n terms.

6) Find the sum $2^2+4^2+6^2+8^2+\ldots$ upto n terms.

7) Find $(70^2 - 69^2) + (68^2-67^2)+(66^2-65^2) + \ldots + (2^2-1^2)$

8) Find the sum $1\times3\times5 + 3\times5\times7 + 5\times7\times9 + \ldots + n(n+1)(n+4)$

9) If $\frac{1\times2 + 2\times3 + 3\times4 + 4\times5 + \ldots \text{ upto n terms}}{1+2+3+4+\ldots \text{ upto n terms}} = \frac{100}{3}$, find n.

10) If S_1, S_2 and S_3 are the sums of first n natural numbers, their squares and their cubes respectively then show that -
$9\,S_2^{\,2} = S_3\,(1+8\,S_1)$.

Let's Remember

1) For A.P. $t_n = a + (n-1)\,d$

2) For G.P. $t_n = a\,r^{\,n-1}$, $S_n = \frac{a(r^n-1)}{(r-1)}$

3) A.M. of two numbers $A = \frac{x+y}{2}$

4) G.M. of two numbers $G = \sqrt{xy}$

5) H.M. of two numbers $H = \frac{2xy}{x+y}$

6) $G^2 = AH$

7) If $x = y$ then $A = G = H$

8) If $x \neq y$ then $H < G < A$.

MISCELLANEOUS EXERCISE - 2

(I) Select the correct answer from the given alternative.

1) The common ratio for the G.P. 0.12,0.24,0.48, is –

A) 0.12 B) 0.2 C) 0.02 D) 2 .

2) The tenth term of the geometric sequence $\frac{1}{4}, \frac{-1}{2}, 1, -2, \ldots$ is –

A) 1024 B) $\frac{1}{1024}$ C) -128 D) $\frac{-1}{128}$

3) If for a G.P. $\frac{t_6}{t_3} = \frac{1458}{54}$ then r = ?

A) 3 B) 2 C) 1 D) −1.

4) Which term of the geometric progression 1, 2, 4, 8, ... is 2048.

A) 10^{th} B) 11^{th} C) 12^{th} D) 13^{th}

5) If common ratio of the G.P is 5 , 5^{th} term is 1875 , the first term is -

A) 3 B) 5 C) 15 D) −5

6) The sum of 3 terms of a G.P. is $\frac{21}{4}$ and their product is 1 then the common ratio is –

A) 1 B) 2 C) 4 D) 8

7) Sum to infinity of a G.P. $5, -\frac{5}{2}, \frac{5}{4}, -\frac{5}{8}, \frac{5}{16}, \ldots$ is –

A) 5 B) $-\frac{1}{2}$ C) $\frac{10}{3}$ D) $\frac{3}{10}$

8) The tenth term of H.P. $\frac{2}{9}, \frac{1}{7}, \frac{2}{19}, \frac{1}{12}, \ldots$ is -

A) $\frac{1}{27}$ B) $\frac{9}{2}$ C) $\frac{5}{2}$ D) 27.

9) Which of the following is not true, where A, G, H are the AM, GM, HM of a and b respectively. (a, b > 0)

A) $A = \frac{a+b}{2}$ B) $G = \sqrt{ab}$

C) $H = \frac{2ab}{a+b}$ D) A = GH .

10) The G.M.of two numbers exceeds their H.M. by $\frac{6}{5}$, the A.M. exceeds G.M. by $\frac{3}{2}$ the two numbers are ...

A) 6, 15/2 B) 15, 25

C) 3, 12 D) 6/5, 3/2 .

(II) Answer the following.

1) In a G.P., the fourth term is 48 and the eighth term is 768. Find the tenth term.

2) Find the sum of the first 5 terms of the G.P. whose first term is 1 and common ratio is $\frac{2}{3}$

3) For a G.P. $a = \frac{4}{3}$ and $t_7 = \frac{243}{1024}$, find the value of r .

4) For a sequence , if $t_n = \frac{5^{n-2}}{7^{n-3}}$, verify whether the sequence is a G.P. If it is a G.P. , find its first term and the common ratio.

5) Find three numbers in G.P. such that their sum is 35 and their product is 1000.

6) Find five numbers in G.P. such that their product is 243 and sum of second and fourth number is 10.

7) For a sequence $S_n = 4(7^n - 1)$ verify that the sequence is a G.P.

8) Find 2+22+222+2222+ ... upto n terms.

9) Find the n^{th} term of the sequence 0.6,0.66,0.666,0.6666, ...

10) Find $\sum_{r=1}^{n} (5r^2 + 4r - 3)$

11) Find $\sum_{r=1}^{n} r(r-3)(r-2)$

12) Find $\sum_{r=1}^{n} \frac{1^2 + 2^2 + 3^2 + \ldots + r^2}{2r+1}$

13) Find $\sum_{r=1}^{n} \frac{1^3 + 2^3 + 3^3 + \ldots r^3}{(r+1)^2}$

14) Find $2 \times 6 + 4 \times 9 + 6 \times 12 + \ldots$ upto n terms.

15) Find $2 \times 5 \times 8 + 4 \times 7 \times 10 + 6 \times 9 \times 12 + \ldots$ upto n terms.

16) Find $\frac{1^2}{1} + \frac{1^2+2^2}{2} + \frac{1^2+2^2+3^2}{3} + \ldots$ upto n terms.

17) Find $12^2 + 13^2 + 14^2 + 15^2 + \ldots 20^2$

18) If $\frac{1+2+3+4+5+\ldots \text{ upto n terms}}{1\times2+2\times3+3\times4+4\times5+\ldots \text{ upto n terms}} = \frac{3}{22}$

Find the value of n .

19) Find $(50^2 - 49^2) + (48^2 - 47^2) + (46^2 - 45^2) + \ldots + (2^2 - 1^2)$.

20) If $\frac{1\times3+2\times5+3\times7+\ldots \text{ upto n terms}}{1^3+2^3+3^3+\ldots\text{upto n terms}} = \frac{5}{9}$, find the value of n.

21) For a G.P. if $t_2 = 7$, $t_4 = 1575$ find a

22) If for a G.P. $t_3 = 1/3$, $t_6 = 1/81$ find r

23) Find $\sum_{r=1}^{n} \left(\frac{2}{3}\right)^r$.

24) Find k so that k–1,k , k+2 are consecutive terms of a G.P.

25) If for a G.P. first term is $(27)^2$ and seventh term is $(8)^2$, find S_8.

26) If p^{th}, q^{th} and r^{th} terms of a G.P. are x,y,z respectively .Find the value of $x^{q-r} . y^{r-p} . z^{p-q}$

27) Which 2 terms are inserted between 5 and 40 so that the resulting sequence is G.P.

28) If p,q,r are in G.P. and $p^{1/x} = q^{1/y} = r^{1/z}$,verify whether x,y,z are in A.P. or G.P. or neither.

29) If a,b,c are in G.P. and $ax^2+2bx+c=0$ and $px^2+2qx+r=0$ have common roots then verify that $p b^2 - 2 q b a + r a^2 = 0$

30) If p,q,r,s are in G.P., show that $(p^2 + q^2 + r^2)(q^2 + r^2 + s^2) = (pq+qr+rs)^2$

31) If p,q,r,s are in G.P. , show that $(p^n + q^n)$, $(q^n + r^n)$, $(r^n + s^n)$ are also in G.P.

32) Find the coefficientof x^6 in the expansion of e^{2x} using series expansion .

33) Find the sum of infinite terms of

$$1 + \frac{4}{5} + \frac{7}{25} + \frac{10}{125} + \frac{13}{625} + \ldots$$

3 PERMUTATIONS AND COMBINATIONS

Let's Study

- Fundamental principles of counting
- Factorial notation
- Permutations
- Permutations of distinct objects
- Permutations when some objects are identical
- Circular permutations
- Combinations

Let's Recall

- The number system.
- The four basic mathematical operations: addition, subtraction, multiplication, division.

3.1 INTRODUCTION

Counting is a fundamental activity in Mathematics. Learning to count was our first step in learning Mathematics. After learning to count objects one by one, we used operations of addition and multiplication to make counting easy. We shall now learn two more methods of counting to make complicated counting easier. These two methods are known as permutations and combinations.

Permutations refer to the number of different arrangements of given objects, when the order of objects is important. Combinations are related to the number of different selections from a given set of objects, when the order of objects in the selections is immaterial.

The theory of permutations and combinations is central in problems of counting a large number of objects that are impossible to count manually. The theory of permutations and combinations enables us to count objects without listing or enumerating them.

Let us begin with the following simple example. Every smartphone requires a passcode to unlock it. A passcode is formed by four of the ten digits on the screen. The order of these four digits cannot be changed for passcode to work. How many distinct passcodes are possible? Note that a passcode consists of four digits. The first digit of a passcode can be any of the ten digits, the second digit can be any of the ten digits, and similarly for the third and fourth digits. This gives a total of $10 \times 10 \times 10 \times 10 = 10{,}000$ as the number of distinct possible passcodes.

Consider one more example which is not as easy as the last example. The school cricket team has eleven players. The school wants a photograph of these players, along with the principal and the two vice principals of the school, for school magazine. Seven chairs are arranged in a row for the photograph. Three chairs in the middle are reserved for the principal and the two vice principals. Four players will occupy the remaining four chairs and seven players will stand behind the chairs. The question then is: "In how many different ways can the eleven players take positions for the photograph?" This example will be considered later in the chapter. Till then, we can try, on our own, to find the number of different ways in which the eleven players can sit or stand for the photo.

Let us understand three principles of counting that are fundamental to all methods of counting, including permutations and combinations.

3.2 Fundamental principles of counting

Tree Diagram

We have learnt in set theory that subsets of a set can be represented in the form of a Venn diagram. An alternative method is to draw a tree diagram if the subsets are disjoint. For example, The games students play at school are of two types 1) Indoor games, 2) Outdoor games. Available indoor games in school are chess, carrom and table tennis. While outdoor, games in school are cricket, volleyball, basketball and badminton. Such information is presented with the help of tree-diagram as follows: (see fig 3.1)

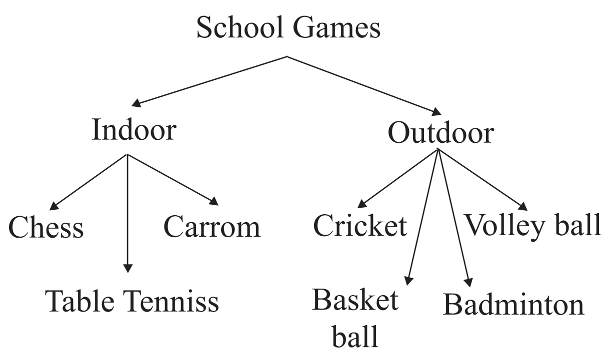


Fig. 3.1

Diagrams of this nature are called tree diagrams. A tree diagram shows the division of a set into disjoint subsets.

The fundamental principles of counting provide an efficient way of finding the number of different ways to carry out two or more activities, either simultaneously or successively (that is, one after another).

3.2.1 Addition Principle:

Consider the situation where a boy wants to go for movie. He has three T−shirts of three different colours: white, green, and blue. He also has four shirts of four different colours: red, green, yellow, and orange. How many choices does he have to wear? This situation can be represented using a tree diagram as follows. (see fig 3.2)

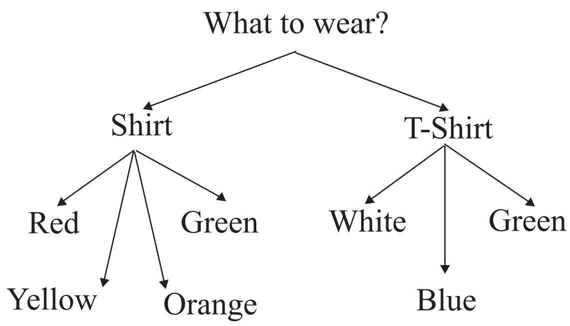


Fig. 3.2

The tree diagram shows that the boy has 4 choices in shirt and 3 choices in T-shirt, i.e. seven choices in all.

The boy can choose from 4 Shirts OR from 3 T-shirts. Hence, the number of ways of choosing are $3 + 4 = 7$ in all.

This example shows that the total number of outcomes is obtained by adding the number of outcomes of each characteristic when only one characteristic is to be chosen.

Statement of The Addition Principle.

Addition Principle : Suppose one operation can be done in m ways and another operation can be done in n ways, with no common way among them. If one of these operations is to be performed then there are $m + n$ ways to do it.

Ex. 1) A restaurant offers five types of fruit juices and three types of milk shakes. If a customer wants to order a drink, how many choices does the customer have?

Solution : Since the restaurant offers five fruit juices, the customer has five ways of selecting a fruit juices. Similarly, since the restaurant offers three types of milk shakes, the customer has three ways of selecting a milk shake. Finally, since the customer wants to select only one drink, there are $5 + 3 = 8$ choices for the customer.

Ex. 2) Consider an experiment of drawing a card from a pack of 52 playing cards. What is the number of ways in which the drawn card is a spade or a club?

Solution : Since there are 13 spade cards and 13 club cards hence the number of ways in which the drawn card is spade or club is $13 + 13 = 26$ ways.

Note : The word 'OR' in the statement suggests addition, i.e. '+'.

3.2.2 Multiplication principle

Now, consider the following situation. An Ice cream is served either in a cup or in a cone. Also, Ice cream is available in three flavours: vanilla, chocolate, and strawberry. This information can be represented in the form of a tree diagram as follows. (see Fig 3.3)

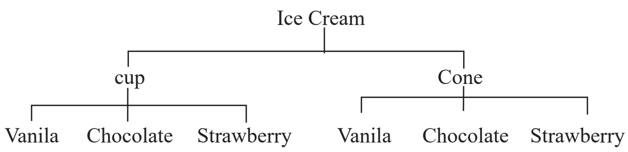


Fig. 3.3

There are 2 ways of choosing mode of serving and 3 ways of choosing flavours.

So, the tree diagram shows that there are six possible outcomes, i.e. $2 \times 3 = 6$ ways to serve Ice cream.

In the example of Ice cream, the final count of six is obtained by multiplying 3 (number of flavours) by 2 (number of serving options), in either order.

Definition of The Multiplication Principle

If one operation can be carried out in m ways, followed by the second operation that can be carried out in n ways, and these two operations are independent then the two operations can be carried out in $m \times n$ ways.

Ex. 1) : Samadhan Bhojanalay offers a thali that has four items: roti, rice, vegetable and dal. Following choices are available and one option is to be selected for each item.

Roti: chapati, tandoor roti

Rice: plain rice, jeera rice, dal khichadi

Vegetable: dum aloo, paneer masala, mixed veg

Dal: dal fry, dal tadka

How many different menus are possible?

Solution : Since one option is to be selected for each item, the number of different possible thali choices are identified as follows.

Roti 2, Rice 3, Vegetable 3, Dal 2 and hence the total number of different possible thali menus is $2 \times 3 \times 3 \times 2 = 36$.

Ex. 2) A company decides to label each of its different products with a code that consists of two letters followed by three digits. How many different products can be labeled in this way?

Solution : Since the first two characters in a label are letters, they can be formed in $26 \times 26 = 676$ ways. The next three characters are digits and can be formed in $10 \times 10 \times 10 = 1000$ ways. The total number of distinct labels is, by the multiplication principle, given by $676 \times 1000 = 6{,}76{,}000$.

Activity 1) Suresh has 4 pencils and 2 erasers. He wants to take one pencil and one eraser for the examination. Can we find the number of ways in which he can select a pencil and an eraser?

Activity 2) Sunil has 4 ballpens of one company and 3 ballpens of another company. In how many ways can he select a ball pen?

In the above activities, can we decide when to use the addition principle and when to use the multiplication principle? Can we give reasons? What are answers in the above examples?

Remark: The addition and multiplication principles can be extended from two to any finite number of activities, experiments, events, or operations.

Extended Addition Principle : Suppose there are three possible choices with no common outcome for any two, the first choice can be made in m ways, second in n ways and third in r ways. If only one of the choices is to be made, it has $m + n + r$ possible ways.

Extended Multiplication Principle: Suppose an experiment consists of three independent activities, where first activity has m possible outcomes, second has n possible outcomes, third has r possible outcomes. Then the total number of different possible outcomes of the experiment is $m \times n \times r$.

3.3 Invariance Principle

The result of counting objects in a set does not depend on the order in which these objects are counted or on the method used for counting these objects.

For example, 1) In earlier example of addition principle of choosing first from 4 Shirts, then from 3 T-Shirts is same as choosing first from 3 T-Shirts and then from 4 Shirts.

2) In earlier example of multiplication principle choosing first from 2 modes of servings and then from 3 flavours is same as choosing first from 3 flavours and then from the 2 modes of servings.

SOLVED EXAMPLES

Ex. 1) : From the figure below, find the total number of routes from A to B. e.g. one upper route is $A \to D \to N \to E \to B$. (See fig 3.4)

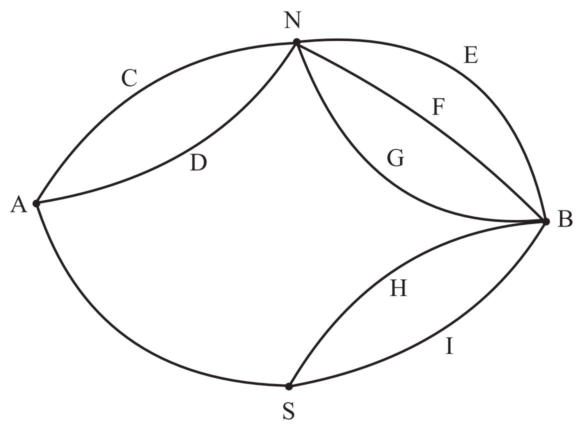


Fig 3.4

Solution: $A \xrightarrow{D} N \xrightarrow{E} B$ is one route through N. $A \xrightarrow{C} N \xrightarrow{F} B$ is another route through N. Thus there are in all $2 \times 3 = 6$ routes through N. There are 2 routes through S. Hence the total number of routes is $6 + 2 = 8$.

Ex. 2) : Suppose 5 chocoloates of different type are to be distributed among 4 children and there is no condition on how many chocolates a child can get (including zero.) How many different ways are possible for doing so?

Solution: The first chocolate can be given to any of the four children. Therefore, there are four different ways of giving the first chocolate. Similarly, the second chocolate can be given in four different ways, and similarly for each of the remaining chocolates. The multiplication principle then gives the total number of different ways as $4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$.

Ex. 3) : How many even numbers can be formed using the digits 2, 3, 7, 8 so that the number formed is less than 1000?

Solution: The condition that the number formed from the given digits is less that 1000 means that this number can have upto 3 digits. Let us therefore consider the three cases separately,

case i) One digit numbers

Since the number must be even, the one–digit number can be only 2 or 8. Hence, there are 2 ways of forming a one digit even number.

case ii) Two digit numbers

Since the required number is even, the units place of a two–digit number must be either 2 or 8 i.e. 2 ways. The ten's place can be filled with any of the four given digits. Therefore, there are $4 \times 2 = 8$ ways of forming a two–digit even number.

case iii) Three digit numbers

Finally,since the requried number is even, the units place can be filled in two ways, ten's place can be filled in four ways, and hundred's place can also be filled in four ways. The number of ways of forming a three digit even number is $4 \times 4 \times 2 = 32$

The addition principle finally gives the total number of ways of forming an even number less than 1000 using digits 2,3,7,8 is $2+8+32=42$.

EXERCISE 3.1

1. A teacher wants to select the class monitor in a class of 30 boys and 20 girls. In how many ways can the monitor be selected if the monitor must be a girl?
2. A Signal is generated from 2 flags by putting one flag above the other. If 4 flags of different colours are available, how many different signals can be generated?
3. How many two letter words can be formed using letters from the word SPACE, when repetition of letters (i) is allowed, (ii) is not allowed?
4. How many three–digit numbers can be formed from the digits 0, 1, 3, 5, 6 if repetitions of digits (i) are allowed, (ii) are not allowed?
5. How many three–digit numbers can be formed using the digits 2, 3,4,5,6 if digits can be repeated?
6. A letter lock has 3 rings and each ring has 5 letters. Determine the maximum number of trials that may be required to open the lock.
7. In a test, 5 questions are of the form 'state, true or false'. No student has got all answers correct. Also, the answer of every student is different. Find the number of students appeared for the test.
8. How many numbers between 100 and 1000 have 4 in the units place?
9. How many numbers between 100 and 1000 have the digit 7 exactly once?
10. How many four digit numbers will not exceed 7432 if they are formed using the digits 2,3,4,7 without repetition?

11. If numbers are formed using digits 2, 3, 4, 5, 6 without repetition, how many of them will exceed 400?

12. How many numbers formed with the digits 0, 1, 2, 5, 7, 8 will fall between 13 and 1000 if digits can be repeated?

13. A school has three gates and four staircases from the first floor to the second floor. How many ways does a student have to go from outside the school to his classroom on the second floor?

14. How many five–digit numbers formed using the digit 0, 1, 2, 3, 4, 5 are divisible by 5 if digits are not repeated?

3.4 Factorial Notation.

The theory of permutations and combinations uses a mathematical notation known as the factorial notation. Let us first understand the factorial notation, which is defined for natural number.

Definition: For a natural number n, the factorial of n, written as n! or $\lfloor\underline{n}$ and read as "n factorial", is the product of n natural numbers from 1 to n.

That is, n! or $\lfloor\underline{n}$ is expressed as

$1 \times 2 \times 3 \times .... \times (n-2) \times (n-1) \times n.$

Note: The factorial nonation can also be defined as the product of the natural numbers from n to 1.

That is, $n! = n \times (n-1) \times .... \times 3 \times 2 \times 1$

For example, $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

It is read as "5 factorial is equal to 120."

Illustrations:

Factorial 1 = 1! = 1

Factorial 2 = $2! = 2 \times 1 = 2$

Factorial 3 = $3! = 3 \times 2 \times 1 = 6$

Factorial 4 = $4! = 4 \times 3 \times 2 \times 1 = 24$, and so on.

Note: Though 0 is not a natural number, we define 0! = 1.

Properties of the factorial notaction.

For any positive integers m,n.,

1) $n! = n \times (n-1)!$
2) $n > 1,\ n! = n \times (n-1) \times (n-2)!$
3) $n > 2,\ n! = n \times (n-1) \times (n-2) \times (n-3)!$
4) $(m+n)!$ is always divisible by m! as well as by n! e.g. $(3+4)!$ is divisible by 3! as well as 4!
5) $(m \times n)! \neq m! \times n!$
6) $(m+n)! \neq m! + n!$
7) $m > n$, $(m-n)! \neq m! - n!$ but m! is divisible by n!
8) $(m \div n)! \neq m! \div n!$

Ex. 1) : Find the value of 6!

Solution : $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

Ex. 2) : Show that $(7-3)! \neq 7! - 3!$

Solution: $(7-3)! = 4! = 4 \times 3 \times 2 \times 1 = 24.$

$7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$

Also, $3! = 3 \times 2 \times 1 = 6.$

$7! - 3! = 5040 - 6 = 5034$

Therefore, $(7-3)! \neq 7! - 3!$

Ex. 3): Find n if (n+6)! = 56 (n+4)!

Solution: (n+6)! = 56 (n+4)!

$\therefore$ (n+6) (n+5) (n+4)! = 56 (n+4)!

$\therefore$ (n+6) (n+5) = 56

Here instead of solving quadratic in n, we write 56 as product of 2 consecutive numbers as 56 = 8×7

$\therefore$ (n+6) (n+5) = 8×7

Equating bigger factors from either side

$\therefore$ (n+6) = 8

n = 8 – 6 = 2

Ex. 4) : Show that $\frac{12!}{5!7!}+\frac{12!}{6!6!}=\frac{13!}{6!7!}$

Solution: Consider,

$$\text{L. H. S.} = \frac{12!}{5!7!}+\frac{12!}{6!6!}$$

$$= 12!\left[\frac{1}{5!\times 7\times 6!}+\frac{1}{5!\times 6\times 6!}\right]$$

$$(\because\ 7! = 7\times 6!\quad 6! = 6\times 5!)$$

$$= \frac{12!}{5!6!}\left[\frac{1}{7}+\frac{1}{6}\right]$$

$$= \frac{12!}{5!6!}\left[\frac{13}{6\times 7}\right]$$

$$= \frac{12!\times 13}{(5!\times 6)\times(6!\times 7)}$$

$$= \frac{13!}{6!\times 7!}$$

= R. H. S.

EXERCISE 3.2

1. Evaluate:

   (i) 8! (ii) 10!

   (iii) 10! – 6! (iv) (10 – 6)!

2. Compute:

   (i) $\frac{12!}{6!}$ (ii) $\left(\frac{12}{6}\right)!$

   (iii) $(3\times 2)!$ (iv) $3!\times 2!$

   (v) $\frac{9!}{3!\,6!}$ (vi) $\frac{6!-4!}{4!}$

   (vii) $\frac{8!}{6!-4!}$ (viii) $\frac{8!}{(6-4)!}$

3. Write in terms of factorials

   (i) $5\times 6\times 7\times 8\times 9\times 10$

   (ii) $3\times 6\times 9\times 12\times 15$

   (iii) $6\times 7\times 8\times 9$

   (iv) $5\times 10\times 15\times 20$

4. Evaluate : $\frac{n!}{r!(n-r)!}$ for

   (i) n = 8, r = 6 (ii) n = 12, r = 12,

   (iii) n = 15, r = 10 (iv) n = 15, r = 8

5. Find n, if

   (i) $\frac{n}{8!}=\frac{3}{6!}+\frac{1!}{4!}$ (ii) $\frac{n}{6!}=\frac{4}{8!}+\frac{3}{6!}$

   (iii) $\frac{1!}{n!}=\frac{1!}{4!}-\frac{4}{5!}$

   (iv) $(n+1)! = 42\times(n-1)!$

   (v) $(n+3)! = 110\times(n+1)!$

6. Find n, if:

   (i) $\frac{(17-n)!}{(14-n)!}=5!$ (ii) $\frac{(15-n)!}{(13-n)!}=12$

   (iii) $\frac{n!}{3!(n-3)!}:\frac{n!}{5!(n-5)!}=5:3$

(iv) $\dfrac{n!}{3!(n-3)!}:\dfrac{n!}{5!(n-7)!}=1:6$

(v) $\dfrac{(2n)!}{7!(2n-7)!}:\dfrac{n!}{4!(n-4)!}=24:1$

7. Show that

$$\frac{n!}{r!(n-r)!}+\frac{n!}{(r-1)!(n-r+1)!}=\frac{(n+1)!}{r!(n-r+1)}$$

8. Show that

$$\frac{9!}{3!6!}+\frac{9!}{4!5!}=\frac{10!}{4!6!}$$

9. Show that

$$\frac{(2n)!}{n!}=2^n(2n-1)(2n-3)...5.3.1$$

10. Simplify

(i) $\dfrac{(2n+2)!}{(2n)!}$ (ii) $\dfrac{(n+3)!}{(n^2-4)(n+1)!}$

(iii) $\dfrac{1}{n!}-\dfrac{1}{(n-1)!}-\dfrac{1}{(n-2)!}$

(iv) $n[n!+(n-1)!]+n^2(n-1)!+(n+1)!$

(v) $\dfrac{n+2}{n!}-\dfrac{3n+1}{(n+1)!}$

(vi) $\dfrac{1}{(n-1)!}+\dfrac{1-n}{(n+1)!}$

(vii) $\dfrac{1}{n!}-\dfrac{3}{(n+1)!}-\dfrac{n^2-4}{(n+2)!}$

(viii) $\dfrac{n^2-9}{(n+3)!}+\dfrac{6}{(n+2)!}-\dfrac{1}{(n+1)!}$

3.5 Permutations: (When all objects are distinct)

If we want to place 3 persons on 3 chairs in a row, in how many ways can we obtain the seating arrangement?

Suppose the 3 seats are numbered 1, 2 and 3 and the 3 persons are named A, B and C.

We can fill the 1st chair in 3 ways. Having done that, we can fill the 2nd chair with any of the remaining two people, hence in 2 ways. The 3rd chair is then filled in a unique way as there is only one person left. Thus the total number of the seating arrangements are

$3 \times 2 \times 1 = 3! = 6$

This can also be checked by listing them as follows

ABC	BAC	CAB
ACB	BCA	CBA.

Extending this result for n persons to be placed in n chairs in a row, we get $n!$ ways of arrangement.

Note that all persons are distinct and chairs have their own ordinal numbers (1st, 2nd, 3rd etc.)

Now we see a different problem. There are 4 persons and 2 chairs in a row to be filled. The 1st chair can be filled in 4 ways. Having done this, the 2nd chair can be filled in 3 ways. So the number of different arrangements is $4 \times 3 = 12$.

See the enumeration

AB	BA	CA	DA
AC	BC	CB	DB
AD	BD	CD	DC

Extending this argument if 7 different objects are available and 3 boxes are in a row. We want to place one object in each box. There are $7 \times (7-1) \times (7-2) = 7 \times 6 \times 5$ different ways to do it.

A permutation is formally defined as follows.

Permutation : A permutation is an arrangement, in a definite order of a number of objects, taken some or all at a time.

The number of distinct permutations of r distinct objects chosen from a given collection of n distinct objects is denoted by nPr, $^{n}P_{r}$, or P(n,r).

3.5.1 Permutations when all objects are distinct [$r \leq n$]:

Theorem 1. The number of permutations of n distinct objects taken r ($r \leq n$) at a time, without repetitions, is

$$n \times (n-1) \times (n-2) \times \ldots \times (n-r+1).$$

Proof. The number of ways of arranging n distinct objects taken r at a time without repetitions is same as the number of ways r places can be filled using n objets ($r \leq n$).

For this, consider the following table.

Place	1^{st}	2^{nd}	3^{rd}	...	$(r-2)^{th}$	$(r-1)^{th}$	r^{th}
Number of ways	n	n–1	n–2	...	[n–(r–3)]	[n–(r–2)]	[n–(r–1)]

The table shows that the first place can be filled with any of the n objects. As the result, there are n ways of filling the the first place. After putting one object in the first place, only n–1 objects are available because repetitions are not allowed. Therefore, the second place can be filled in n–1 ways. After putting two distinct objects in the first two places, only n–2 objects are available for the third place, so that the third place can filled in n–2 ways.

Continuing in this way, after putting r–1 objects in the first r–1 places, the number of available objects is [n – (r–1)] for the r^{th} place. Hence, the r^{th} place can be filled in [n–(r–1)] ways.

Now, using the multiplication principle of counting, the total number of ways of filling r places using n distinct objects is denoted by $^{n}P_{r}$ and given by the product

$$n \times (n-1) \times (n-2) \times \ldots \times ([n-(r-2)] \times [n-(r-1)]$$

$$= n \times (n-1) \times (n-2) \times \ldots \times (n-r+2) \times (n-r+1)$$

Hence $^{n}P_{r} = n \times (n-1) \times (n-2) \times \ldots \times (n-r+2) \times (n-r+1)$

If we multiply and divide this product by

$(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1$,

we find that

$$^{n}P_{r} = n \times (n-1) \times \ldots \times (n-r+1) \times \frac{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}$$

$$= \frac{n \times (n-1) \times (n-2) \times \ldots \times (n-r+1) \times (n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}{(n-r) \times (n-r-1) \times \ldots \times 3 \times 2 \times 1}$$

$$= \frac{n!}{(n-r)!}$$

We have thus found that

$$^{n}P_{r} = \frac{n!}{(n-r)!} \quad (\text{for } r \leq n)$$

Note:

When $r = n$ i.e. all n objects are placed in a row.

$$^{n}P_{n} = n \times (n-1) \times (n-2) \times \ldots \times [n-(n-1)]$$

$$= n \times (n-1) \times (n-2) \times \ldots \times 1 = n!$$

Alternatively, from the above formula, we obtain

$$^{n}P_{n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

(since 0!=1, by definition)

SOLVED EXAMPLES

Ex. 1) Find the value of ${}^5P_2$.

Solution: $5P_2 = \frac{5!}{(5-2)!} = \frac{5!}{3!} = \frac{5\times4\times3!}{3!}$

$= 5 \times 4 = 20.$

Ex. 2) How many different ways are there to arrange letters of the word 'WORLD'? How many of these arrangements begin with the letter R? How many arrangements can be made taking three letters at a time?

Solution: The word WORLD has 5 letters W, O, R, L, D. These can be arranged among themselves in ${}^5P_5$=5!=120! different ways.

If an arrangement begins with R, the remaining four letters can be arranged among themselves in ${}^4P_4$=4!=24 ways.

The Number of arrangements of 5 letters, taken 3 at a time, is ${}^5P_3 = \frac{5!}{2!} = 60$

Ex. 3) How many three digit numbers can be formed from the digits 2,4,5,6,7 if no digit is repeated?

Solution: Every arrangement of digits gives a different number. Therefore, the problem is to find the number of arrangements of five digits taken two at a time This is given by ${}^5P_3 = \frac{5!}{2!} = 60$

Ex 4) How many numbers can be formed with the digits 3, 4, 6, 7, 8 taken all at a time? Find the sum of all such numbers.

Solution : The five digits can be arranged in ${}^5P_5$ = 5! = 120 ways.

Now, consider any one of the five given digits, say 3. Suppose the digit 3 is in the unit's place. The other four digits can be arranged in ${}^4P_4$ = 4! = 24 ways to form numbers that have 3 in the unit's place. This shows that 24 of the 120 numbers have 3 in the unit's place.

Similarly, each of the other four digits is in the unit's place in 24 of the 120 numbers.

The sum of the digits in the units place among all 120 numbers is 24 (3+4+6+7+8) = 24×28 = 672.

Similary, the sum of the digits in the ten's place among all 120 numbers is 672. The same is also the sum of the digits in each place among all 120 numbers.

The required sum is 672 units + 672 tens + 672 hundreds + 672 thousands + 672 ten thousands.

$= 672 \times (1+10+100+1000+10000)$

$= 672\times11111 = 74{,}66{,}592$

Ex 5) A teacher has 2 different books on English, 3 different books on Physics, and 4 different books on Mathematics. These books are to be placed in a shelf so that all books on any one subjects are together. How many different ways are there to do this?

Solution : First, let us consider all books on each subjects to be one set, so that there are three sets, say E,P,M. These three can be arranged in ${}^3P_3$ = 3! = 6 different ways. Now, in each of these ways, the 2 books on English can be arranged in ${}^2P_2$ = 2! = 2 different ways the 3 books on physics can be arragned in ${}^3P_3$ =3! = 6 different ways, and the four books on Mathematics can be arranged in ${}^4P_4$ = 4! = 24 different ways.

The required number of arrangments is then given by 6×2×6×24 = 1728

Ex 6) Find n if ${}^{n}P_{5} = 42 \times {}^{n}P_{3}$

Solution: We are given that ${}^{n}P_{5} = 42 \times {}^{n}P_{3}$

That is, $\dfrac{n!}{(n-5)!} = 42\dfrac{n!}{(n-3)!}$

$$\therefore \frac{n!}{(n-5)!} = 42\frac{n!}{(n-3)(n-4)(n-5)!}$$

$\therefore (n-3)(n-4) = 42$

$\therefore (n-3)(n-4) = 7\times 6$

$\therefore n-3 = 7$

$\therefore n = 3+7 = 10$

3.5.2 Permutations when repetitions are allowed:

We now consider problems of arranging n objects taken r at a time when repetitions are allowed.

Theorem 2. The number of arrangements of n distinct objects taken r at a time, when repetitions are allowed, is same as the number of ways of filling r places using n distinct objects when repetitions are allowed. Consider the following table.

Place	1st	2nd	3rd	…	(r–2)th	(r–1)th	rth
Number of ways	n	n	n	…	n	n	n

Because repetitions are allowed, each place can be filled in n different ways.

Using multiplication principle, it can be concluded that the number of permutations of n distinct objects taken r at a time, when repetitions are allowed, is given by

$n\times n\times \ldots \times n(r \text{ times}) = n^r$

SOLVED EXAMPLES

Ex 1: It is required to arrange 8 books on a shelf. Find the number of ways to do this if two specified books are

(i) always together (ii) never together.

Solution:

(i) Consider 2 books as one single set. These 2 books can be arrannged among themselves in 2! ways. The total 7 books can be arranged among themselves in 7! ways.

The total number of ways = 2! 7! = 2 × 5040 = 10080.

(ii) We take one of the two books and remaining 6 books and arrange these in 7! ways. Then we can place the remaining book in such a way to avoid 2 places adjacent to the 1st book, so there are 6 places available hence 6 × 7! = 6 × 5040 = 30240.

Alternative Method :

The number is also obtained by subtracting from the total arrangements those when these books are together i.e.

8! – 2 × 7! = (8 – 2) × 7! = 6 × 7! = 6 × 5040 = 30240.

Ex 2: In how many ways can 7 examination papers be arranged so that papers 6 and 7 are never together?

Solution : By the same argument as in example above, the number of ways in which any two papers are never together is.

(7–2) (7–1)! = 5×6! = 3600

Ex 3: A family of 3 brothers and 5 sisters is to be arranged for a photograph in such a ways that, (i) all brothers sit together. (ii) no two brothers sit together.

Solution :

(i) Since all 3 brothers are together, treat them as one person, so that there are 5+1=6 persons. the number of arranging them is

$^6P_6 = 6! = 720.$

Once this is done, the three brothers can be arranged among themselves in $^3P_3 = 3! = 6$ ways.

The total number of arrangements is then given by $6!\times3! = 720 \times 6 = 4320$.

(ii) 5 sisters can be arranged among themselves in $^5P_5 = 5! = 120$ ways.

Consider the following arrangement.

$$* \, S_1 * S_2 * S_3 * S_4 * S_5 *$$

Where * indicates a position where one brother can be placed so that no two brothers are together. Since there are 6 such positions and 3 brothers, the number of arrangements is

$$^6P_3 = \frac{6!}{3!} = 120$$

The required number of arrangements is then given by

$^5P_5 \times {}^6P_3 = 120 \times 120$

$= 14400.$

If some of the n objects are always kept together in a permutation problem, then the following theorem is useful for such cases.

Theorem. The number of permutations taken all at a time, when m specified objects among n always come together, is m!(n–m+1)!

Proof. Since the specified m objects always come together, let us consider them as a single object. This makes the number of distinct objects n–m+1 for the purpose of permutations. The number of permutations of n–m+1 objects taken all at a time is (n–m+1)!. The m specified objects are together, but can be rearranged among themselves. The number of permutations of these m objects taken all at a time is m!. Since this argument holds for each of the (n–m+1)! permutations, the required number of permutations of n objects taken all together, when m specified objects are always together is m! (n–m+1)!

Remarks:

1. The number of permutations of n distinct objects taken all at a time, when 2 specified objects are always together, is $2\times(n-1)!$.

2. The number of permutations of n distinct objects taken r at a time, when a specified object is always to be included, is.

   $r \times {}^{(n-1)}P_{(r-1)}$.

   First keep the specified object aside. Arrange (r–1) from the remaining (n–1) objects in $^{(n-1)}P_{(r-1)}$ ways. Then place the specified object in r possible ways. Hence the total number of arrangements is $r \times {}^{(n-1)}P_{(r-1)}$.

3. The number of permutations of n distinct objects taken r at a time, when a specified object is not be included in any permutation, is $^{(n-1)}P_r$.

EXERCISE 3.3

1. Find n, if $^nP_6 : {}^nP_3 = 120:1$

2. Find m and n, if $^{(m+n)}P_2 = 56$ and

   $^{(m-n)}P_2 = 12$

3. Find r, if $^{12}P_{r-2} : {}^{11}P_{r-1} = 3:14$

4. Show that $(n+1)\,({}^nP_r) = (n-r+1)\,[{}^{(n+1)}P_r]$

5. How many 4 letter words can be formed using letters in the word MADHURI if (a) letters can be repeated (b) letters cannot be repeated.

6. Determine the number of arrangements of letters of the word ALGORITHM if.

   (a) vowels are always together.

   (b) no two vowels are together.

   (c) consonants are at even positions.

   (d) O is the first and T is the last letter.

7. In a group photograph, 6 teachers are in the first row and 18 students are in the second row. There are 12 boys and 6 girls among the students. If the middle position is reserved for the principal and if no two girls are together, find the number of arrangements.

8. Find the number of ways so that letters of the word HISTORY can be arranged as,

   (a) Y and T are together

   (b) Y is next to T.

   (c) there is no restriction

   (d) begin and end with vowel

   (e) end in ST

   (f) begin with S and end with T

9. Find the number of arrangements of the letters in the word SOLAPUR so that consonents and vowels are placed alternately.

10. Find the number of 4–digit numbers that can be formed using the digits 1, 2, 4, 5, 6, 8 if

    (a) digits can be repeated

    (b) digits cannot be repeated

11. How many numbers can be formed using the digits 0, 1, 2, 3, 4, 5 without repetition so that resulting numbers are between 100 and 1000?

12. Find the number of 6–digit numbers using the digits 3,4,5,6,7,8 without repetition. How many of these numbers are

    (a) divisible by 5, (b) not divisible by 5.

13. A code word is formed by two different English letters followed by two non–zero distinct digits. Find the number of such code words. Also, find the number of such code words that end with an even digit.

14. Find the number of ways in which 5 letters can be posted in 3 post boxes if any number of letters can be posted in a post box.

15. Find the number of arranging 11 distinct objects taken 4 at a time so that a specified object.

    (a) always occurs (b) never occurs.

16. In how many ways can 5 different books be arranged on a shelf if

    (i) there are no restrictions

    (ii) 2 books are always together

    (iii) 2 books are never together

17. 3 boys and 3 girls are to sit in a row. How many ways can this be done if

    (i) there are no restrictions

    (ii) there is a girl at each end

    (iii) boys and girls are at alternate places

    (iv) all boys sit together

3.5.3 Permutations when some objects are identical

Consider the problem of arranging letters in the words like GOOD, INDIA, GEOLOGY, MATHEMATICS, or PHILOSOPHY, where some letters occur more than once and hence all letters are not distinct.

Consider the word ODD. First we consider the 2 D's as distinct objects D_1 and D_2.

The total number of words is OD_1D_2, OD_2D_1, D_1OD_2, D_2OD_1, D_1D_2O, $D_2D_1O = 6 = 3!$

But D_1 and D_2 are identical so, $OD_1D_2 = OD_2D_1$, $D_1D_2O = D_2D_1O$, $D_2OD_1 = D_1OD_2$ and as D_1 and D_2 can be arranged among themselves in 2! ways, (D_1D_2, D_2D_1)

Thus there are $\frac{6}{2} = \frac{3!}{2!} = 3$ different words formed (ODD,DOD,DDO)

Theorem : Consider a set of n objects where n_1 objects are identical and the remaining $n - n_1$ are distinct. The number of permutations of these n objects is $\frac{n!}{n_1!}$. Note that $n_1 < n$.

Proof : Let m be the total number of arrangements, where n_1 out of n objects are identical. The number of permutations of n different objects is $n!$. Now n_1 objects can be rearranged in $n_1!$ ways among themselves if they were all different. Thus each arrangement where these n_1 objects are identical corresponds to $n_1!$ different arrangements if they were all different.

Hence $m \times n_1! = n!$. Therefore $m = \frac{n!}{n_1!}$

Remarks.

1. The number of permutations of n objects, not all distinct, where n_1 objects are of one type and n_2 objects are of a second type, taken all at a time is $\frac{n!}{n_1!n_2!}$ and note that $n_1 + n_2 \leq n$.

   The proof is similar to the proof of the above theorem.

2. The number of permutations of n objects, not all distinct, where n_i objects are of type i, i=1, 2, …, k, taken all at a time is $\frac{n!}{n_1!n_2!...n_k!}$, $n_1 + n_2 + ... n_k \leq n$.

SOLVED EXAMPLES

Ex 1 : Find he number of permutations of the letters of the word UBUNTU.

Solution : The word UBUNTU consists of 6 letters, in which letter 'U' is repeated 3 times.

Therefore, number of permutations of the letters of the word UBUNTU = $\frac{6!}{3!} = 120$.

Ex 2 : How many arrangements can be made, with the letters of the word CALCULATOR? In how many of these arrangements, vowels occur together?

Solution : The word CALCULATOR consists of 10 letters, in which 'C' is repeated two times, 'A' is repeated two times, 'L' is repreated two times and rest all are different.

Therefore, number of permutations of the letters of the word CALCULATOR = $\frac{10!}{2!\,2!\,2!}$.

The word CALCULATOR consists of 4 vowels A, U, A, O. Let us consider them as a single letter say P.

Therefore, now we have 7 letters P, C, L, C, L, T, R in which 'C' is repeated two times, 'L' is repeated two times. The number of arrangement of these 7 letters is given $\frac{7!}{2!\,2!} = 1260$. After this is done, 4 vowels (in which 'A' is repeated 2 times) can be arranged in $\frac{4!}{2!} = 12$ ways

Therefore, number of arrangements of the letters of the word CALCULATOR in which vowels are together = $1260 \times 12 = 15120$.

EXERCISE 3.4

1. Find the number of permutations of letters in each of the following words.
   (i) DIVYA
   (ii) SHANTARAM
   (iii) REPRESENT
   (iv) COMBINE
   (v) BALBHARATI

2. You have 2 identical books on English, 3 identical books on Hindi, and 4 identical books on Mathematics. Find the number of distinct ways of arranging them on a shelf .

3. A coin is tossed 8 times. In how many ways can we obtain. (a) 4 heads and 4 tails? (b) at least 6 heads?

4. A bag has 5 red, 4 blue, and 4 green marbles. If all are drawn one by one and their colours are recorded, how many different arrangements can be found?

5. Find the number of ways of arranging letters of the word MATHEMATICAL How many of these arrangements have all vowels together?

6. Find the number of different arrangements of letters in the word MAHARASHTRA. How many of these arrangements have (a) letters R and H never together? (b) all vowels together?

7. How many different words are formed if the letters R is used thrice and letters S and T are used twice each?

8. Find the number of arrangements of letters in the word MUMBAI so that the letter B is always next to A.

9. Find the number of arrangements of letters in the word CONSTITUTION that begin and end with N.

10. Find the number of different ways of arranging letters in the word ARRANGE. How many of these arrangement do not have the two R's nor A's together?

11. How many distinct 5 digit numbers can be formed using the digits 3, 2, 3, 2, 4, 5.

12. Find the number of distinct numbers formed using the digits 3, 4, 5, 6, 7, 8, 9, so that odd positions are occupied by odd digits.

13. How many different 6–digit numbers can be formed using digits in the number 659942? How many of them are divisible by 4?

14. Find the number of distinct words formed from letters in the word INDIAN. How many of them have the two N's togethers?

15. Find the number of different ways of arranging letters in the word PLATOON if. (a) the two O's are never together. (b) consonants and vowels occupy alternate positions.

3.5.4 Circular permutation:

We can imagine a circular arrangement of n different objects to be transformed into an arrangement in a line by cutting the circle in a place. This cut can be made at n different places.

For example, consider $n = 4$

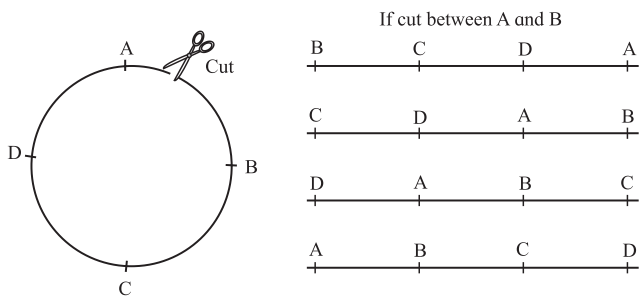


Fig. 3.5

Thus given a single circular arrangement of 4 objects, it corresponds to 4 different linear arrangements (arrangements in a row).

Let the circular arrangements be m.

We know that the total number of linear arrangement of 4 different objects is 4!

Each arrangement in row corresponds to some circular arrangement.

Therefore number of circular arrangements is $m = \frac{4!}{4} = 3!$

Similarly, each circular arrangement of n objects corresponds to n different arrangements in a row.

Therefore number of circular arrangements is $\frac{n!}{n} = (n-1)!$

Note:

1) A circular arrangement does not have a fixed starting point and any rotation of it is considered to be the same circular arrangement. This arrangement will be the same with respect to each other if a rotation is made but clockwise and anticlockwise arrangements are different.

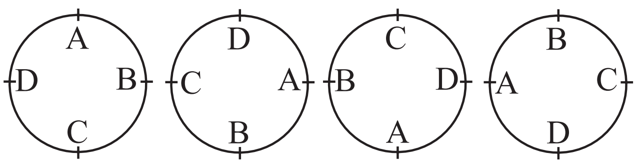


Fig. 3.5 (a)

In circular arrangement, (with respect to each other) all these count as one arrangement.

2) If clockwise and anticlockwise circular arrangements are considered to be the same then each circular arrangement corresponds to 2n different linear arrangements. Thus the numbers of such circular arrangement is $\frac{n!}{2n} = \frac{(n-1)!}{2}$.

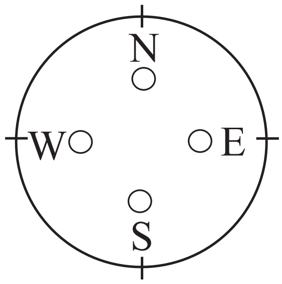


Fig. 3.5 (b)

Consider the 4 directions, E, N, W, S on Mariner's compass. If 4 stickers of different colours are to be placed at the different letters.

The number of arrangements is 4!. Here rotation of W, S, E, N is not allowed. Hence arrangement is like a linear arrangement. Such arrangement is called as an arrangement with respect to round table.

Now consider the situation where m out of the n objects are alike and cannot be distinguished from one another. The following result gives the formula for the number of circular arrangements of these n objects.

Theorem. The number of circular arrangements of n objects, of which m objects are alike (identical), is given by $\frac{(n-1)!}{m!}$

Proof. As argued earlier, m like (indistinguishable) objects can be rearranged among themselves in m! ways without affecting the arrangements of the n objects. The number of circular arrangements of n objects is (n−1)! and the number of arrangements of m like (indistinguishable) objects among themselves is m!, the required number of arrangements is given by $\frac{(n-1)!}{m!}$

Does this argument remind us of a similar argument that we came across earlier?

Remark: The number of circular permutations of r objects taken from n distinct objects can be found under two different conditions as follows.

(a) When clockwise and anticlockwise arrangements are considered to be different, then the required number of circular arrangements is given by $\dfrac{{}^{n}p_{r}}{r}$

(b) When clockwise and anticlockwise arrangements are not to be considered different, then the required number of circular arrangements is given by $\dfrac{{}^{n}p_{r}}{2r}$

Verify these two statements for n = 6, r = 3.

SOLVED EXAMPLES

Ex 1 : In how many ways can 8 students be arranged at a round table so that 2 particular students are together, if

(i) students are arranged with respect to each other? (That is the seats are not numbered.)

(ii) students are arranged with respect to the table? (That is the seats are numbered serially.)

Solution : Considering those 2 particular students as one student, we have 7 students.

(i) In circular arrangement 7 students can be arranged at a round table, in 6! ways and 2 students can be arranged among themselves in ${}^{2}p_{2} = 2!$ ways. Hence, the required number of ways in which two particular students come together = 6! × 2! = 1440.

(ii) Here the arrangement is like an arrangement in a row.

So 7 students can be arranged in 7! ways and 2 students can be arranged amongst themselves in ${}^{2}p_{2} = 2!$ ways. Hence, the required number of ways in which two particular students come together = 7! × 2! = 10080.

Ex 2 : In how many ways 6 men and 3 women can be seated at a round table so that every man has woman by his side.

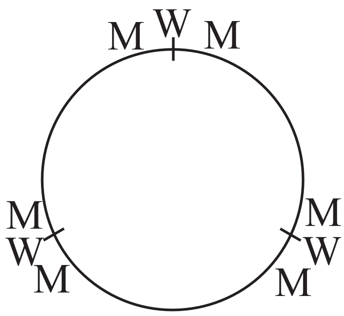


Fig. 3.6

Solution : 3 women have (3 – 1)! = 2 ways of circular seatings.

In each seating a women has one place on each side for a man. Thus there are 6 different places for 6 men which can be filled in 6! ways.

Hence, the total number of required arrangements = 6! × 2 = 720 × 2 = 1440.

Ex 3 : Find the number of ways in which 12 different flowers can be arranged in a garland so that 4 particular flowers are always together.

Solution : Considering 4 particular flowers as a single flower, we have 9 flowers which can be arranged to form a garland in 8! ways. But 4 particular flowers can be arranged in 4! ways. Hence, the required number of ways $= \dfrac{1}{2}$ (8! × 4!) = 483840.

Ex 4 : How many necklaces of 12 bead each, can be made from 18 beads of different colours?

Solution : Here, clockwise and anticlockwise arrangements are same. Hence, total number of circular permutations = $\frac{^{18}P_{12}}{2\times12} = \frac{^{18}P_{12}}{24}$.

Ex 5 : Three boys and three girls are to be seated around a table in a circle. Among them, the boy X does not want any girl as neighbour and girl Y does not want any boy as neighbour. How many such distinct arrangements are possible?

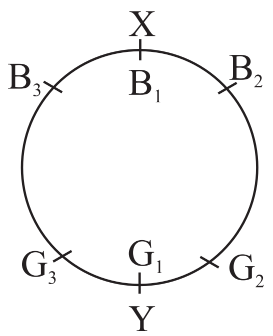


Fig. 3.7

Solution : The arrangement is as shown in the figure (fig. 3.7). The boy X will have B_2 and B_3 as neighbours. The girl Y will have G_2, G_3 as neighbours. The two boys B_2, B_3 can be arranged in two ways. The two girls G_2, G_3 can be arranged in two ways.

Hence, the total number of arrangements
$= 2 \times 2 = 4$.

Ex 6 : In how many different arrangements can 6 gentlemen and 6 ladies sit around a table if (i) there is no restriction. (ii) no two ladies sit side by side?

Solution : (i) There is no restriction.

Here, the total number = 6 + 6 = 12.

12 persons can be arranged in circular permutation in (12–1)! = 11! ways.

(ii) No two ladies sit side by side.

When 6 gentlemen are arranged around a table, there are 6 positions, each being between two gentlemen, for 6 ladies when no two ladies sit side by side. Now, the number of ways in which 6 gentlemen can be seated around a table $= (6 - 1)! = 5!$

Now, corresponding to each seating arrangement for the gentlemen, the 6 ladies can be seated in 6! ways.

Thus, the required number of arrangements $= (5!)\ (6!) = 86400$.

Ex 7 : In how many different ways can 4 married couples occupy seats around a circular table if (i) Spouses sit opposite to each other? (ii) Men and women alternate?

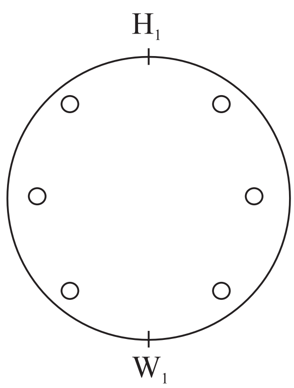


Fig. 3.8

Solution : (i) Spouses sit opposite to each other. Since there is no fixed starting point let one woman occupy one seat. Let, W_1H_1, W_2H_2, W_3H_3 and W_4H_4 be the four couples. Since there is no fixed starting point of the circle, Let W_1 occupy one seat then H_1 has to sit exactly opposite. There are 3 seats on either side of W_1. Then W_2 can occupy any of the 6 seats which confirms the seat of her husband H_2. Now there are 4 seats left. W_3 can occupy anyone of them so that her husband's seat is fixed. Now W_4 has 2 choices, fixing the seat of her husband. In all there are $6 \times 4 \times 2 =$ 48 ways.

ii) Let 4 women sit on alternate seats around a table starting at any place. This is done in 3! ways. It leaves 4 alternate seats empty. They are fixd by 4 men in 4! ways. Hence, the total number of ways is 4! × 3! = 144.

EXERCISE 3.5

1. In how many different ways can 8 friends sit around a table?
2. A party has 20 participants. Find the number of distinct ways for the host to sit with them around a circular table. How many of these ways have two specified persons on either side of the host?
3. Delegates from 24 countries participate in a round table discussion. Find the number of seating arrangments where two specified delegates are. (a) always together, (b) never together.
4. Find the number of ways for 15 people to sit around the table so that no two arrangements have the same neighbours.
5. A committee of 10 members sits around a table. Find the number of arrangements that have the president and the vice president together.
6. Five men, two women, and a child sit around a table. Find the number of arrangements where the child is seated (a) between the two women. (b) between two men.
7. Eight men and six women sit around a table. How many of sitting arrangements will have no two women together?
8. Find the number of seating arrangements for 3 men and 3 women to sit around a table so that exactly two women are together.
9. Four objects in a set of ten objects are alike. Find the number of ways of arranging them in a circular order.
10. Fifteen persons sit around a table. Find the number of arrangements that have two specified persons not sitting side by side.

Properties of Permutations

(i) ${}^{n}P_{n} = n!$

(ii) ${}^{n}P_{0} = 1$

(iii) ${}^{n}P_{1} = n$

(iv) ${}^{n}P_{r} = n \times {}^{(n-1)}P_{(r-1)}$

$= n(n-1) \times {}^{(n-2)}P_{(r-2)}$

$= n(n-1)(n-2) \times {}^{(n-3)}P_{(r-3)}$ and so on.

(v) $\dfrac{{}^{n}p_{r}}{{}^{n}p_{(r-1)}} = n - r + 1$

3.6 Combinations

Permutations involve ordered arrangements of objects. We shall now consider situations where the order of objects in an arrangement is immaterial, but only selection of a set of objects in the arrangement is considered. A selection of objects without any consideration of the order is called a combination.

Consider the earlier example from permutations where 2 chairs were filled from a group of 4 persons. We make a little change in the problem. We want to select a group of 2 people and not consider the order. So the arrangements AB and BA correspond to the same group. Similarly BC and CB are given in the same group. The list is given as follows.

$$\begin{pmatrix} AB \\ BA \end{pmatrix} \begin{pmatrix} BC \\ CB \end{pmatrix} \begin{pmatrix} CA \\ AC \end{pmatrix} \begin{pmatrix} AD \\ DA \end{pmatrix} \begin{pmatrix} BD \\ DB \end{pmatrix} \begin{pmatrix} CD \\ DC \end{pmatrix}$$

Thus there are altogether $\dfrac{{}^{4}P_{2}}{2} = \dfrac{4 \times 3}{2 \times 1} = 6$ different groups selected. This is called the combination number of selecting a group of 2 from 4 persons denoted by ${}^{4}C_{2}$.

Definition Combination. A combination of a set of n distinct objects taken r at a time without repetition is an r–element subset of the n objects.

Note: The order of arrangement of the elements is immaterial in a combination.

If we want to choose a team of 3 players from a set of 8 different players, we first get the number $^8P_3$, i.e. different ordered sets of 3 players and since any set of 3 gives 3! ordered sets, we divided $^8P_3$ by 3!. Thus the number of combinations of 3 players from 8 players is $\frac{^8P_3}{3!} = \frac{8!}{(8-3)!3!}$

Combination (^nC_r) : From n different objects, the number of ways of selecting a group or a set of r objects (without considering order) is denoted by ^nC_r or C(n,r) or nCr. It is the number of combinations of r ^nC_r objects from n distinct objects.

Theorem : $^nC_r = \frac{n!}{(n-r)!r!}$

Proof : First we find the number of 'ordered' sets of r objects from n distinct objects.

$\therefore\ ^nP_r = \frac{n!}{(n-r)!}$. Now, each set of r objects corresponds to $r!$ different ordered arrangements. So, we count the number of different sets or r objects, without considering the order among them, is $= \frac{^nP_r}{r!} = \frac{n!}{(n-r)!r!}$

3.6.1 Properties of combinations.

1. Consider $^nC_{n-r} = \frac{n!}{(n-r)![n-(n-r)]!}$

$$= \frac{n!}{(n-r)!r!}$$

$$= {}^nC_r.$$

Thus, $^nC_{n-r} = {}^nC_r$ for $0 \le r \le n$.

2. $^nC_0 = \frac{n!}{0!(n-0)!} = \frac{n!}{n!} = 1$, because $0! = 1$ as has been stated earlier.

3. If $^nC_r = {}^nC_s$, then either s = r or s = n–r.

4. $^nC_r = \frac{^nP_r}{r!}$

5. $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

6. $^nC_0 + {}^nC_1 + \ldots {}^nC_n = 2^n$

7. $^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{(n-1)}$

8. $^nC_r = \left(\frac{n}{r}\right) {}^{(n-1)}C_{(r-1)} = \left(\frac{n}{r}\right)\left(\frac{n-1}{r-1}\right) {}^{(n-2)}C_{(r-2)} = \ldots$

9. ^nC_r has maximum value if (a) r = $\frac{n}{2}$ when n is even (b) r = $\frac{n-1}{2}$ or $\frac{n+1}{2}$ when n is odd.

SOLVED EXAMPLES

Ex 1 : Find the value of (i) $^7C_3$ (ii) $^{10}C_7$ (iii) $^{52}C_3$

Solution : We know that

$$^nC_r = \frac{n!}{r!(n-r)!}$$

(i) $$^7C_3 = \frac{7!}{3!(7-3)!} = \frac{7!}{3!(4)!} = \frac{7\times6\times5\times4!}{3\times2\times1\times4!} = \frac{7\times6\times5}{3\times2\times1} = 35$$

(ii) $$^{10}C_7 = \frac{10!}{7!(10-7)!} = \frac{10!}{7!(3)!} = \frac{10\times9\times8}{3\times2} = 120$$

(iii) $^{52}C_3 = \dfrac{52!}{3!(52-3)!} = \dfrac{52!}{3!(49)!}$

$= \dfrac{52\times51\times50}{3\times2\times1} = 22100$

Ex 2 : Find n and r

if $^nC_{r-1} : {}^nC_r : {}^nC_{r+1} = 14:8:3$

Solution : $\dfrac{^nC_{r-1}}{^nC_r} = \dfrac{14}{8} = \dfrac{7}{4}$ and $\dfrac{^nC_r}{^nC_{r+1}} = \dfrac{8}{3}$

$\therefore \quad \dfrac{r}{n-r+1} = \dfrac{7}{4}$ and $\dfrac{r+1}{n-r} = \dfrac{8}{3}$

$\therefore \quad (n-r) = \left(\dfrac{4}{7}r\right) - 1$ and $(n-r) = \dfrac{3}{8}(r+1)$

$\therefore \quad \left(\dfrac{4}{7}r\right) - 1 = \dfrac{3}{8}(r+1)$

Solving this we get r = 7 and n = 10

Ex 3 : There are n points in a plane. Find the number of straight lines and triangles that can be obtained by joining points on a plane if

(i) no three points are collinear

(ii) p-points are collinear ($p\geq3$)

Solution : (i) Straight line can be drawn by joining any two points and traingle can be drawn by joining any three non-collinear points.

From n-points, any two points can be selected in $^nC_2$ ways.

$\therefore$ Number of straight lines = $^nC_2$

Since no three points are collinear, any three non-collinear points can be selected in $^nC_3$ ways.

$\therefore$ Number of triangles = $^nC_3$

(ii) If P points are non-collinear then, we can obtain $^PC_2$ straight lines and $^PC_3$ triangles from those p points. But we are given that p points are collinear, therefore they form only one line and no triangle i.e. we have counted ($^pC_2 - 1$) extra lines and $^pC_3$ extra triangles.

$\therefore$ If p-points ($p\geq3$) are collinear, then number of straight lines = $^nC_2$ – ($^pC_2 - 1$) and number of triangles = $^nC_3 - {}^pC_3$

Ex 4 : Four cards are drawn from a pack of 52 playing cards. In how many different ways can this be done? How many selections will contain

(i) exactly one card of each suit?

(ii) all cards of the same suit?

(iii) all club cards?

(iv) at least one club card?

(v) three kings and one queen?

(vi) three black and one red cards?

Solution : A pack of 52 cards contains 4 different suits, viz. Club, Spade, Diamond and Heart. Each suit contains 13 cards. Club and Spade are black coloured cards. Diamond and Heart are red colour cards. i.e. pack of 52 cards contains 26 black and 26 red colour cards.

From a pack of 52 cards, any 4 cards can be drawn in $^{52}C_4 = \dfrac{52!}{4!\times48!}$ ways.

(i) Exactly one card of each suit.

One club card can be selected in $^{13}C_1 = 13$ ways. One heart card can be selected in $^{13}C_1 = 13$ ways. One spade card can be selected in $^{13}C_1 = 13$ ways. One diamond card can be selected in $^{13}C_1 =$ 13 ways.

$\therefore$ Using fundamental principle, exactly one card of each suit can be selected in $13 \times 13 \times 13 \times 13 = (13)^4$ ways.

(ii) All cards are of the same suit.

From 4 suits, any one suit can be selected in ${}^4C_1 = 4$ ways. After this is done, any four cards from selected suit can be drawn in ${}^{13}C_4 = 715$ ways.

$\therefore$ Using fundamental principle, all 4 cards of the same suit can be selected in $4 \times 715 = 2860$ ways.

(iii) All club cards

From 13 club cards, any 4 club cards can be drawn in ${}^{13}C_4 = 715$ ways.

(iv) At least one club card

From a pack of 52 cards, any 4 cards can be drawn in ${}^{52}C_4$ ways if there is no condition.

From ${}^{52}C_4$ selections, remove those selections that do not contain any club card, so that in the remaining selection we have at least one club card.

If the selection does not contain any club card i.e. 4 cards are drawn from remaining 39 non-club cards, then this can be done in ${}^{39}C_4$ ways.

$\therefore$ Number of selections which contain at least one club card = ${}^{52}C_4 - {}^{39}C_4$.

(v) Three king cards and one queen card

From 4 kings, any 3 king cards can be selected in ${}^4C_3 = 4$ ways and from 4 queen cards 1 queen card can be selected in ${}^4C_1 = 4$ ways.

$\therefore$ Using fundamental principle, three king cards and one queen card can be selected in $4 \times 4 = 16$ ways.

(vi) Three black cards and one red card

From 26 black cards, any 3 black cards can be selected in ${}^{26}C_3$ ways and from 26 red cards, any one red card can be selected in ${}^{26}C_1$ ways.

Using fundamental principle, three black and one red card can be selected in ${}^{26}C_3 \times {}^{26}C_1$ ways.

Ex 5 : Find n, if ${}^nC_8 = {}^nC_6$

Solution : If ${}^nC_x = {}^nC_y$, then either $x = y$ or $x = n-y$

$\therefore \quad 8 = n-6 \qquad (\because x \neq y)$

$\therefore \quad n = 14$

Ex 6 : Find r, if ${}^{16}C_4 + {}^{16}C_5 + {}^{17}C_6 + {}^{18}C_7 = {}^{19}C_r$

Solution : $({}^{16}C_4 + {}^{16}C_5) + {}^{17}C_6 + {}^{18}C_7 = {}^{19}C_r$

$\therefore \quad ({}^{17}C_5 + {}^{17}C_6) + {}^{18}C_7 = {}^{19}C_r$

$\therefore \quad ({}^{18}C_6 + {}^{18}C_7) = {}^{19}C_r$

$\therefore \quad {}^{19}C_7 = {}^{19}C_r$

$\therefore \quad r = 7 \text{ or } r = 19-7 = 12$

Ex 7 : Find the difference between the maximum values of ${}^8C_r$ and ${}^{11}C_r$

Solution :

Maximum value of ${}^8C_r$ occurs at $r = \dfrac{8}{2} = 4$

$\therefore \quad$ maximum value of ${}^8C_r = {}^8C_4 = 70$

Maximum value of ${}^{11}C_r$ occurs at $r = \dfrac{10}{2} = 5$

or at $\dfrac{12}{2} = 6$

$\therefore \quad$ Maximum value of ${}^{11}C_r = {}^{11}C_5 = {}^{11}C_6 = 462$

$\therefore \quad$ difference between the maximum values of ${}^8C_r$ and ${}^{11}C_r = {}^{11}C_5 - {}^8C_4 = 462 - 70 = 392$

EXERCISE 3.6

1. Find the value of (a) ${}^{15}C_4$ (b) ${}^{80}C_2$

   (c) ${}^{15}C_4 + {}^{15}C_5$ (d) ${}^{20}C_{16} - {}^{19}C_{16}$

2. Find n if

   (a) ${}^6P_2 = n\,{}^6C_2$

   (b) ${}^{2n}C_3 : {}^nC_2 = 52 : 3$

   (c) ${}^nC_{n-3} = 84$

3. Find r if ${}^{14}C_{3r} : {}^{10}C_{2r-4} = 143:10$

4. Find n and r if.

   (a) ${}^{n}P_{r} = 720$ and ${}^{n}C_{n-r} = 120$

   (b) ${}^{n}C_{r-1} : {}^{n}C_{r} : {}^{n}C_{r+1} = 20: 35:42$

5. If ${}^{n}P_{r} = 1814400$ and ${}^{n}C_{r} = 45$, find ${}^{n+4}C_{r+3}$

6. If ${}^{n}C_{r-1} = 6435$, ${}^{n}C_{r} = 5005$, ${}^{n}C_{r+1} = 3003$, find ${}^{r}C_{5}$.

7. Find the number of ways of drawing 9 balls from a bag that has 6 red balls, 8 green balls, and 7 blue balls so that 3 balls of every colour are drawn.

8. Find the number of ways of selecting a team of 3 boys and 2 girls from 6 boys and 4 girls.

9. After a meeting, every participant shakes hands with every other participants. If the number of handshakes is 66, find the number of participants in the meeting.

10. If 20 points are marked on a circle, how many chords can be drawn?

11. Find the number of diagonals of an n–sided polygon. In particular, find the number of diagonals when.

    (a) n = 10 (b) n = 15

    (c) n = 12 (d) n = 8

12. There are 20 straight lines in a plane so that no two lines are parallel and no three lines are concurrent. Determine the number of points of intersection.

13. Ten points are plotted on a plane. Find the number of straight lines obtained by joining these points if

    (a) no three points are collinear.

    (b) four points are collinear.

14. Find the number of triangles formed by joining 12 points if

    (a) no three points are collinear

    (b) four points are collineas.

15. A word has 8 consonants and 3 vowels. How many distinct words can be formed if 4 consonants and 2 vowels are chosen?

16. Find n if,

    (i) ${}^{n}C_{8} = {}^{n}C_{12}$

    (ii) ${}^{23}C_{3n} = {}^{23}C_{2n+3}$

    (iii) ${}^{21}C_{6n} = {}^{21}C_{(n^2+5)}$

    (iv) ${}^{2n}C_{r-1} = {}^{2n}C_{r+1}$

    (v) ${}^{n}C_{n-2}=15$

17. Find x if ${}^{n}P_{r} = x\,{}^{n}C_{r}$

18. Find r if ${}^{11}C_{4}+ {}^{11}C_{5} + {}^{12}C_{6} + {}^{13}C_{7} = {}^{14}C_{r}$

19. Find the value of $\sum_{r=1}^{4} {}^{(21-r)}C_{4}$

20. Find the differences between the greatest values in the following:

    (a) ${}^{14}C_{r}$ and ${}^{12}C_{r}$, (b) ${}^{13}C_{r}$ and ${}^{8}C_{r}$,

    (c) ${}^{15}C_{r}$ and ${}^{11}C_{r}$,

21. In how many ways can a boy invite his 5 friends to a party so that at least three join the party?

22. A group consists of 9 men and 6 women. A team of 6 is to be selected. How many of possible selections will have at least 3 women?

23. A committee of 10 persons is to be formed from a group of 10 women and 8 men. How many possible committees will have at least 5 women? How many possible committees will have men in majority?

24. A question paper has two sections. section I has 5 questions and section II has 6 questions. A student must answer at least two question from each section among 6 questions he answers. How many different choices does the student have in choosing questions?

25. There are 3 wicketkeepers and 5 bowlers among 22 cricket players. A team of 11 players is to be selected so that there is exactly one wicketkeeper and at least 4 bowlers in the team. How many different teams can be formed?

26. Five students are selected from 11. How many ways can these students be selected if.

    (a) two specified students are selected?

    (b) two specified students are not selected?

Let's Remember

- **Factorial notation :**

  n! or $\lfloor\underline{n}$ = 1,2,3.... (n–2) (n–1) n,

  (0! = 1)

- **Principle of Addition :**

  If an event can occur either in m or n mutually exclusive alternate ways, then the total number of ways in which the event can occur is m + n.

- **Principle of Multiplication :**

  If an event has m possible outcomes, and another independent event has n possible outcomes, then there are m.n possible outcomes for the two events together.

- **Permutation :**

  A permutation is an arrangement, in a definite order, of a number of objects, taken some or all at a time.

(1) Linear Permutation :

(a) The number of permutation of n different objects taken r at a time when repetition of r objects in the permutation is not allowed is given by

$$^{n}P_{r} = \frac{n!}{(n-r)!} \text{ where } r \le n$$

(b) The number of permutations of n different objects, taken r objects at a time, when repetition of r objects in the permutation is allowed, is given by n^r.

(c) The number of permutations of n objects, when p objects are of one kind, q objects are of second kind, r objects are of third kind and the rest, (if any), are of different kind is $\frac{n!}{p!q!r!}$

(ii) Circular Permutation:

The arrangements in a circle are called circular permutations.

(a) The number of circular permutations of n different objects = (n–1)!

(b) The number of ways in which n things of which p are alike, can be arranged in a circular order is $\frac{(n-1)!}{p!}$

- **Combination :**

  A combination is a selection. Total number of selections of 'n' different objects, taken 'r' at a time is denoted by $^{n}C_{r}$ or nCr or C(n,r), or or $\binom{n}{r}$ and is given by

$$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{^{n}P_{r}}{r!}$$

- **Properties of ${}^{n}C_{r}$:**

(a) ${}^{n}C_{r} = {}^{n}C_{n-r}$

(b) If ${}^{n}C_{x} = {}^{n}C_{y}$, then either $x = y$ or $x + y = n$

(c) ${}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r}$ $\quad 1 \leq r \leq n$ (Pascal's rule)

(d) The number of ways of selecting one or more things from n different things is given by 2^{n-1}.

(e) ${}^{n}C_{r}$ has maximum value, if

(i) $r = \dfrac{n}{2}$ when n is even

(ii) $r = \dfrac{n-1}{2}$ or $\dfrac{n+1}{2}$ when n is odd.

MISCELLANEOUS EXERCISE - 3

(I) Select the correct answer from the given alternatives.

1) A college offers 5 courses in the morning and 3 in the evening. The number of ways a student can select exactly one course, either in the morning or in the evening.

A) 5 B) 3 C) 8 D) 15

2) A college has 7 courses in the morning and 3 in the evening. The possible number of choices with the student if he wants to study one course in the morning and one in the evening is-.

A) 21 B) 4 C) 42 D) 10

3) In how many ways can 8 Indians and, 4 American and 4 Englishmen can be seated in a row so that all person of the same nationality sit together?

A) 3! 8! B) 3! 4! 8! 4!

C) 4! 4! D) 8! 4! 4!

4) In how many ways can 10 examination papers be arranged so that the best and the worst papers never come together?

A) 9×8! B) 8×8! C) 8×9! D) 8×9!

5) In how many ways 4 boys and 3 girls can be seated in a row so that they are alternate.

A) 12 B) 288 C) 144 D) 256

6) Find the number of triangles which can be formed by joining the angular points of a polygon of 8 sides as vertices.

A) 16 B) 56 C) 24 D) 8

7) A question paper has two parts, A and B, each containing 10 questions. If a student has to choose 8 from part A and 5 from part B, In how many ways can he choose the questions?

A) 320 B) 750 C) 40 D) 11340

8) There are 10 persons among whom two are brothers. The total number of ways in which these persons can be seated around a round table so that exactly one person sits between the brothers, is equal to:

A) 2!×7! B) 2!×8! C) 3!×7! D) 3!×8!

9) The number of arrangements of the letters of the word BANANA in which two N's do not appear adjacently.

A) 80 B) 60 C) 40 D) 100

10) The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two females are not seated together is

A) 840 B) 600 C) 720 D) 480

(II) Answer the following.

1) Find the value of r if ${}^{56}C_{r+2}$: ${}^{54}P_{r-1}$ = 30800:1

2) How many words can be formed by writing letters in the word CROWN in different order?

3) Find the number of words that can be formed by using all the letters in the word REMAIN If these words are written in dictionary order, what will be the 40^{th} word?

4) Capital English alphabet has 11 symmetric letters that appear same when looked at in a mirror. These letters are A, H, I, M, O, T, U, V, W, X, and Y. How many symmentric three letter passwords can be formed using these letters?

5) How many numbers formed using the digits 3,2,0,4,3,2,3 exceed one million?

6) Ten students are to be selected for a project from a class of 30 stdudents. There are 4 students who want to be together either in the project or not in the project. Find the number of possible selections.

7) A student finds 7 books of his interest, but can borrow only three books. He wants to borrow Chemistry part II book only if Chemistry Part I can also be borrowed. Find the number of ways he can choose three books that he wants to borrow.

8) 30 objects are to be divided in three groups containing 7,10,13 objects. Find the number of distinct ways for doing so.

9) A student passes an examination if he secures a minimum in each of the 7 subjects. Find the number of ways a student can fail.

10) Nine friends decide to go for a picnic in two groups. One group decides to go by car and the other group decides to go by train. Find the number of different ways of doing so if there must be at least 3 friends in each group.

11) A hall has 12 lamps and every lamp can be switched on independently. Find the number of ways of illuminating the hall.

12) How many quadratic equations can be formed using numbers from 0,2,4,5 as coefficients if a coefficient can be repeated in an equation.

13) How many six–digit telephone numbers can be formed if the first two digits are 45 and no digit can appear more than once?

14) A question paper has 6 questions. How many ways does a student have to answer if he wants to solve at least one question?

15) Find the number of ways of dividing 20 objects in three groups of sizes 8,7,and 5.

16) There are 4 doctors and 8 lawyers in a panel. Find the number of ways for selecting a team of 6 if at least one doctor must be in the team.

17) Four parallel lines intersect another set of five parallel lines. Find the number of distinct parallelograms formed.

18) There are 12 distinct points A,B,C,.....,L, in order, on a circle. Lines are drawn passing through each pair of points

   i) How many lines are there in total.

   ii) How many lines pass through D.

   iii) How many triangles are determined by lines.

   iv) How many triangles have on vertex C.

❖ ❖ ❖

4 METHOD OF INDUCTION AND BINOMIAL THEOREM

Let's Study

- Mathematical Induction
- Binomial Theorem
- General term of expansion
- Expansion for negative and fractional index
- Binomial coefficients

Let's Learn

Introduction :

The earliest implicit proof by induction was given by Al Karaji around 100 AD. The first explicit formulation of the principle was given by Pascal in 1665. The Mathematical Induction is a powerful method, easy to use for proving many theorems.

4.1 Principle of Mathematical Induction :

Principle of Mathematical Induction consists of the following four 4 steps:

Step 1 : (Foundation) To prove $P(n)$ is true for $n = 1$

(It is advisable to check if $P(n)$ is true for $n = 2, 3$ also if P(1) is trivial).

Step 2 : (Assumption) To assume $P(n)$ is true for $n = k$.

Step 3 : (Succession) To prove that $P(n)$ is true for $n = k + 1$.

Step 4 : (Induction) To conclude that $P(n)$ is true for all $n \in N$

Row of dominos standing close to each other gives us the idea of how the Principle of Mathematical Induction works.

Step 1 : (Foundation) The 1st domino falls down.

(followed by it 2^{nd} also falls down. Then 3^{rd}, 4^{th} and so on.)

Step 2 : (Assumption) Assume if k^{th} domino falls down.

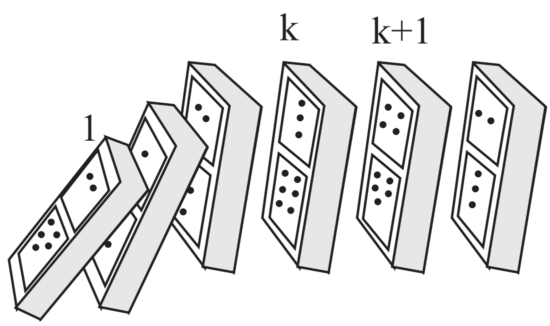


Fig. 4.1

Step 3 : (Succession) Followed by k^{th} domino, $(k + 1)^{th}$ domino will also fall down.

Step 4 : (Induction) It is true that all the dominos will fall down.

We will see how to use the principle of mathematical induction to prove statements.

Stepwise Explanation :

Step 1. (Foundation) Formulate the statement of the theorem as P(n) say, for any positive integer n and verify it for integer $n = 1$. In fact, it is often instructive, though not necessary, to verify the statement for $n = 2$ and 3. This gives better insight into the theorem.

Step 2. (Assumption) Assume that the statement P(n) is true for a positive integer k.

Step 3. (Succession) Prove the statement for $n = k + 1$.

Step 4. (Induction) Now invoke the principle of Mathematical induction. Conclude that the theorem is true for any positive integer n.

Illustration :

Let us prove a theorem with this method. The theorem gives the sum of the first n positive integers.

It is stated as P(n) : 1 + 2 + 3 . . . + n = n(n+1)/2.

Step 1 : (Foundation)

To prove *P(n)* is true for n = 1

L.H.S = 1 R.H.S = $\frac{1(1+1)}{2}$ = 1 which is trivially true.

Check that $1 + 2 = \frac{2\times(2+1)}{2}$ and

$1 + 2 + 3 = \frac{3\times(3+1)}{2}$, so P(2) and P(3) are also true.

Step 2 : (Assumption) Assume that P(*n*) is true for n = k and in particular,

$$1 + 2 + 3 + . . . + k = \frac{k(k+1)}{2}.$$

Step 3 : (Succession) To prove P(*n*) is true for n = k +1 that is

$$1 + 2 + 3 + . . . + k + (k+1) = \frac{(k+1)(k+2)}{2}$$

Here L.H.S. = 1 + 2 + 3 + . . . + k + (k+1)

$$= \frac{k(k+1)}{2} + (k+1) \quad \text{(by step 2)}$$

$$= (k + 1)\left(\frac{k}{2} + 1\right)$$

$$= \frac{(k+1)(k+2)}{2} = \text{R.H.S.}$$

Thus, P(k + 1) is proved.

Step 4 : (Induction) Now by the Principle of Mathematical induction, the statement P(n) is proved for all positive integers n.

SOLVED EXAMPLES

Ex.1 By method of induction, prove that.

$1.3 + 2.5 + 3.7 + ... + n(2n+1) = \frac{n}{6}(n+1)(4n+5)$ for all $n \in N$

Solution :

Let $P(n) \equiv 1.3 + 2.5 + 3.7 + ... + n(2n+1)$,

for all $n \in N$

$$= \frac{n}{6}(n+1)(4n+5)$$

Step (I) : (Foundation) To prove P(1) is true

Let n = 1

L. H. S. = 1.3 = 3

R. H. S. $= \frac{1}{6}(1 + 1)(4.1 + 5)$

$= \frac{1}{6}(2)(9) = 3$

∴ L. H. S. = R. H. S.

∴ P(1) is true

Step (II) : (Assumption) Assume that let P(k) is true

$$\therefore 1.3 + 2.5 + 3.7 + ... + k(2k+1)$$

$$= \frac{k}{6}(k+1)(4k+5) \qquad ...(i)$$

Step (III) : (Succession) To prove that P(k+1) is true.

i.e. $1.3 + 2.5 + 3.7 + ... + (k+1)[2(k+1)+1]$

$$= \frac{(k+1)}{6}(k+1+1)[4(k+1)+5]$$

i.e. $1.3 + 2.5 + 3.7 + ... + (k+1)(2k+3)$

$$= \frac{(k+1)}{6}(k+2)(4k+9)$$

Now

$$\text{L.H.S.} = 1.3 + 2.5 + 3.7 + ... + (k+1)(2k+3)$$
$$= 1.3+2.5+...+k(2k+1)+(k+1)(2k+3)$$
$$= \frac{k}{6}(k+1)(4k+5)+(k+1)(2k+3)$$
... from (i)
$$= (k+1)\left[\frac{k(4k+5)}{6} + 2k + 3\right]$$
$$= (k+1)\frac{\left[4k^2+5k+12k+18\right]}{6}$$
$$= \frac{(k+1)(k+2)(4k+9)}{6}$$
$$= \text{R.H.S.}$$

$\therefore$ P (k+1) is true.

Step (IV) : (Induction) From all steps above by the principle of mathematical induction, P(n) is true for all $n \in N$.

$$\therefore\ 1.3 + 2.5 + 3.7 + ... + n(2n+1)$$
$$= \frac{n}{6}(n+1)(4n+5), \text{ for all } n \in N.$$

Ex.2 By method of induction, prove that.

$$\sum_{r=1}^{n} ax^{r-1} = a\left(\frac{1-x^n}{1-x}\right), \text{ for all } n \in N, x \neq 1.$$

Solution : Let $P(n) \equiv \sum_{r=1}^{n} ax^{r-1}$

$$= a + ax + ax^2 + ... + ax^{n-1} = a\left(\frac{1-x^n}{1-x}\right)$$

Step (I) : To prove that P(1) is true

Let n = 1

$\therefore$ L. H. S. $= a$

$$\text{R. H. S.} = a\left(\frac{1-x}{1-x}\right) = a$$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P(1) is true

Step (II) : Assume that P(k) is true.

$$\sum_{r=1}^{k} ax^{r-1} = a + ax + ax^2 + ... + ax^{k-1}$$
$$= a\left[\frac{1-x^k}{1-x}\right] \quad ....(i)$$

Step (III) : To prove that P(k+1) is true

i.e. $a + ax + ax^2 + ... + ax^k = a\left[\frac{1-x^{k+1}}{1-x}\right]$

Now, L.H.S. $= a + ax + ax^2 + ... + ax^{k-1} + ax^k$

$$= a\left[\frac{1-x^k}{1-x}\right] + ax^k \quad \text{[by (i)]}$$
$$= \frac{a(1-x^k)+ax^k(1-x)}{(1-x)}$$
$$= \frac{a[1-x^k+x^k-x^{k+1}]}{(1-x)}$$
$$= a\left[\frac{1-x^{k+1}}{1-x}\right]$$
$$= \text{R. H. S.}$$

$\therefore$ P(k+1) is true.

Step (IV) : From all steps above by the principle of mathematical induction, P(n) is true for all $n \in N$.

$$\therefore \sum_{r=1}^{n} ax^{r-1} = a\left(\frac{1-x^n}{1-x}\right), \text{ for all } n \in N, x \neq 1.$$

Ex.3 By method of induction, prove that.

$5^{2n} - 1$ is divisible by 6, for all $n \in N$.

Solution : $5^{2n} - 1$ is divisible by 6, if and only if

$5^{2n} - 1$ is a multiple of 6

Let P(n) be $5^{2n} - 1 = 6m$, $m \in N$.

Step (I) : To prove that P(1) is true, Let n = 1

$\therefore\ 5^{2n} - 1 = 25 - 1 = 24 = 6{\cdot}4$

$\therefore\ 5^{2n} - 1$ is a multiple of 6

$\therefore$ P(1) is true.

Step (II) : Assume that P(k) is true.

$\therefore\ 5^{2k} - 1 = 6a,$ where $a \in N$

$\therefore\ 5^{2k} = 6a + 1$ ...(i)

Step (III) : To prove that $P(k+1)$ is true

i.e. to prove that $5^{2(k+1)} - 1$ is a multiple of 6

i.e. $5^{2k+2} - 1 = 6b,$ $b \in N$

Now $5^{2k+2} - 1 = 5^{2k}.5^2 - 1$

$= (6a + 1)\ 25 - 1$ by (i)

$= 150a + 24 = 6(25a + 4)$

$= 6b$

Step (IV) : From all the steps above

$P(n) = 5^{2n} - 1$ is divisible by 6,

for all $n \in N$

Note :

1) 5<5 is not a true statement, whereas $5 \leq 5$, $5 \geq 5$ are true statements.

2) $2 = 3$, $2 > 3$, $2 \geq 3$ are not true statements, whereas $2 < 3$, $2 \leq 3$are true statements,

Ex. 4) By method of induction, prove that

$n\,! \geq 2^n$; $\forall\ n \in N, n \geq 4$.

Solution : Step I : (Foundation) Since $P(n)$ is stated for $n \geq 4$. Take $n = 4$

L.H.S. = 4! = 24, R.H.S. = 2^4 = 16.

Since 24≥16, $P(n)$ is true for $n = 4$

[$P(n)$ is not true for $n = 1, 2, 3$ (Verify!)]

Step (II) : (Assumption) Assume that let P(k) is true.

i.e. $k\,! \geq 2^k$; $k \in N$, $k \geq 4$.

Step (III) : (Succession) To prove that $P(k+1)$ is true.

i.e. to prove that $(k + 1)! \geq 2^{k+1}$, $k + 1 \geq 4$.

L.H.S. = $(k+1)! = (k + 1)k!$

Since $k \geq 4$, $k + 1 > 4+1$, i.e. $k + 1 \geq 5$,
also $k + 1 \geq 2$ (why?)

and from Step II, $k\,! \geq 2^k$; $k \geq 4$.

Therefore, L.H.S. = $(k+1)k! \geq 2.2^k = 2^{k+1}$ =R.H.S.

i.e. $(k+1)! \geq 2^{k+1}$, $k + 1 \geq 4$

Therefore $P(k+1)$ is true.

Step (IV) : (Induction) From all steps above, P(n) is true for $\forall\ n \in N$, $n \geq 4$.

Ex. 5) Given that (recurrence relation) $t_{n+1} = 3t_n + 4$, $t_1 = 1$, prove by induction that (general statement) $t_n = 3^n - 2$.

Solution : The statement $P(n)$ has L.H.S. a recurrence relation $t_{n+1} = 3t_n + 4$, $t_1 = 1$ and R.H.S. a general statement $t_n = 3^n - 2$.

Step I : (Foundation) To prove $P(1)$ is true.

L.H.S. = 1, R.H.S = $3^1 - 2 = 3 - 2 = 1$ So $P(1)$ is true.

For n = 2, L.H.S. = $t_2 = 3t_1 + 4 = 3(1) + 4 = 7$

Now R.H.S. = $t_2 = 3^2 - 2 = 9 - 2 = 7$. P(2) is also true.

Step II : (Assumption) Assume that $P(k)$ is true.

i.e. for $t_{k+1} = 3t_k + 4$, $t_1 = 1$, then $t_k = 3^k - 2$

Step III : (Sucession) To prove that $P(k + 1)$ is ture.

i.e. to prove $t_{k+1} = 3^{k+1} - 2$

Since $t_{k+1} = 3t_k + 4$, and $t_k = 3^k - 2$ (From Step II) $t_{k+1} = 3(3^k - 2) + 4 = 3^{k+1} - 6 + 4$

$= 3^{k+1} - 2$.

Therefore $P(k + 1)$ is true.

Step IV: (Induction) From all the steps above $P(n)$, $t_n = 3^n - 2$ is true for $\forall$ n $\in$ N, where $t_{n+1} = 3t_n + 4$, $t_1 = 1$.

Ex.6 By method of induction, prove that.

$2^n > n$, for all $n \in N$.

Solution : Let $P(n) = 2^n > n$

Step (I) : To prove that P(1) is true, Let n = 1

L.H.S. = $2^1 = 2$

R.H.S. = 1

2 > 1 Which is true

$\therefore$ P(1) is true

Step (II) : Assume that P(k) is true, $k \in N$

$\therefore 2^k > k$ ...(i)

Step (III) : To prove that P(k+1) is true

i.e. $2^{k+1} > k + 1$

Now $2^{k+1} = 2^k.2^1 > k.2$ ...by (i)

$\therefore 2^{k+1} > 2k$

$\therefore 2^{k+1} > k + k$

$\therefore 2^{k+1} > k + 1$ ($\therefore k \geq 1$)

$\therefore$ P(k + 1) is true.

Step (IV) : From all steps above and by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore 2^n > n$, for all $n \in N$.

Remarks : (1) In the proof of P(n) by method of induction, both the conditions viz. (i) P(1) is true and (ii) P(k+1) is true when P(k) is true, must be satisfied. (2) In some problems, second step is satisfied but the first step is not satisfied. Hence the result is not valid for all $n \in N$.

for example,

let $P(n) \equiv 1.6+2.9+3.12+....+n(3n + 3) = n^3 + 3n^2 + 2n + 3$

Let us assume that P(k) is true.

$\therefore 1\cdot6+2\cdot9+3\cdot12+....+k(3k + 3) = k^3 + 3k^2 + 2k + 3$ ...(i)

We have to prove that P(k+1) is true,

i.e. to prove that

$1\cdot6 + 2\cdot9+ 3\cdot12 +....+ (k + 1) (3k + 6) = (k+1)^3+3(k+1)^2+2(k+1)+3$

L.H.S. = $1\cdot6 + 2\cdot9+ 3\cdot12 +....+ (k + 1) (3k + 6)$

$= 1\cdot6+2\cdot9+3\cdot12 +....+ k(3k+3) + (k+1)(3k+6)$

$= k^3 + 3k^2 + 2k + 3 + (k + 1) (3k + 6)$by(i)

$= k^3 + 3k^2 + 2k + 3 + 3k^2 + 6k+ 3k + 6$

$= k^3 + 3k^2 + 3k + 1 + 3k^2+6k+3+2k+2+3$

$= (k + 1)^3 + 3(k^2 + 2k + 1) + 2(k + 1) + 3$

$= (k + 1)^3 + 3(k + 1)^2 + 2(k + 1) + 3$

= R. H. S.

$\therefore$ P(k + 1) is true.

If P(k) is true then P (k+1) is true.

Now we examine the result for n = 1

L.H.S. = $1\cdot6 = 6$

R. H. S. = $1^3 + 3(1)^2 + 2 (1) + 3$

= 9

$\therefore$ L. H. S. $\neq$ R. H. S.

$\therefore$ P(1) is not true

$\therefore$ P (n) is not true for all $n \in N$.

EXERCISE 4.1

Prove by method of induction, for all $n \in N$.

(1) $2 + 4 + 6 + ... + 2n = n (n+1)$

(2) $3 + 7 + 11 + ...$ + to n terms = $n(2n+1)$

(3) $1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$

(4) $1^2+3^2 + 5^2 + ... + (2n-1)^2= \frac{n}{3}(2n-1)(2n+1)$

(5) $1^3+3^3 + 5^3 + ...$ to n terms = $n^2 (2n^2-1)$

(6) $1\cdot2+2\cdot3+3\cdot4+.....+n(n+1) = \frac{n}{3}(n+1)(n+2)$

(7) $1\cdot3+3\cdot5+5\cdot7+.....$ to n terms = $\frac{n}{3}(4n^2+6n-1)$

(8) $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} +....+ \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$

(9) $\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+\ldots \text{ to n terms} = \frac{n}{3(2n+3)}$

(10) $(2^{3n}-1)$ is divisible by 7.

(11) $(2^{4n}-1)$ is divisible by 15.

(12) $3^n - 2n - 1$ is divisible by 4.

(13) $5 + 5^2 + 5^3 + ... + 5^n = \frac{5}{4}(5^n - 1)$

(14) $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(15) Given that $t_{n+1} = 5t_n + 4$, $t_1=4$, prove by method of induction that $t_n = 5^n - 1$

(16) Prove by method of induction

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} \quad \forall\, n \in N$$

4.2 Binomial Theorem for positive integral index :

We know that

$(a + b)^0 =$ 1

$(a + b)^1 =$ 1a + 1b

$(a + b)^2 =$ $1a^2 + 2ab + 1b^2$

$(a + b)^3 =$ $1a^3 + 3a^2b + 3ab^2 + 1b^3$

$(a + b)^4 =$ $1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$

The coefficients of these expressions are arranged by Pascal's triangle as follows and are expressed in the form of ^nC_r

Index	
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1

Index	
0	$^0C_0 = 1$
1	$^1C_0 = 1$ $^1C_1 = 1$
2	$^2C_0 = 1$ $^2C_1 = 2$ $^2C_2 = 1$
3	$^3C_0 = 1$ $^3C_1 = 3$ $^3C_2 = 3$ $^3C_3 = 1$
4	$^4C_0=1$ $^4C_1=4$ $^4C_2= 6$ $^4C_3= 4$ $^4C_4=1$

Now, we will study how to expand binomials of higher powers.

Theorem : If a, b $\in$ R and n $\in$ N, then

$(a+b)^n = {}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n\, a^0\, b^n$

Proof : We prove this theorem by method of induction.

Let P(n) be $(a+b)^n =$

${}^nC_0\, a^nb^0 + {}^nC_1\, a^{n-1}\, b^1 + {}^nC_2\, a^{n-2}b^2 + \ldots + {}^nC_n\, a^0\, b^n$

Step (I) : Let n = 1

$\therefore$ L. H. S. = $(a + b)^1 = a + b$

R. H. S. = ${}^1C_0\, a^1b^0 + {}^1C_1\, a^0b^1 = a + b$

$\therefore$ L. H. S. = R. H. S.

$\therefore$ P (1) is true.

Step (II) : Let P(k) be true.

$\therefore (a+b)^k = {}^kC_0\, a^kb^0 + {}^kC_1 a^{k-1}\, b^1 + {}^kC_2 a^{k-2}b^2 + .. + {}^kC_k a^0\, b^k$ ....(i)

Step (III) : We have to prove that P(k+1) is true.

i.e. to prove that

$(a+b)^{k+1} =$

${}^{k+1}C_0\, a^{k+1}b^0 + {}^{k+1}C_1 a^kb^1 + {}^{k+1}C_2 a^{k-1}b^2 + \ldots$

$+ {}^{k+1}C_{k+1} a^0\, b^{k+1}$

Now L. H. S. = $(a + b)^{k+1}$

$= (a+b)\,(a+b)^k$

$= (a+b)\,[{}^kC_0\, a^kb^0 + {}^kC_1 a^{k-1}\, b^1 + {}^kC_2 a^{k-2}b^2 + ..$

$+ {}^kC_k a^0\, b^k]$ by (i)

$= a[{}^kC_0 a^kb^0 + {}^kC_1 a^{k-1}b + {}^kC_2 a^{k-2}b^2 + \ldots + {}^kC_k a^0b^k]$

$+ b[{}^kC_0 a^kb^0 + {}^kC_1 a^{k-1}\, b + {}^kC_2 a^{k-2}b^2 + \ldots + {}^kC_k a^0\, b^k]$

$= [{}^kC_0a^{k+1}b^0+{}^kC_1a^kb^1+{}^kC_2a^{k-1}b^2+\ldots+{}^kC_kab^k]+$
$[{}^kC_0a^kb+{}^kC_1a^{k-1}\,b^2+{}^kC_2a^{k-2}b^3+\ldots+{}^kC_ka^0\,b^{k+1}]$
$= {}^kC_0a^{k+1}b^0+{}^kC_1a^kb^1+{}^kC_0a^kb^1+{}^kC_2a^{k-1}b^2+$
${}^kC_1a^{k-1}b^2+\ldots+{}^kC_ka^1\,b^k+{}^kC_{k-1}a^1b^k+{}^kC_ka^0\,b^{k+1}$
$= {}^kC_0a^{k+1}b^0+({}^kC_1+{}^kC_0)a^kb^1+({}^kC_2+{}^kC_1)a^{k-1}b^2+$
$\ldots+({}^kC_k+{}^kC_{k-1})\,a^1b^k+{}^kC_ka^0\,b^{k+1}$

But we know that

${}^kC_0 = 1 = {}^{k+1}C_{k+1},\ {}^kC_1+{}^kC_0 = {}^{k+1}C_1.$

${}^kC_2+{}^kC_1 = {}^{k+1}C_2, \ldots\ {}^kC_k+{}^kC_{k-1} = {}^{k+1}C_k, \ldots$

$${}^kC_k = 1 = {}^{k+1}C_{k+1}$$

$\therefore$ L.H.S. $= {}^{k+1}C_0a^{k+1}b^0+{}^{k+1}C_1a^kb^1+{}^{k+1}C_2a^{k-1}b^2 + \ldots\ldots + {}^{k+1}C_{k+1}a^0b^{k+1}$

= R.H.S.

$\therefore$ P (k+1) is true.

Step (IV) : From all steps above and by the principle of mathematical induction, P(n) is true for all $n \in N$.

$\therefore (a+b)^n={}^nC_0\,a^nb^0+{}^nC_1a^{n-1}\,b^1+{}^nC_2a^{n-2}b^2\ldots+{}^nC_na^0\,b^n$, for all $n \in N$.

Remarks :

(1) The expansion of $(a+b)^n$ contains n+1 terms.

(2) First term is a^n and last term is b^n.

(3) In each term, the sum of indices of a and b is always n.

(4) In successive terms, the index of a decreases by 1 and index of b increases by 1.

(5) Coefficients of the terms in binomial expansion equidistant from both the ends are equal. i.e. coefficients are symmetric.

(6) $(a-b)^n = {}^nC_0a^nb^0 - {}^nC_1a^{n-1}\,b^1+{}^nC_2a^{n-2}b^2 - \cdots + (-1)^n C_na^0\,b^n$.

In the RHS, the first term is positive and consequent terms are alternately negative and positive.

SOLVED EXAMPLES

Ex. 1 : Expand $(x^2 + 3y)^5$

Solution : Here $a = x^2$, $b = 3y$ and $n = 5$ using binomial theorem,

$$(x^2+3y)^5 = {}^5C_0(x^2)^5(3y)^0 + {}^5C_1(x^2)^4(3y)^1 + {}^5C_2(x^2)^3(3y)^2 + {}^5C_3(x^2)^2(3y)^3 + {}^5C_4(x^2)^1(3y)^4 + {}^5C_5(x^2)^0(3y)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\therefore (x^2+3y)^5 = 1(x^{10})(1)+5(x^8)(3y)+10(x^6)(9y^2)+10(x^4)(27y^3)+5(x^2)(81y^4)+1(1)(243y^5)$$

$$\therefore (x^2+3y)^5 = x^{10}+15x^8y+90x^6y^2+270x^4y^3+405x^2y^4+243y^5$$

Ex. 2 : Expand $\left(2x - \frac{y}{2}\right)^5$

Solution : Here $a = 2x$, $b = \frac{y}{2}$ and $n = 5$

Using binomial theorem,

$$\left(2x - \frac{y}{2}\right)^5 = {}^5C_0(2x)^5\left(\frac{y}{2}\right)^0 - {}^5C_1(2x)^4\left(\frac{y}{2}\right)^1 + {}^5C_2(2x)^3\left(\frac{y}{2}\right)^2 - {}^5C_3(2x)^2\left(\frac{y}{2}\right)^3 + {}^5C_4(2x)^1\left(\frac{y}{2}\right)^4 - {}^5C_5(2x)^0\left(\frac{y}{2}\right)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$,

$${}^5C_2 = {}^5C_3 = \frac{5\cdot 4}{2\cdot 1} = 10$$

$$\therefore \left(2x - \frac{y}{2}\right)^5 = 1(32x^5)(1) - 5(16x^4)\left(\frac{y}{2}\right) + 10(8x^3)\left(\frac{y^2}{4}\right) - 10(4x^2)\left(\frac{y^3}{8}\right) + 5(2x)\left(\frac{y^4}{16}\right) - 1(1)\left(\frac{y^5}{32}\right)$$

$\therefore \left(2x - \frac{y}{2}\right)^5 = 32x^5 - 40x^4y + 20x^3y^2 - 5x^2y^3$

$+ \frac{5}{8}xy^4 - \frac{y^5}{32}$

Ex.3 : Expand $\left(\sqrt{5}+\sqrt{3}\right)^4$

Solution : Here a = $\sqrt{5}$, b = $\sqrt{3}$ and n = 4

Using binomial theorem,

$$\left(\sqrt{5}+\sqrt{3}\right)^4 = {}^4C_0\left(\sqrt{5}\right)^4\left(\sqrt{3}\right)^0 + {}^4C_1\left(\sqrt{5}\right)^3\left(\sqrt{3}\right)^1$$

$$+{}^4C_2\left(\sqrt{5}\right)^2\left(\sqrt{3}\right)^2 + {}^4C_3\left(\sqrt{5}\right)^1\left(\sqrt{3}\right)^3$$

$$+ {}^4C_4\left(\sqrt{5}\right)^0\left(\sqrt{3}\right)^4$$

Now ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = \frac{4.3}{2.1} = 6$,

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 1(25)(1) + 4\left(5\sqrt{5}\right)\left(3\sqrt{3}\right) + 6(5)(3)$$

$$+ 4\left(\sqrt{5}\right)\left(3\sqrt{3}\right) + 1(1)(9)$$

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 25 + \left(20\sqrt{15}\right) + 90 + \left(12\sqrt{15}\right) + 9$$

$$\therefore \left(\sqrt{5}+\sqrt{3}\right)^4 = 124 + \left(32\sqrt{15}\right)$$

Ex.4 : Evaluate $\left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5$

Solution : $\left(\sqrt{2}+1\right)^5 = {}^5C_0\left(\sqrt{2}\right)^5 + {}^5C_1\left(\sqrt{2}\right)^4 +$

$${}^5C_2\left(\sqrt{2}\right)^3 + {}^5C_3\left(\sqrt{2}\right)^2 + {}^5C_4\left(\sqrt{2}\right)^1$$

$$+ {}^5C_5\left(\sqrt{2}\right)^0$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_4 = {}^5C_1 = 5$, ${}^5C_2 = {}^5C_3 = \frac{5.4}{2.1}$ $= 10$

$$\therefore \left(\sqrt{2}+1\right)^5 = 1\left(4\sqrt{2}\right) + 5(4) + 10\left(2\sqrt{2}\right) + 10(2)$$

$$+ 5\left(\sqrt{2}\right) + 1$$

$$\therefore \left(\sqrt{2}+1\right)^5 = \left(4\sqrt{2}\right) + 20 + \left(20\sqrt{2}\right) + 20$$

$$+\left(5\sqrt{2}\right) + 1 \qquad \text{... (i)}$$

Similarly,

$$\left(\sqrt{2}-1\right)^5 = \left(4\sqrt{2}\right) - 20 + \left(20\sqrt{2}\right) - 20 + \left(5\sqrt{2}\right) - 1 \qquad \text{... (ii)}$$

Subtracting (ii) from (i) we get,

$$\left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5$$

$$= \left(4\sqrt{2} + 20 + 20\sqrt{2} + 20 + 5\sqrt{2} + 1\right)$$

$$- \left(4\sqrt{2} - 20 + 20\sqrt{2} - 20 + 5\sqrt{2} - 1\right)$$

$$= 2(20 + 20 + 1)$$

$$= 82$$

$$\therefore \left(\sqrt{2}+1\right)^5 - \left(\sqrt{2}-1\right)^5 = 82$$

Ex. 5 (Activity) : Using binomial theorem, find the value of $(99)^4$

Solution : We have $(99)^4 = (\square - 1)^4$

$$\therefore \quad (99)^4 = {}^4C_0\,(\square)^4 - {}^4C_1\,(\square)^3 + {}^4C_2\,(\square)^2$$

$$- {}^4C_3\,(\square)^1 + {}^4C_4\,(\square)^0$$

Now ${}^4C_0 = {}^4C_4 = 1$, ${}^4C_1 = {}^4C_3 = 4$, ${}^4C_2 = \frac{4.3}{2.1} = 6$

$$\therefore \quad (99)^4 = 1(10)^{\square} - 4(10)^{\square} + 6(10)^{\square}$$

$$- 4(10)^{\square} + 1(1)$$

$$= \square - \square + \square - \square + \square = \square$$

Ex. 6 : Find the value of $(2.02)^5$ correct upto 4 decimal places.

Solution : $(2.02)^5 = [2+0.02]^5$

$$= {}^5C_0\,(2)^5(0.02)^0 + {}^5C_1\,(2)^4(0.02)^1 +$$

$${}^5C_2\,(2)^3(0.02)^2 + {}^5C_3\,(2)^2(0.02)^3 +$$

$${}^5C_4\,(2)^1(0.02)^4 + {}^5C_5\,(2)^0(0.02)^5$$

Now ${}^5C_0 = {}^5C_5 = 1$, ${}^5C_1 = {}^5C_4 = 5$, ${}^5C_2 = {}^5C_3 = 10$

$$\therefore \quad (2.02)^5 = 1(32)(1) + 5(16)(0.02)$$

$$+ 10(8)(0.0004) + 10(4)(0.000008)$$

$$+ 5(2)(0.00000016)$$

$$+ 1(0.0000000032)$$

Ignore last two terms for four decimal places

$\therefore \quad (2.02)^5 = 32 + 1.60 + 0.0320 + 0.0003$

$\therefore \quad (2.02)^5 = 33.6323.$

Ex. 7 : Without expanding, find the value of

$(2x-1)^5 + 5(2x-1)^4(1-x) + 10(2x-1)^3(1-x)^2 + 10(2x-1)^2(1-x)^3 + 5(2x-1)(1-x)^4 + (1-x)^5$

Solution : We notice that 1, 5, 10, 10, 5, 1 are the values of ${}^5C_0$, ${}^5C_1$, ${}^5C_2$, ${}^5C_3$, ${}^5C_4$ and ${}^5C_5$ respectively.

Hence, given expression can be written as

$${}^5C_0(2x-1)^5 + {}^5C_1(2x-1)^4(1-x)$$
$$+ {}^5C_2(2x-1)^3(1-x)^2 + {}^5C_3(2x-1)^2(1-x)^3$$
$$+ {}^5C_4(2x-1)(1-x)^4 + {}^5C_5(1-x)^5$$
$$= [(2x-1) + (1-x)]^5$$
$$= (2x - 1 + 1 - x)^5$$
$$= x^5$$

$\therefore \quad (2x-1)^5+5(2x-1)^4(1-x) + 10(2x-1)^3(1-x)^2 + 10(2x-1)^2(1-x)^3 + 5(2x-1)(1-x)^4 + (1-x)^5 = x^5$

EXERCISE 4.2

(1) Expand (i) $\left(\sqrt{3}+\sqrt{2}\right)^4$ (ii) $\left(\sqrt{5}-\sqrt{2}\right)^5$

(2) Expand (i) $(2x^2 + 3)^4$ (ii) $\left(2x-\frac{1}{x}\right)^6$

(3) Find the value of

(i) $\left(\sqrt{3}+1\right)^4 - \left(\sqrt{3}-1\right)^4$

(ii) $\left(2+\sqrt{5}\right)^5 + \left(2-\sqrt{5}\right)^5$

(4) Prove that

(i) $\left(\sqrt{3}+\sqrt{2}\right)^6 + \left(\sqrt{3}-\sqrt{2}\right)^6 = 970$

(ii) $\left(\sqrt{5}+1\right)^5 - \left(\sqrt{5}-1\right)^5 = 352$

(5) Using binomial theorem, find the value of

(i) $(102)^4$ (ii) $(1.1)^5$

(6) Using binomial theorem, find the value of

(i) $(9.9)^3$ (ii) $(0.9)^4$

(7) Without expanding, find the value of

(i) $(x+1)^4 - 4(x+1)^3(x-1) + 6(x+1)^2(x-1)^2 - 4(x+1)(x-1)^3 + (x-1)^4$

(ii) $(2x-1)^4 + 4(2x-1)^3(3-2x) + 6(2x-1)^2(3-2x)^2 + 4(2x-1)^1(3-2x)^3 + (3-2x)^4$

(8) Find the value of $(1.02)^6$, correct upto four places of decimals.

(9) Find the value of $(1.01)^5$, correct upto three places of decimals.

(10) Find the value of $(0.9)^6$, correct upto four places of decimals.

4.3 General term in expansion of $(a+b)^n$

In the expansion of $(a+b)^n$, we denote the terms by $t_1, t_2, t_3, \ldots, t_r, t_{r+1}, \ldots t_n \ldots$ then

$t_1 = {}^nC_0\, a^n b^0$

$t_2 = {}^nC_1\, a^{n-1} b^1$

$t_3 = {}^nC_2\, a^{n-2} b^2$

. . .

. . .

. . .

$t_r = {}^nC_{r-1}\, a^{n-r+1} b^{r-1}$

$t_{r+1} = {}^nC_r\, a^{n-r} b^r$

t_{r+1} is called a general term for all $r \in N$ and $0 \leq r \leq n$. Using this formula, we can find any term of the expansion.

4.3 Middle term (s) in the expansion of $(a+b)^n$:

(i) In $(a+b)^n$ if n is even then the number of terms in the expansion is odd. So the only middle term is $\left(\frac{n+2}{2}\right)^{th}$ term.

(ii) In $(a+b)^n$ if n is odd then the number of terms in the expansion is even. So the two middle terms are $\left(\frac{n+1}{2}\right)^{th}$ term and $\left(\frac{n+3}{2}\right)^{th}$ term.

SOLVED EXAMPLES

Ex. 1 : Find the fifth term in the expansion of

$$\left(2x^2+\frac{3}{2x}\right)^8$$

Solution : Here $a = 2x^2$, $b = \frac{3}{2x}$, $n = 8$

For t_5, $r = 4$

Since, $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_5 = {}^8C_4 (2x^2)^{8-4}\left(\frac{3}{2x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}(2x^2)^4\left(\frac{3}{2x}\right)^4$$

$$= 70(16x^8)\left(\frac{81}{16x^4}\right)$$

$$= 5670x^4$$

$\therefore$ The fifth term in the expansion of $\left(2x^2+\frac{3}{2x}\right)^8$ is $5670x^4$

Ex. 2 : Find the middle term(s) in the expansion of $\left(x^2+\frac{2}{x}\right)^8$

Solution : Here $a = x^2$, $b = \frac{2}{x}$, $n = 8$

Now n is even, hence $\left(\frac{n+2}{2}\right) = \left(\frac{8+2}{2}\right) = 5$

$\therefore$ Fifth term is the only middle term.

For t_5, $r = 4$

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

$$t_5 = {}^8C_4 (x^2)^{8-4}\left(\frac{2}{x}\right)^4$$

$$= \frac{8.7.6.5}{4.3.2.1}(x^2)^4\left(\frac{2}{x}\right)^4$$

$$= 70\,(x^8)\,\frac{16}{x^4}$$

$$= 1120x^4$$

Ex. 3 : Find the middle terms in the expansion of

$$\left(2x-\frac{1}{4x}\right)^9$$

Solution : Here $a = 2x$, $b = -\frac{1}{4x}$, $n = 9$

Now n is odd $\left(\frac{n+1}{2}\right) = 5$ $\left(\frac{n+3}{2}\right) = 6$

$\therefore$ Fifth and sixth terms are the middle terms.

We have $t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$,

For t_5, $r = 4$

$$\therefore\quad t_5 = {}^9C_4\,(2x)^{9-4}\left(-\frac{1}{4x}\right)^4$$

$$= \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}(32x)^5\left(-\frac{1}{4x}\right)^4$$

$$= 126\,(2x^5)\left(\frac{1}{256x^4}\right)$$

$$= \left(\frac{63x}{4}\right)$$

For t_6, $r = 5$

$$\therefore\quad t_6 = {}^9C_5\,(2x)^{9-5}\left(\frac{-1}{4x}\right)^5$$

$$= \frac{9\cdot8\cdot7\cdot6}{4\cdot3\cdot2\cdot1}(2x)^4\left(\frac{-1}{4x}\right)^5$$

$$= 126\,(16x^4)\left(\frac{-1}{1024x^5}\right)$$

$$= -\frac{63}{32x}$$

$\therefore$ The middle terms are $\left(\frac{63x}{4}\right)$ and $-\frac{63}{32x}$

Ex. 4 : Find the coefficient of x^7 in the expansion of $\left(x^2+\frac{1}{x}\right)^{11}$

Solution : $a = x^2$, $b = \frac{1}{x}$, $n = 11$

We have $t_{r+1} = {}^{n}C_r\, a^{n-r}\, b^r$,

$$t_{r+1} = {}^{11}C_r\,(x^2)^{11-r}\left(\frac{1}{x}\right)^r$$

$$= {}^{11}C_r\, x^{22-2r}\, x^{-r}$$

$$= {}^{11}C_r\, x^{22-3r}$$

To get coefficient of x^7, we must have

$$x^{22-3r} = x^7$$

$\therefore \quad 22 - 3r = 7$

$\therefore \quad r = 5$

$\therefore \quad {}^{11}C_5 = \frac{11\cdot10\cdot9\cdot8\cdot7}{5\cdot4\cdot3\cdot2\cdot1} = 462$

$\therefore$ Coefficient of x^7 is 462.

Ex. 5 : Find the coefficient of x^{-2} in the expansion of $\left(2x-\frac{1}{\sqrt{3}x^2}\right)^{10}$

Solution : Here $a = 2x$, $b = \frac{-1}{\sqrt{3}x^2}$, $n = 10$

We have $t_{r+1} = {}^{n}C_r\, a^{n-r}\, b^r$,

$$= {}^{10}C_r\,(2x^2)^{10-r}\left(\frac{-1}{\sqrt{3}x^2}\right)^r$$

$$= {}^{10}C_r\,(2)^{10-r}x^{10-r}\left(\frac{-1}{\sqrt{3}}\right)^r x^{-2r}$$

$$= {}^{10}C_r\,(2)^{10-r}\left(\frac{-1}{\sqrt{3}}\right)^r x^{10-3r}$$

To get coefficient of x^{-2}, we must have

$$x^{10-3r} = x^{-2}$$

$\therefore\ 10 - 3r = -2$

$\therefore\ -3r = -12$

$\therefore\ r = 4$

$\therefore$ coefficient of x^{-2}

$$= {}^{10}C_4\,(2)^{10-4}\left(\frac{-1}{\sqrt{3}}\right)^4 = \frac{10.9.8.7}{4.3.2.1}\,(2)^6\left(\frac{-1}{\sqrt{3}}\right)^4$$

$$= 210\ (64)\left(\frac{1}{9}\right)$$

$$= \frac{4480}{3}$$

$\therefore$ Coefficient of x^{-2} is $\frac{4480}{3}$

Ex. 6 : Find the term independent of x, in the expansion of $\left(\sqrt{x}-\frac{2}{x^2}\right)^{10}$

Solution : Here $a = \sqrt{x}$, $b = \frac{-2}{x^2}$, $n = 10$

We have $t_{r+1} = {}^{n}C_r\, a^{n-r}\, b^r$

$$= {}^{10}C_r\left(\sqrt{x}\right)^{10-r}\left(\frac{-2}{x^2}\right)^r$$

$$= {}^{10}C_r\, x^{\left(\frac{10-r}{2}\right)}\,(-2)^r x^{-2r}$$

$$= {}^{10}C_r\,(-2)^r\, x^{\frac{10-5r}{2}}$$

To get the term independent of x, we must have

$$x^{\frac{10-5r}{2}} = x^0$$

$\therefore\ \frac{10-5r}{2} = 0$

$\therefore\ 10 - 5r = 0$

$\therefore\ r = 2$

$\therefore$ the term independent of x is

$${}^{10}C_2\,(-2)^2 = \frac{10.9}{2.1}\,(-2)^2 = 45(4) = 180$$

$\therefore$ the term independent of x is 180.

EXERCISE 4.3

(1) In the following expansions, find the indicated term.

(i) $\left(2x^2+\dfrac{3}{2x}\right)^8$, 3rd term

(ii) $\left(x^2-\dfrac{4}{x^3}\right)^{11}$, 5th term

(iii) $\left(\dfrac{4x}{5}-\dfrac{5}{2x}\right)^9$, 7th term

(iv) In $\left(\dfrac{1}{3}+a^2\right)^{12}$, 9th term

(v) In $\left(3a+\dfrac{4}{a}\right)^{13}$, 10th term

(2) In the following expansions, find the indicated coefficients.

(i) x^3 in $\left(x^2+\dfrac{3\sqrt{2}}{x}\right)^9$ (ii) x^8 in $\left(2x^5-\dfrac{5}{x^3}\right)^8$

(iii) x^9 in $\left(\dfrac{1}{x}+x^2\right)^{18}$ (iv) x^{-3} in $\left(x-\dfrac{1}{2x}\right)^5$

(v) x^{-20} in $\left(x^3-\dfrac{1}{2x^2}\right)^{15}$

(3) Find the constant term (term independent of x) in the expansion of

(i) $\left(2x+\dfrac{1}{3x^2}\right)^9$ (ii) $\left(x-\dfrac{2}{x^2}\right)^{15}$

(iii) $\left(\sqrt{x}-\dfrac{3}{x^2}\right)^{10}$ (iv) $\left(x^2-\dfrac{1}{x}\right)^9$

(v) $\left(2x^2-\dfrac{5}{x}\right)^9$

(4) Find the middle terms in the expansion of

(i) $\left(\dfrac{x}{y}+\dfrac{y}{x}\right)^{12}$ (ii) $\left(x^2+\dfrac{1}{x}\right)^7$

(iii) $\left(x^2-\dfrac{2}{x}\right)^8$ (iv) $\left(\dfrac{x}{a}-\dfrac{a}{x}\right)^{10}$

(v) $\left(x^4-\dfrac{1}{x^3}\right)^{11}$

(5) In the expansion of $(k+x)^8$, the coefficient of x^5 is 10 times the coefficient of x^6. Find the value of k.

(6) Find the term containing x^6 in the expansion of $(2-x)(3x+1)^9$

(7) The coefficient of x^2 in the expansion of $(1+2x)^m$ is 112. Find m.

4.4 Binomial Theorem for Negative Index or Fraction.

If n is negative then n! is not defined.

We state binomial theorem in another form.

$$(a+b)^n = a^n + \frac{n}{1!}a^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \ldots..$$

$$\frac{n(n-1)\ldots.(n-r+1)}{r!}a^{n-r}b^r + \ldots..+ b^n$$

Here $t_{r+1} = \dfrac{n(n-1)(n-2)\ldots.(n-r+1)}{r!}a^{n-r}b^r$

Consider the binomial theorem

$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \ldots. + x^n$

$= 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots.+ x^n$

This is a finite sum.

The theorem has an extension to the case where 'n' is negative or faction. We state it here without proof.

For $|x|<1$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots \frac{n(n-1)....(n-r+1)}{r!}x^r + ...$$

Here n is not an integer and the terms on the RHS are infinite, the series does not terminate.

Here there are infinite number of terms in the expansion. The general term is given by

$$t_{r+1} = \frac{n(n-1)(n-2)...(n-r+1)x^r}{r!},\ r \geq 0$$

Remarks : (1) If $|x|<1$ and n is any real number, not a positive integer, then

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

The general term is given by

$$t_{r+1} = \frac{(-1)^r n(n-1)(n-2)...(n-r+1)}{r!}x^r$$

(ii) If n is any real number and $|b|<|a|$, then

$$(a+b)^n = \left[a\left(1+\frac{b}{a}\right)\right]^n = a^n\left(1+\frac{b}{a}\right)^n$$

Note : While expanding $(a+b)^n$ where n is a negative integer or a fraction, reduce the binomial to the form in which the first term is unity and the second term is numerically less than unity.

Particular expansion of the binomials for negative index, fraction. $|x|<1$

(1) $\dfrac{1}{1+x} = (1+x)^{-1} = 1-x+x^2-x^3+x^4-x^5+\ldots$

(2) $\dfrac{1}{1-x} = (1-x)^{-1} = 1+x+x^2+x^3+x^4+x^5+\ldots$

(3) $\dfrac{1}{(1+x)^2} = (1+x)^{-2} = 1-2x+3x^2-4x^3+\ldots$

(4) $\dfrac{1}{(1-x)^2} = (1-x)^{-2} = 1+2x+3x^2+4x^3+\ldots$

(5) $\sqrt{1+x} = (1+x)^{1/2} = 1+\dfrac{x}{2}-\dfrac{x^2}{8}+\dfrac{x^3}{16}+\ldots$

(6) $\sqrt{1-x} = (1-x)^{1/2} = 1-\dfrac{x}{2}-\dfrac{x^2}{8}-\dfrac{x^3}{16}-\ldots$

SOLVED EXAMPLES

Ex.1 : State first four terms in the expansion of $\dfrac{1}{(a-b)^4}$ where $|b|<|a|$

Solution : We have $\dfrac{1}{(a-b)^4} = (a-b)^{-4}$

$$= \left[a\left(1-\frac{b}{a}\right)\right]^{-4}$$

$$= a^{-4}\left[1+(-4)\left(-\frac{b}{a}\right)+\frac{(-4)(-5)}{2!}\left(-\frac{b}{a}\right)^2+\frac{(-4)(-5)(-6)}{3!}\left(-\frac{b}{a}\right)^3+\ldots\right]$$

$$= a^{-4}\left[1+4\frac{b}{a}+\frac{20}{2}\frac{b^2}{a^2}+\frac{120}{6}\frac{b^3}{a^3}+\ldots\right]$$

$$= a^{-4}\left[1+\frac{4b}{a}+\frac{10b^2}{a^2}+\frac{20b^3}{a^3}+\ldots\right]$$

Ex. 2 : State first four terms in the expansion of $\dfrac{1}{(a+b)}$, $|b|<|a|$

Solution : $= \dfrac{1}{(a+b)} = (a+b)^{-1}$

$$= a^{-1}\left(1+\frac{b}{a}\right)^{-1}$$

$$a^{-1}\left[1+(-1)\left(\frac{b}{a}\right)+\frac{(-1)(-2)}{2!}\left(\frac{b}{a}\right)^2+\frac{(-1)(-2)(-3)}{3!}\left(\frac{b}{a}\right)^3+\ldots\right]$$

$$= a^{-1}\left[1-\frac{b}{a}+\frac{b^2}{a^2}-\frac{b^3}{a^3}+\ldots\right]$$

Ex. 3 : State first four terms in the expansion of $(2-3x)^{-1/2}$ if $|x|< \frac{2}{3}$

Solution : $|x|< \frac{2}{3}$

$\therefore \left|\frac{3x}{2}\right| < 1$

We have $(2-3x)^{-1/2}$

$$= 2^{-1/2}\left(1-\frac{3x}{2}\right)^{-\frac{1}{2}}$$

$$= 2^{-\frac{1}{2}}\left[1+\left(-\frac{1}{2}\right)\left(\frac{-3x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)}{2!}\left(\frac{-3x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\left(-\frac{1}{2}-2\right)}{3!}\left(\frac{-3x}{2}\right)^3+\ldots\right]$$

$$= 2^{-\frac{1}{2}}\left[1+\frac{3x}{4}+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{9x^2}{4}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(\frac{-27x^3}{8}\right)+\ldots.\right]$$

$$= 2^{-\frac{1}{2}}\left[1+\frac{3x}{4}+\frac{27x^2}{32}+\frac{135x^3}{128}+\ldots\right]$$

Ex. 4 : Find the value of $\sqrt{30}$ upto 4 decimal places.

Solution:

$$\sqrt{30} = (25+5)^{1/2}$$

$$= (25)^{1/2}\left(1+\frac{5}{25}\right)^{\frac{1}{2}}$$

$$= 5\left(1+\frac{1}{5}\right)^{\frac{1}{2}}$$

$$= 5\left[1+\frac{1}{10}+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2}\left(\frac{1}{25}\right) + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(\frac{-3}{2}\right)}{6}\left(\frac{1}{125}\right)+\ldots\right]$$

$$= 5\left[1+\frac{1}{10}-\frac{1}{200}+\frac{1}{2000}-\ldots\right]$$

$= 5\,[\,1 + 0.1 - 0.005 + 0.0005]$

(upto 4 decimal places)

$= 5[1.0955]$

$= 5.4775$

EXERCISE 4.4

(1) State, by writing first four terms, the expansion of the following, where $|x|<1$

(i) $(1+x)^{-4}$ (ii) $(1-x)^{-1/3}$

(iii) $(1-x^2)^{-3}$ (iv) $(1+x)^{-1/5}$

(v) $(1+x^2)^{-1}$

(2) State, by writing first four terms, the expansion of the following, where $|b|<|a|$

(i) $(a-b)^{-3}$ (ii) $(a+b)^{-4}$

(iii) $(a+b)^{1/4}$ (iv) $(a-b)^{-1/4}$

(v) $(a+b)^{-1/3}$

(3) Simplify first three terms in the expansion of the following

(i) $(1+2x)^{-4}$ (ii) $(1+3x)^{-1/2}$

(iii) $(2-3x)^{1/3}$ (iv) $(5+4x)^{-1/2}$

(v) $(5-3x)^{-1/3}$

(4) Use binomial theorem to evaluate the following upto four places of decimals.

(i) $\sqrt{99}$ (ii) $\sqrt[3]{126}$

(iii) $\sqrt[4]{16.08}$ (iv) $(1.02)^{-5}$

(v) $(0.98)^{-3}$

4.5 Binomial Coefficients :

The coefficients $^nC_0$, $^nC_1$, $^nC_2$, ... ^nC_n in the expansion of $(a+b)^n$ are called the binomial coefficients and denoted by C_0, C_1, C_2, ... C_n respectively

Now $(1+x)^n = {}^nC_0x^0 + {}^nC_1x^1 + {}^nC_2x^2 + ... + {}^nC_nx^n$ ...(i)

Put $x = 1$ we get

$(1 + 1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore 2^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n$

$\therefore {}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$

$\therefore C_0 + C_1 + C_2 + ... + C_n = 2^n$

$\therefore$ The sum of all binomial coefficients is 2^n

(ii) Put $x = -1$, in equation (i) we get

$(1 - 1)^n = {}^nC_0 - {}^nC_1 + {}^nC_2 - ... + (-1)^n \, {}^nC_n$

$\therefore 0 \quad = {}^nC_0 - {}^nC_1 + {}^nC_2 - ..... + (-1)^n \, {}^nC_n$

$\therefore {}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + ... + (-1)^n \, {}^nC_n = 0$

$\therefore {}^nC_0 + {}^nC_2 + {}^nC_4 + ... = {}^nC_1 + {}^nC_3 + {}^nC_5 + ...$

$\therefore C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ...$

C_0, C_2, C_4,..... are called as even coefficients.

C_1, C_3, C_5,.... are called as odd coefficients.

Let $C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = k$

Now $C_0 + C_1 + C_2 \div C_3 + ... + C_n = 2^n$

$\therefore (C_0 + C_2 + C_4 + ...) + (C_1 + C_3 + C_5 + ...) = 2^n$

$\therefore k + k = 2^n$

$\therefore 2k = 2^n$

$\therefore k = \dfrac{2^n}{2}$, $k = 2^{n-1}$

$\therefore C_0 + C_2 + C_4 + ... = C_1 + C_3 + C_5 + ... = 2^{n-1}$

$\therefore$ The sum of even coefficients = The sum of odd coefficients = 2^{n-1}

SOLVED EXAMPLES

Ex.1 : Show that $C_0 + C_1 + C_2 + ... + C_{10} = 1024$

Solution : We have $C_0 + C_1 + C_2 + ... \; C_n = 2^n$

Put n = 10, we get

$C_0 + C_1 + C_2 + ... C_{10} = 2^{10}$

$\therefore C_0 + C_1 + C_2 + ... + C_{10} = 1024$

Ex. 2 : Show that

$C_0 + C_2 + C_4 + ... + C_{12} = C_1 + C_3 + C_5 + ... + C_{11} = 2048$

Solution : We have

$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + ... + C_{n-1} + C_n = 2^n$

Put n = 12, we get

$C_0 + C_1 + C_2 + C_3 + C_4 + C_5 + ... + C_{11} + C_{12} = 2^{12} = 4096$ (i)

We know that

The sum of even coefficients = The sum of odd coefficents.

$\therefore \quad C_0 + C_2 + C_4 + ... + C_{12} = C_1 + C_3 + C_5 + ... + C_{11} = k$ ...(ii)

Now from (i)

$(C_0 + C_2 + C_4 + ... + C_{12}) + (C_1 + C_3 + C_5 + ... + C_{11}) = 4096$

$\therefore k + k = 4096$

$\therefore 2k = 4096$

$\therefore k = 2048$

$\therefore C_0 + C_2 + C_4 + ... + C_{12} = C_1 + C_3 + C_5 + ... + C_{11} = 2048$

Ex. 3 : Prove that

$C_1 + 2C_2 + 3C_3 + 4C_4 + ... + nC_n = n.2^{n-1}$

Solution :

L.H.S. = $C_1 + 2C_2 + 3C_3 + 4C_4 + ... + nC_n$

$$= n + 2\frac{n(n-1)}{2!} + \frac{3n(n-1)(n-2)}{3!} + ... + n.1$$

$$= n\left[1+(n-1)+\frac{(n-1)(n-2)}{2!}+....+1\right]$$

$$= n\,[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + ...+ {}^{n-1}C_{n-1}]$$

$$= n\,[C_0 + C_1 + C_2 + ...+ C_{n-1}]$$

$$= n.2^{n-1}$$

$$= \text{R.H.S.}$$

$$\therefore\ C_1 + 2C_2 + 3C_3 + ... + nC_n = n.2^{n-1}$$

Ex. 4 : Prove that

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

Solution : L.H.S.

$$= \frac{C_0}{1} + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} .... + \frac{C_n}{n+1}$$

$$= 1 + \frac{n}{2} + \frac{1}{3}\frac{n(n-1)}{2!} + \frac{1}{4}\frac{n(n-1)(n-2)}{3!} + ... + \frac{1}{n+1}$$

$$= \frac{1}{(n+1)}\left[(n+1) + \frac{n(n+1)}{2!} + \frac{(n+1)n(n-1)}{3!} + ... + 1\right]$$

$$= \frac{1}{(n+1)}\ [{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + ...+ {}^{n+1}C_{n+1}]$$

$$= \frac{1}{(n+1)}\ [1 + {}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + ...+ {}^{n+1}C_{n+1} - 1]$$

$$= \frac{1}{(n+1)}\ [{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + ...+ {}^{n+1}C_{n+1} - 1]$$

$$= \frac{1}{(n+1)}\ [C_0 + C_1 + C_2 + ...+ C_{n+1} - 1]$$

$$= \frac{1}{(n+1)}\ (2^{n+1}-1)$$

$$= \frac{(2^{n+1}-1)}{n+1} = \text{R.H.S.}$$

$$\therefore\ C_0 + \frac{C_1}{2} + \frac{C_2}{3} + ... + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$$

EXERCISE 4.5

Show That

(1) $C_0 + C_1 + C_2 + ... C_8 = 256$

(2) $C_0 + C_1 + C_2 + ... C_9 = 512$

(3) $C_1 + C_2 + C_3 + ... C_7 = 127$

(4) $C_1 + C_2 + C_3 + ... C_6 = 63$

(5) $C_0 + C_2 + C_4 + C_6 + C_8 = C_1 + C_3 + C_5 + C_7 = 128$

(6) $C_1 + C_2 + C_3 + ... C_n = 2^n - 1$

(7) $C_0 + 2C_1 + 3C_2 + 4C_3 + ... + (n+1)C_n = (n+2)2^{n-1}$

Let's Remember

- Step (I) Foundation : To prove P(1) is true

  Step (II) Assumption : To assume P(k) is true.

  Step (III) Succession : To prove that P(k+1) is true.

  Step (IV) Induction :P(n) is true for all $n \in N$.

- If a, b $\in$ R and n $\in$ N, then

  $(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + ... + {}^nC_n\, a^0 b^n$

  $(a-b)^n = {}^nC_0\, a^n b^0 - {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 .... + (-1)^n\, {}^nC_n\, a^0 b^n$

- General term in the expansion of $(a+b)^n$ is

  $t_{r+1} = {}^nC_r\, a^{n-r} b^r$

- If $|x| < 1$ and n is any real number then

  $$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

  $$(1+x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

- $C_0 + C_1 + C_2 + ... + C_n = 2^n$

- $C_0 + C_2 + C_4 + C_6 + ... = C_1 + C_3 + C_5 + C_7 + ... = 2^{n-1}$

MISCELLANEOUS EXERCISE - 4

(I) Select the correct anwsers from the given alternatives.

(1) The total number of terms in the expression of $(x+y)^{100}+(x-y)^{100}$ after simplification is :

A) 50 B) 51 C) 100 D) 202

(2) The middle term in the expansion of $(1+x)^{2n}$ will be :

A) $(n-1)^{th}$ B) n^{th} C) $(n+1)^{th}$ D) $(n+2)^{th}$

(3) In the expansion of $(x^2-2x)^{10}$, the coefficient of x^{16} is

A) −1680 B) 1680 C) 3360 D) 6720

(4) The term not containing x in expansion of $(1-x)^2\left(x+\frac{1}{x}\right)^{10}$ is

A) ${}^{11}C_5$ B) ${}^{10}C_5$ C) ${}^{10}C_4$ D) ${}^{10}C_7$

(5) The number of terms in expansion of $(4y+x)^8-(4y-x)^8$

A) 4 B) 5 C) 8 D) 9

(6) The value ${}^{14}C_1+{}^{14}C_3+{}^{14}C_5+\ldots\ldots+{}^{14}C_{11}$ is

A) $2^{14}-1$ B) $2^{14}-14$ C) 2^{12} D) $2^{13}-14$

(7) The value ${}^{11}C_2+{}^{11}C_4+{}^{11}C_6+{}^{11}C_8$ is equal to

A) $2^{10}-1$ B) $2^{10}-11$ C) $2^{10}+12$ (D) $2^{10}-12$

(8) In the expansion of $(3x+2)^4$, the coefficient of middle term is

A) 36 B) 54 C) 81 D) 216

(9) The coefficient of the 8^{th} term in the expansion of $(1+x)^{10}$ is :

A) 7 B) 120 C) ${}^{10}C_8$ D) 210

(10) If the coefficient of x^2 and x^3 in the expansion of $(3+ax)^9$ are the same, then the value of a is

A) $-\frac{7}{9}$ B) $-\frac{9}{7}$ C) $\frac{7}{9}$ D) $\frac{9}{7}$

(II) Answer the following.

(1) Prove, by method of induction, for all $n \in N$

(i) $8+17+26+\ldots.+(9n-1)=\frac{n}{2}(9n+7)$

(ii) $1^2+4^2+7^2+\ldots+(3n-2)^2=\frac{n}{2}(6n^2-3n-1)$

(iii) $2+3.2+4.2^2+\ldots+(n+1)2^{n-1}=n.\ 2^n$

(iv) $\frac{1}{3.4.5}+\frac{2}{4.5.6}+\frac{3}{5.6.7}+\ldots+\frac{n}{(n+2)(n+3)(n+4)}$

$=\frac{n(n+1)}{6(n+3)(n+4)}$

(2) Given that $t_{n+1}=5t_n-8$, $t_1=3$, prove by method of induction that $t_n=5^{n-1}+2$

(3) Prove by method of induction

$$\begin{pmatrix}3 & -4\\ 1 & -1\end{pmatrix}^n=\begin{pmatrix}2n+1 & -4n\\ n & -2n+1\end{pmatrix},\ \forall\ n\in N$$

(4) Expand $(3x^2+2y)^5$

(5) Expand $\left(\frac{2x}{3}-\frac{3}{2x}\right)^4$

(6) Find third term in the expansion of $\left(9x^2-\frac{y^3}{6}\right)^4$

(7) Find tenth term in the expansion of $\left(2x^2+\frac{1}{x}\right)^{12}$

(8) Find the middle term (s) in the expansion of

(i) $\left(\frac{2a}{3}-\frac{3}{2a}\right)^6$ (ii) $\left(x-\frac{1}{2y}\right)^{10}$

(iii) $(x^2+2y^2)^7$ (iv) $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$

(9) Find the coefficients of

(i) x^6 in the expansion of $\left(3x^2 - \frac{1}{3x}\right)^9$

(ii) x^{40} in the expansion of $\left(\frac{1}{x^2} + x^4\right)^{18}$

(10) Find the constant term in the expansion of

(i) $\left(\frac{4x^2}{3} + \frac{3}{2x}\right)^9$ (ii) $\left(2x^2 - \frac{1}{x}\right)^{12}$

(11) Prove by method of induction

(i) $\log_a x^n = n \log_a x, x > 0.$ n $\in$ N

(ii) $15^{2n-1}+1$ is divisible by 16, for all n $\in$ N.

(iii) $5^{2n} - 2^{2n}$ is divisible by 3, for all n $\in$ N.

(12) If the coefficient of x^{16} in the expansion of $(x^2 + ax)^{10}$ is 3360, find a.

(13) If the middle term in the expansion of $\left(x + \frac{b}{x}\right)^6$ is 160, find b.

(14) If the coefficient of x^2 and x^3 in the expansion of $(3 + kx)^9$ are equal, find k.

(15) If the constant term in the expansion of $\left(x^3 + \frac{k}{x^8}\right)^{11}$ is 1320, find k.

(16) Show that there is no term containing x^6 in the expansion of $\left(x^2 - \frac{3}{x}\right)^{11}$.

(17) Show that there is no constant term in the expansion of $\left(2x - \frac{x^2}{4}\right)^9$

(18) State, first four terms in the expansion of $\left(1 - \frac{2x}{3}\right)^{-1/2}$

(19) State, first four terms in the expansion of $(1-x)^{-1/4}$

(20) State, first three terms in the expansion of $(5 + 4x)^{-1/2}$

(21) Using binomial theorem, find the value of $\sqrt[3]{995}$ upto four places of decimals.

(22) Find approximate value of $\frac{1}{4.08}$ upto four places of decimals.

(23) Find the term independent of x in the in expansion of $(1 - x^2)\left(x + \frac{2}{x}\right)^6$

(24) $(a + bx)(1 - x)^6 = 3 - 20x + cx^2 + ...$ then find a, b, c.

(25) The 3rd term of $(1+x)^n$ is $36x^2$. Find 5th term.

(26) Suppose $(1+kx)^n = 1 - 12x + 60x^2 - ....$ find k and n.

5 SETS AND RELATIONS

Let's Study

- Representation of a set
- Types of sets
- Intervals
- Operations on sets
- Ordered pair
- Types of relations

Let's Recall

5.1 Introduction:

The concept of a set was developed by German mathematician George Cantor (1845–1918)

You have already learnt about sets and some basic operations involving them in the earlier standards.

We often talk about group or collection of objects. Surely you must have used the words such as team, bouquet, bunch, flock, family for collection of different objects.

It is very important to determine whether a given object belongs to a given collection or not. Consider the following collections:

i) Successful persons in your city.

ii) Happy people in your town

iii) Clever students in your class.

iv) Days in a week.

v) First five natural numbers.

First three collections are not examples of sets, but last two collections represent sets. This is because in first three collections, we are not sure of the objects. The terms 'successful persons,' 'Happy people', 'Clever student' are all relative terms. Here, the objects are not well–defined. In the last two collections. We can determine the objects clearly. Thus, we can say that objects are well–defined.

5.1.1 Set : Definition:

A collection of well–defined objects is called a set.

The object in a set is called its element or member.

We denote sets by capital letters A,B,C. etc. The elements of a set are represented by small letters a, b, c, x, y, z etc. If x is an element of a set A we write $x \in A$, and read as 'x belongs to A'. If x is not an element of a set A, we write $x \notin A$, and read as 'x does not belong to A.'

e.g. zero is a whole number but not a natural number.

$\therefore$ $0 \in W$ (Where W is the set of whole numbers) and $0 \notin N$ (Where N is the set of natural numbers)

5.1.2 Representation of a set:

There are two methods of representing a set.

1) Roster or Tabular method or List method

2) Set–Builder or Rule Method

3) Venn Diagram

1. Roster Method:

In the Roster method, we list all the elements of the set within braces {,} and separate the elements by commas.

Ex : State the sets using Roster method.

i) B is the set of all days in a week.

B = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

ii) C is the set of all vowels in English alphabets.
C = {a, e, i, o, u}

Let's Note:

1) If the elements are repeated, they are written only once.

2) While listing the elements of a set, the order in which the elements are listed is immaterial.

2. Set-Builder Method:

In the set builder method, we describe the elements of the set by specifying the property which determines the elements of the set uniquely.

Ex : State the sets using set–Builder method.

i) Y = {Jan, Feb, Mar, Apr, ..., Dec}
Y = {x/x is a month of a year}

ii) B = {1, 4, 9, 16, 25, ...}
B = {x/$x \in$N and x is a square}

3) Venn Diagram:

The pictorial representation of a set is called Venn diagram. English Logician John Venn introduced such diagrams. We can use triangles, circles, rectangles or any closed figure to represent a set.

In a Venn diagram the elements of the sets are shown as points

A = {1,2,3} B = {a,b,c,d,e,f} C = {4,5,6}

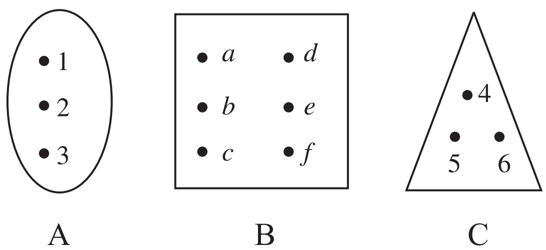


Fig. 5.1

5.1.3 Number of elements of a set:

The number of distinct elements contained in a finite set A is denoted by n(A).

Thus, if A = {5, 2, 3, 4}, then n(A) = 4

n(A) is also called the cardinality of set A.

5.1.4 Types of Sets:

1) Empty Set:

A set containing no element is called an empty or a null set and is denoted by the symbol ϕ or { } or void set.

e.g. A = {x/$x \in$N, 1<x<2} = { }

Here n(A) = 0

2) Singleton set:

A Set containing only one element is called a singleton set.

e.g. Let A be a set of all integers which are neither positive nor negative.

$\therefore$ A = {0} Here n (A) = 1

3) Finite set:

The empty set or set which contains finite number of objects is called a finite set.

e.g set of letters in the word 'BEAUTIFUL'

A = {B, E,A ,U, T, I, F, L}, n(A) = 7

A is a finite set

4) Infinite set:

A set which is not finite, is called an infinite set.

e.g. set of natural numbers, set of rational numbers.

Note :

1) An empty set is a finite set.

2) N, Z, set of all points on a circle, are infinite sets.

5) Subset:

A set A is said to be a subset of set B if every element of A is also an element of B and we write A $\subseteq$ B.

Note: 1) ϕ is subset of every set.

2) A $\subseteq$ A, Every set is subset of itself.

6) **Superset:** If $A \subseteq B$, then B is called a superset of A and we write, $B \supseteq A$.

7) **Proper Subset:** A nonempty set A is said to be a proper subset of the set B, if all elements of set A are in set B and at least one element of B is not in A.

i.e. If $A \subseteq B$ and $A \neq B$ then A is called a proper subset of B and we write $A \subset B$.

e.g. Let $A = \{1, 3, 5\}$ and $B = \{1, 3, 5, 7\}$. Then, evey element of A is an element of B but $A \neq B$.

$\therefore$ $A \subset B$, i.e. A is a proper subset of B.

Remark: If there exists even a single element in A which is not in B then A is not a subset of B and we wrtie, $A \not\subset B$.

8) **Power Set:**

The set of all subsets of a given set A is called the power set of A and is denoted by P(A), Thus, every element of power set A is a set.

e.g. consider the set $A=\{a,b\}$, let us write all subsets of the set A. We know that ϕ is a subset of every set, so ϕ is a subset of A. Also $\{a\}$, $\{b\}$, $\{a,b\}$ are also subsets of A. Thus, the set A has in all four subsets viz. ϕ, $\{a\}$, $\{b\}$, $\{a,b\}$

$\therefore$ $P(A) = \{\phi, \{a\}, \{b\}, \{a,b\}\}$

9) **Equal sets:**

Two sets are said to be equal if they contain the same elements i.e. if $A \subseteq B$ and $B \subseteq A$.

e.g. Let X be the set of letters in the word 'ABBA' and Y be the set of letters in the word 'BABA'.

$\therefore$ $X = \{A, B\}$, $Y = \{B, A\}$

Thus the sets X and Y are equal sets and we denote it by $X = Y$

10) **Equivalent sets:**

Two finite sets A and B are said to be equivalent if $n(A) = n(B)$

e.g. $A = \{d, o, m, e\}$

$B = \{r, a, c, k\}$

$n(A) = n(B) = 4$

$\therefore$ A and B are equivalent sets.

11) **Universal set:** If in a particular discussion all sets under consideration are subsets of a set, say U, then U is called the universal set for that discussion.

e.g. The set of natural numbers N, the set of integers Z are subsets of real numbers R. Thus, for this discussion R is a universal set.

In general universal set is denoted by U or X.

5.1.5 Operations on sets:

1) **Complement of a set:**

The complement of the set A is denoted by

A' or A^c. It is defined as

$A' = \{x / x \in U, x \notin A\}$ = set of all elements in U which are not in A.

Ex. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ be the Universal Set and $A = \{2, 4, 6, 8\}$

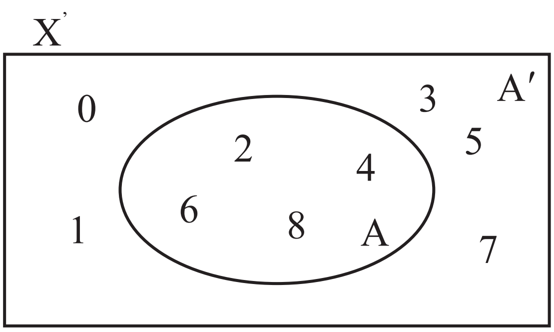


Fig. 5.2

$\therefore$ The complement of the set A is

$A' = \{0, 1, 3, 5, 7\}$

Properties:

i) $(A')' = A$ ii) $\phi' = U$ iii) $U' = \phi$

2) Union of Sets:

The union of two sets A and B is the set of all elements which are in A or in B, (here 'or' is taken in the inclusive sense) and is denoted by $A \cup B$

Thus, $A \cup B = \{x / x \in A \textbf{ or } x \in B\}$

The Union of two sets A and B i.e. $A \cup B$ is represented by a shaded part in Venn–diagram as in fig. 5.3 and fig. 5.4.

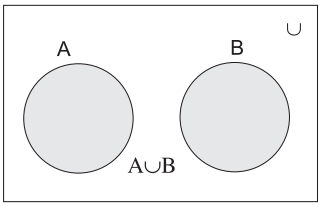


Fig. 5.3

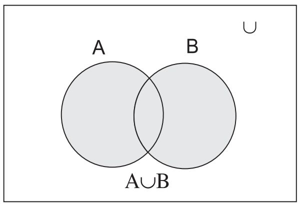


Fig. 5.4

Ex. A = $\{x / x$ is a prime number less than 10$\}$

B= $\{x / x \in N$, x is a factor of 8$\}$

find $A \cup B$.

Solution : We have A = {2,3,5,7}

B = {1, 2, 4, 8}

$\therefore A \cup B = \{1, 2, 3, 4, 5, 7, 8\}$

Properties:

i) $A \cup B = B \cup A$ (Commutativity)

ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associativity)

iii) $A \cup \phi = A$ Identity for union

iv) $A \cup A = A$ Idempotent law

v) If $A \cup A' = U$

vi) If $A \subset B$ then $A \cup B = B$

vii) $U \cup A = U$

viii) $A \subset (A \cup B)$, $B \subset (A \cup B)$

3) Intersection of sets:

The intersection of two sets A and B is the set of all elements which are both in A and B is denoted by $A \cap B$

Thus, $A \cap B = \{x / x \in A \textbf{ and } x \in B\}$

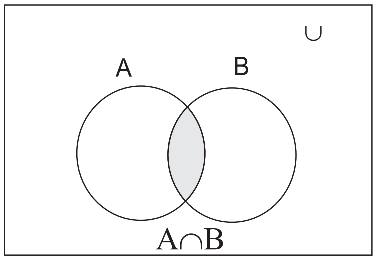


Fig. 5.5

The shaded portion in Fig. 5.5 represents the intersection of A and B i.e. $A \cap B$

Ex. A={1,3,5,7,9}

B = {1,2,3,4,5,6,7,8}

Find $A \cap B$.

Solution: $A \cap B$ = {1,3,5,7}

Ex. (Activity) : If A = $\{x / x \in N$, x is a factor of 12$\}$

B = $\{x / x \in N$, x is a factor of 18$\}$

Find $A \cap B$

Solution:

A={□, □, □, □, □, □}

B = {□, □, □, □, □, □}

$\therefore$ $A \cap B$ = {□, □, □, □}

= common factors of 12 & 18

Ex. : A = {1,3,5,7,9}

B = {2,4,6,8, 10}, $A \cap B$ = ?

Solution : $\therefore A \cap B = \{\ \} = \phi$

If $A \cap B = \phi$, A and B are disjoint sets.

Properties:

i) $A \cap B = B \cap A$ (Commutativity)

ii) $(A \cap B) \cap C = A \cap (B \cap C)$ (Associativity)

iii) $\phi \cap A = \phi$

iv) $A \cap A = A$ Idempotent law

v) $A \cap A = \varnothing$

vi) if $A \subset B$ then $A \cap B = A$

vii) $U \cap A = A$ (Identity for intersection)

viii) $(A \cap B) \subset A$, $(A \cap B) \subset B$

ix) a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

b) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

..... Distributivity

De Morgan's Laws:

For any two sets A and B

1) $(A \cup B)' = (A' \cap B')$

2) $(A \cap B)' = (A' \cup B')$

Verify the above laws by taking

$U = \{1,2,3,4,5\}$ $A = \{1,3,4\}$ $B = \{4,5\}$

4) Difference of Sets:

Difference of set A and set B is the set of elements which are in A but not in B and is denoted by $A-B$, or $A \cap B'$

The shaded portion in fig. 5.6 represents $A-B$. Thus, $A-B = \{x / x \in A, x \notin B\}$

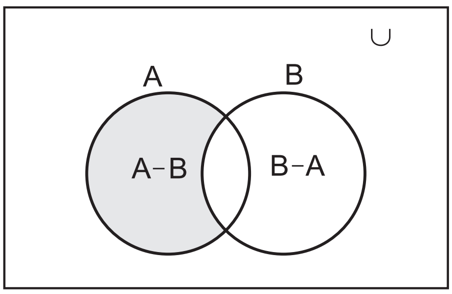


Fig. 5.6

Similary $B-A = \{y / y \in B, y \notin A\}$

Note:

i) $A-B \subseteq A$ and $B-A \subseteq B$.

ii) The sets $A-B$, $A \cap B$ and $B-A$ are mutually disjoint sets, i.e. the intersection of any of these two sets is the null (empty) set.

iii) $A-B = A \cap B'$, $B-A = A' \cap B$

iv) $A \cup B = (A-B) \cup (A \cap B) \cup (B-A)$

Shaded portion in fig. 5.7 represents $A \cup B$

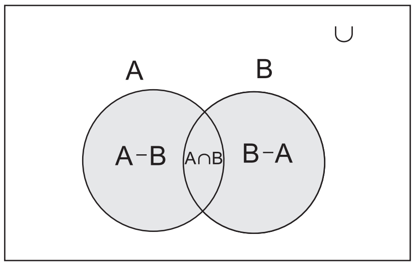


Fig. 5.7

v) $(A-B) \cup (B-A) = A \Delta B$ is called symmetric difference of sets A and B

Illustration :

If $A = \{4,5,6,7,8\}$ and $B = \{3,5,6,8,9\}$ then $A-B = \{4,7\}$, $B-A = \{3,9\}$

and $A \Delta B = (A-B) \cup (B-A) = \{4,7\} \cup \{3,9\} = \{3,4,7,9\}$

Properties:

i) $A \Delta B = (A \cup B) - (A \cap B)$

ii) $A \Delta A = \phi$ iii) $A \Delta \phi = A$

iv) If $A \Delta B = A \Delta C$ then $B = C$

v) $A \Delta B = B \Delta A$

vi) $A \cap (B \Delta C) = (A \cap B) \, \Delta \, (A \cap B)$

Properties of Cardinalityof Sets:

For give sets A, B

1) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

2) When A and B are disjoint sets, then

$n(A \cup B) = n(A) + n(B)$, as $A \cap B = \varnothing$,

$n(A \cap B) = 0$

3) $n(A \cap B') + n(A \cap B) = n(A)$

4) $n(A' \cap B) + n(A \cap B) = n(B)$

5) $n(A \cap B') + n(A \cap B) + n(A' \cap B) = n(A \cup B)$

6) For any sets A, B, C.

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B)$

$- n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$

7) If $n(A) = m$ $n[P(A)] = 2^m$ Where p(A) is power set of A

8) $n(A \Delta B) = n(A) + n(B) - 2n(A \cap B)$

5.1.6 Intervals:

1) Open Interval: Let a, b $\in$ R and a < b then the set $\{x / x \in R, a < x < b\}$ is called open interval and is denoted by (a,b). All the numbers between a and b belong to the open interval (a,b) but a, b themselves do not belonging to this interval.

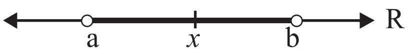


Fig. 5.8

$\therefore (a,b) = \{x / x \in R, a < x < b\}$

2) Closed Interval: Let a, b$\in$R and a < b then the set $\{x / x \in R, a \le x \le b\}$ is called closed interval and is denoted by [a,b]. All the numbers between a and b belong to the closed interval [a, b]. Also a and b belong to this interval.

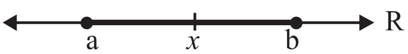


Fig. 5.9

$[a, b] = \{x / x \in R, a \le x \le b\}$

3) Semi-closed Interval:

$[a, b) = \{x / x \in R, a \le x < b\}$

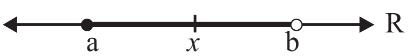


Fig. 5.10

Note that $a \in [a, b)$ but $b \notin [a, b)$

4) Semi–open Interval:

$(a, b] = \{x / x \in R, a < x \le b\}$

Fig. 5.11

(a, b] excludes a but includes b.

i.e. $a \notin (a, b]$ but $b \in (a, b]$

5) i) The set of all real numbers greater than a i.e. $(a,\infty) = \{x / x \in R, x > a\}$

Fig. 5.12

ii) The set of all real numbers greater than or equal to a

$[a,\infty) = \{x / x \in R, x \ge a\}$

Fig. 5.13

6) i) The set of all real numbers less than b. ie. $(-\infty, b)$

$\therefore (-\infty, b) = \{x / x \in R, x < b\}$

Fig. 5.14

ii) The set of all real numbers less than or equal to b i.e. $(-\infty, b]$

Fig. 5.15

$\therefore (-\infty, b] = \{x / x \in R, x \le b\}$

7) The set of all real numbers R is $(-\infty, \infty)$

Fig. 5.16

$R = (-\infty,\infty) = \{x / x \in R, -\infty < x < \infty\}$

Rules of inequality of real numbers.

1) If 0<a<b and $k>0$ then $a \pm k < b \pm k$, $ka < kb$, $a^k < b^k$

2) If $0 < a < b$ and $k < 0$ then $ka > kb$, $a^k > b^k$

Ex. : Solve the following inequalities and write the solution set using interval notation.

a) $-7 < 2x + 5 \le 9$

: Subtracting 5, we get

$-7-5 < 2x \le 9-5$

$-12 < 2x \le 4$

Dividing by 2, we get

$-6 < x \le 2$

So, $x \in (-6,2]$.

b) $x^2 + 2x < 15$

Subtracting 15, we get

$x^2 + 2x - 15 < 0$

Factor $x^2 + 2x - 15$, we get

Say $p(x) = (x - 3)(x + 5) < 0$

As $x-3$ and $x + 5$ is 0, for $x = 3, -5$, consider them as critical points. They divide number line into 3 regions as follow.

Regions	$x<-5$	$(-5<x<3)$	$(x>3)$
Test Points	-6	0	4
Value $(x-3)(x+5)$ of at test points	$(-6-3)$ $(-6+5)$ $= (-9)(-1)$ $= 9$	$(0-3)$ $(0+5)$ $= (-3)(5)$ $= -15$	$(4-3)$ $(4+5)$ $= (1)(9)$ $= 9$
Sign of $p(x)$	$p(x)>0$	$p(x)<0$	$p(x)>0$

Therefore the solution set is $(-5 < x < 3)$

So, x $(-5, 3)$.

c) $\dfrac{x}{x-3} \ge 4$

subtracting, 4, we get

[Note that, as we do not know value of x, we do not multiply by $(x - 3)$]

$\dfrac{x}{x-3} - 4 \ge 0$

$\dfrac{x-4(x-3)}{x-3} \ge 0$

$\dfrac{x-4x+12}{x-3} \ge 0$

$\dfrac{-3x+12}{x-3} \ge 0$

$\dfrac{-3(x-4)}{x-3} \ge 0$

Dividing by -3, we get

Say $p(x) = \dfrac{x-4}{x-3} \le 0$

Take x = 3, 4 as critical points. It divides number line into 3 regions as follows.

Regions	$(x \le 3)$	$(3 \le x \le 4)$	$(x \ge 4)$
Test Points	2	3.5	5
Sign of $x-4$	$-$ve	$-$ve	+ve
Sign of $x-3$	$-$ve	+ve	+ve
Sign of $p(x)$	$p(x)>0$ Except at $x = 3$	$p(x)<0$	$p(x)>0$

The solution set is $(x < 3)$ and $(x \ge 4)$. So $x \in (-\infty, 3) \cup [4, \infty)$.

Here $x = 3$ is not in the solution set, as it makes denominator 0.

Ex. 8 : If A = [–5,3], B = (3,7), and (5,8]. Find (a) A∪B (b) B∪C (c) A∪C (d) A∩B (e) B∩C (f) A∩C (g) A∪C' (h) B – C (i) C – B

Solution :

(a) $A \cup B = [-5, 3] \cup (3, 7) = [-5, 7)$

(b) $B \cup C = (3, 7) \cup (5, 8] = (3, 8]$

(c) $A \cup C = [-5, 3] \cup (5, 8]$

(d) $A \cap B = [-5, 3] \cap (3, 7) = \phi$

(e) $B \cap C = (3, 7) \cap (5, 8] = (5, 7)$

(f) $A \cap C = [-5, 3] \cap (5, 8] = \phi$

(g) $A \cup C' = [-5, 3] \cup (5, 8]' = [-5, 3] \cup \{(-\infty, 5] \cup (8, \infty)\} = (-\infty, 5] \cup (8, \infty)$

(h) $B - C = (3, 7) - (5, 8] = (3, 5]$

(i) $C - B = (5, 8] - (3, 7) = [7, 8]$

SOLVED EXAMPLES

Ex. 1: If $A = \{x / x \in N, x \text{ is a factor of } 6\}$

$B = \{x / x \in N, x \text{ is a factor of } 8\}$

find the A–B and B–A

Solution : $A = \{1, 2, 3, 6\}$

$B = \{1, 2, 4, 8\}$

$\therefore A - B = \{3, 6\}$

$B - A = \{4, 8\}$

Ex.2: $A = \left\{\frac{1}{x} / x \in N, x < 8\right\}$

$B = \left\{\frac{1}{2x} / x \in N, x \leq 8\right\}$ Find A–B and B–A

Solution: $A = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}\right\}$

$B = \left\{\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right\}$

$\therefore A - B = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}\right\}$

and $B - A = \left\{\frac{1}{8}, \frac{1}{10}, \frac{1}{12}, \frac{1}{14}, \frac{1}{16}\right\}$

Ex. 3 : If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$

$C = \{5, 6, 7, 8\}$, $D = \{7, 8, 9, 10\}$

find i) $A \cup B$ ii) $A \cup B \cup C$ iii) $B \cup C \cup D$

Are the sets A, B, C, D equivalent?

Solution: We have

i) $A \cup B = \{1, 2, 3, 4, 5, 6\}$

ii) $A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8\}$

iii) $B \cup C \cup D = \{3, 4, 5, 6, 7, 8, 9, 10\}$

As $n(A) = n(B) = n(C) = n(D) = 4$, the sets A, B, C, D are equivalent.

Ex. 4 : Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ be the universal set, $A = \{1, 3, 5, 7, 9\}$

$B = \{2, 3, 4, 6, 8, 10\}$, $C = \{6, 7, 8, 9\}$

find i) A' ii) $(A \cap C)'$ iii) (A') iv) $(B - C)'$

Solution: We have

i) $A' = \{2, 4, 6, 8, 10\}$

ii) $(A \cap C) = \{7, 9\}$

$\therefore (A \cap C)' = \{1, 2, 3, 4, 5, 6, 8, 10\}$

iii) $(A')' = \{2, 4, 6, 8, 10\} = \{1, 3, 5, 7, 9\} = A$

iv) $B - C = \{2, 3, 4, 10\}$

$\therefore (B - C)' = \{1, 5, 6, 7, 8, 9\}$

Ex. 5 : Let X be the universal set, for the non–empty sets A and B, verify the De Morgan's laws

i) $(A \cup B)' = A' \cap B'$

ii) $(A \cap B)' = A' \cup B'$

Where $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

$A = \{1, 2, 3, 4, 5\}$ $B = \{1, 2, 5, 6, 7\}$

Solution: $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$

$(A \cap B) = \{1, 2, 5\}$

$A' = \{6, 7, 8, 9, 10\}$

$B' = \{3, 4, 8, 9, 10\}$

i) $(A\cup B)' = \{8, 9, 10\}$ ...(1)

$A'\cap B' = \{8, 9, 10\}$ ...(2)

from (1) and (2),

$(A\cup B)' = A'\cap B'$

ii) $(A\cap B)' = \{3,4,6,7,8,9,10\}$ ...(3)

$A'\cup B' = \{3,4,6,7,8,9,10\}$ ...(4)

from (3) and (4)

$(A\cap B)' = A'\cup B'$

Ex.6: If $P = \{x/x^2 + 14x + 40 = 0\}$

$Q = \{x/x^2 - 5x + 6 = 0\}$

$R = \{x/x^2 + 17x - 60 = 0\}$ and the universal set $X = \{-20, -10 - 4, 2, 3, 4\}$, find

i) $P\cup Q$ ii) $Q\cap R$ iii) $P\cup(Q\cap R)$

iv) $P\cap(Q\cup R)$ v) $(P\cup Q)'$ vi) $Q'\cap R'$

Solution: $P = \{x/x^2 + 14x + 40 = 0\}$

$\therefore$ $P = \{-10, -4\}$

Similarly $Q = \{3, 2\}$, $R = \{-20, 3\}$ and

$X = \{-20, -10, -4, 2, 3, 4\}$

i) $P\cup Q = \{-10, -4, 3, 2\}$

ii) $Q\cap R = \{3\}$

iii) $P\cup(Q\cap R) = \{-10, -4, 3\}$

iv) $P\cap(Q\cup R) = \phi$

v) $(P\cup Q)' = \{-20, 4\}$

vi) $Q' = \{-20, -10, -4, 4\}$, $R' = \{-10, -4, 2, 4\}$

$Q'\cap R' = \{-10, -4, 4\}$

Ex. 7: If A and B are the subsets of the universal set X and $n(x) = 50$, $n(A) = 35$, $n(B) = 22$ and $n(A'\cap B') = 3$, find

i) $n(A\cup B)$ ii) $n(A\cap B)$

iii) $n(A'\cap B)$ iv) $n(A\cup B')$

Solution:

i) $n(A\cup B) = n(X) - n[(A\cup B)']$

$= n(X) - n(A'\cap B')$ (De Morgan's Law)

$= 50 - 3$

$= 47.$

ii) $n(A\cap B) = n(A) + n(B) - n(A\cup B)$

$n(A\cap B) = 35 + 22 - 47$

$= 10$

iii) $n(A'\cap B) = n(B) - n(A\cap B)$

$= 22 - 10$

$= 12$

iv) $n(A\cup B') = n(X) - n[(A\cup B)']'$

$= n(X) - n(A'\cap B)$

$= 50 - 12$

$= 38$

Ex. 8 : In a board examination, 40 students failed in Physics, 40 in Chemistry and 35. In Maths, 20 failed in Maths and Physics, 17 in Physics and Chemistry, 15 in Maths and Chemistry and 5 in all the three subjects. If 350 students appeared in the examination, how many of them did not fail in any subject?

Solution:

P = set of students failed in Physics

C = set of students failed in Chemistry

M = set of students failed in Maths

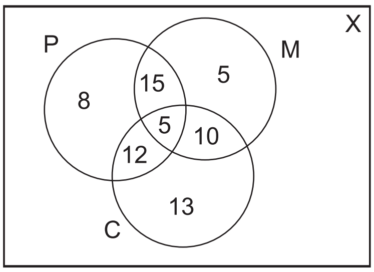


Fig. 5.17

From figure 5.17, we have

$n(X) = 350$, $n(P) = 40$, $n(C) = 40$, $n(M) = 35$

$n(M\cap P) = 20$, $n(P\cap C) = 17$, $n(M\cap C) = 15$

and $n(M\cap P\cap C) = 5$

The number of students who failed in at least one subject $= n(M\cup P\cup C)$

we have,

$n(M\cup P\cup C) = n(M) + n(P) + n(C) - n(M\cap P) - n(P\cap C) - n(M\cap C) + n(M\cap P\cap C)$

$= 35 + 40 + 40 - 20 - 17 - 15 + 5 = 68$

The number of students who did not fail in any subject $= n(X) - n(M\cup P\cup C) = 350 - 68 = 282$

Ex. 9 : A company produces three kinds of products A, B and C. The company studied the perference of 1600 consumers for these 3 products. It was found that the product A was liked by 1250, the product B was liked by 930 and product C was liked by 1000. The proudcts A and B were liked by 650, the products B and C were liked by 610 and the products C and A were liked by 700 consumers. None of the products were liked by 30 consumers.

Find number of consumers who liked.

i) All the three products

ii) Only two of these products.

Solution: Given that totally 1600 consumers were studied. $\therefore n(X) = 1600$, Let A be the set of all consumers who liked product A. Let B the set of all consumers who liked product B and C be the set of all consumers who liked product C.

$n(A) = 1250$,

$n(B) = 930$, $n(C) = 1000$.

$n(A\cap B) = 650$, $n(B\cap C) = 610$

$n(A\cap C) = 700$, $n(A'\cap B'\cap C') = 30$

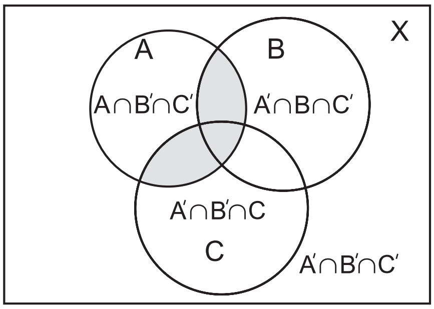


Fig. 5.18

i) $n(A\cup B\cup C) = n(X) - n[(A\cup B\cup C)]'$

$= n(X) - n(A'\cap B'\cap C') = 1600 - 30 = 1570$

Since, $(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(B\cap C) - n(C\cap A) + n(A\cap B\cap C)$

$\therefore 1570 = 1250 + 930 + 1000 - 650 - 610 - 700 + n(A\cap B\cap C)$

$\therefore n(A\cap B\cap C) = 1570 + 1960 - 3180$

$= 350$

$\therefore$ The number of consumers who liked all the three products is 350.

ii) $n[(A\cap B)\cap C'] = n(A\cap B) - n[(A\cap B)\cap C]$

$\because n(P\cap Q') = n(P) - n[P\cap Q]$

$= 650 - 350 = 300$

Similarly $n(A'\cap B\cap C) = 610 - 350 = 260$

and $n(A\cap B'\cap C) = 700 - 350 = 350$

$\therefore$ The number of consumers who liked only two of the three products is

$= n(A\cap B\cap C') + n(A\cap B'\cap C) + n(A'\cap B\cap C)$

$= 300 + 350 + 260 = 910$

Maximum & Minimum of Sets:

(a) $\min\{n(A\cup B)\} = \max\{n(A), n(B)\}$

(b) $\max\{n(A\cup B)\} = n(A) + n(B)$

(c) $\min\{n(A\cap B)\} = 0$

(d) $\max\{n(A\cap B)\} = \min\{n(A), n(B)\}$

Illustration:

e.g. If $n(A) = 10$, $n(B) = 20$ then $\min\{n(A\cup B)\} = \max\{10,20\} = 20$ and $\max\{n(A\cup B)\} = 10+20 = 30$, so $20 \le n(A\cup B) \le 30$.

Also $\min\{n(A\cap B)\} = 0$ and $\max\{n(A\cap B)\} = \min\{10,20\} = 10$, so $0 \le n(A\cap B) \le 10$.

Ex. 10) In a survey of 100 consumers 72 like product A and 45 like product B. Find the least and the most number that must have liked both products A and B.

Soln.: Let $n(A)$ and $n(B)$ be number of consumers who like proudct A and B respectively, i.e. $n(A) = 72$, $n(B) = 45$.

Since

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$ $= 72 + 45 - n(A \cap B) = 117 - n(A \cap B)$.

But as only 100 consumers are surveyed. Therefore $n(A \cup B) \leq 100$ i.e. $117 - n(A \cap B) \leq 100$.

$\therefore 117 - 100 \leq n(A \cap B)$, $17 \leq n(A \cap B)$.

So the least number of consumers who like both the products is 17.

Since $n(A \cap B) \leq n(A)$, $n(A \cap B) \leq n(B)$, then

$n(A \cap B) \leq \min\{n(A), n(B)\} = \min\{72, 45\} = 45$. So the most number of consumers who like both products is 45.

EXERCISE 5.1

1) Describe the following sets in Roster form

i) A = {x/x is a letter of the word 'MOVEMENT'}

ii) $B = \{x/x \text{ is an integer}, -\frac{3}{2} < x < \frac{9}{2}\}$

iii) $C = \{x/x = 2n + 1, n \in N\}$

2) Describe the following sets in Set–Builder form

i) $\{0\}$ ii) $(0, \pm 1, \pm 2, \pm 3\}$

iii) $\left\{\frac{1}{2}, \frac{2}{5}, \frac{3}{10}, \frac{4}{17}, \frac{5}{26}, \frac{6}{37}, \frac{7}{50}\right\}$

iv) $\{0, -1, 2, -3, 4, -5, 6, ...\}$

3) If $A = \{x/6x^2+x-15 = 0\}$

$B = \{x/2x^2-5x-3 = 0\}$

$C = \{x/2x^2-x-3 = 0\}$ then

find i) $(A \cup B \cup C)$ ii) $(A \cap B \cap C)$

4) If A, B, C are the sets for the letters in the words 'college', 'marriage' and 'luggage' respective, then verify that $[A-(B \cup C)] = [(A-B) \cap (A-C)]$

5) If $A = \{1, 2, 3, 4\}$, $B = \{3, 4, 5, 6\}$, $C = \{4, 5, 6, 7, 8\}$ and universal set $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, then verify the following:

i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cup C)$

iii) $(A \cup B)' = (A' \cap B)'$

iv) $(A \cap B)' = A' \cup B'$

v) $A = (A \cap B) \cup (A \cap B')$

vi) $B = (A \cap B) \cup (A' \cap B)$

vii) $(A \cup B) = (A-B) \cup (A \cap B) \cup (B-A)$

viii) $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$

ix) $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

x) $n(B) = (A' \cap B) + n(A \cap B)$

6) If A and B are subsets of the universal set X and $n(X) = 50$, $n(A) = 35$, $n(B) = 20$, $n(A' \cap B') = 5$, find

i) $n(A \cup B)$ ii) $n(A \cap B)$ iii) $n(A' \cap B)$ iv) $n(A \cap B')$

7) In a class of 200 students who appeared certain examinations, 35 students failed in CET, 40 in NEET and 40 in JEE, 20 failed in CET and NEET, 17 in NEET and JEE, 15 in CET and JEE and 5 failed in all three examinations. Find how many students,

i) did not fail in any examination.

ii) failed in NEET or JEE entrance.

8) From amongst 2000 literate individuals of a town, 70% read Marathi newspapers, 50% read English newspapers and 32.5% read both Marathi and English newspapers. Find the number of individuals who read.

i) at least one of the newspapers.

ii) neither Marathi and English newspaper.

iii) Only one of the newspapers.

9) In a hostel, 25 students take tea, 20 students take coffee, 15 students take milk, 10 student take bot tea and coffee, 8 students take both milk and coffee. None of them take tea and milk both and everyone takes atleast one beverage, find the total number of students in the hostel.

10) There are 260 persons with a skin disorder. If 150 had been exposed to the chemical A, 74 to the chemical B, and 36 to both chemicals A and B, find the number of persons exposed to

i) Chemical A but not Chemical B

ii) Chemical B but not Chemical A

iii) Chemical A or Chemical B

11) Write down the power set of A $= \{1,2,3)$

12) Write the following intervals in Set–Builder form

i) $(-3, 0)$, ii) $[6,12]$,

iii) $(6, \infty)$, iv) $(-\infty, 5]$

v) $(2, 5]$ vi) $[-3, 4)$

13) A college awarded 38 medals in volley ball, 15 in football and 20 in basket ball. The medals awarded to a total of 58 players and only 3 players got medals in all three sports. How many received medals in exactly two of the three sports?

14) Solve the following inequalities and write the solution set using interval notation.

i) $-9 < 2x + 7 \leq 19$ ii) $x^2 - x > 20$

iii) $\dfrac{2x}{x-4} \leq 5$ iv) $6x^2 + 1 \leq 5x$

15) If $A = (-7, 3]$, $B = [2, 6]$ and $C = [4, 9]$ then find (i) $A\cup B$ (ii) $B\cup C$ (iii) $A\cup C$ (iv) $A\cap B$ (v) $B\cap C$ (vi) $A\cap C$ (vii) $A'\cap B$ (viii) $B'\cap C'$ (ix) $B-C$ (x) $A-B$

5.2 Relations:

5.2.1 Ordered Pair:

A pair (a, b) of numbers, such that the order, in which the numbers appear is important, is called an ordered pair. Ordered pairs (a,b) and (b, a) are different. In ordered pair (a,b), 'a' is called first component and 'b' is called second component.

Note : i) (a,b)=(c, d), if and only if a = c and b = d.

ii) (a, b) = (b, a) if and only if a = b

Ex. : Find x and y when $(x + 3, 2) = (4, y - 3)$

Solution: Using the definition of equality of two ordered pairs, we have

$(x + 3, 2) = (4, y - 3)$

$\Rightarrow x + 3 = 4$ and $2 = y - 3$

$\Rightarrow x = 1$ and $y = 5$

5.2.2 Carstesian Product of two sets:

Let A and B be two non–empty sets then, the cartensian product of A and B is denoted by $A\times B$ and is defined as set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$

Thus, $A\times B = \{(a, b) / a\in A, b\in B\}$

e.g. 1) If $A = \{1, 2\}$ and $B = \{a, b, c\}$ then $A\times B = \{(1, a) (1, b), (1, c), (2, a), (2, b), (2, c)\}$

2) If $A = \phi$ or $B = \phi$, we define $\therefore A\times B = \phi$

Note:

Let A and B be any two finite sets with $n(A) = m$ and $n(B) = n$ then the number of elements in the Cartesian product of two finite sets A and B is given by $n(A\times B) = mn$

Ex. 2 : Let $A = \{1. 3\}$; $B = \{2, 3, 4\}$ Find the number of elements in the Cartesian product of A and B.

Solution: Given $A = \{1, 3\}$ and $B = \{2, 3, 4\}$

$\therefore$ $n(A) = 2$ and $n(B) = 3$

$\therefore$ $n(A\times B) = 2\times 3 = 6$

5.2.3 Cartesian product of a set with itself:

$A \times A = A^2 = \{(a, b) / a, b \in A\}$

Where (a, b) is an ordered pair.

$A^3 = A \times A \times A = \{(a, b, c) / a, b, c \in A\}$

Where (a, b, c) is an ordered triplet.

e.g. If $A = \{4, 5\}$ then we have

$A^2 = A \times A = \{(4, 4), (4, 5), (5, 4), (5, 5)\}$

$A^3 = A \times A \times A = \{(4, 4, 4), (4, 4, 5), (4, 5, 4)$ $(5, 5, 5), (5, 4, 4) (5, 4, 5), (5, 5, 4), (4, 5, 5)\}$

Note: If $n(A) = m$ then $n(A \times A \times A) = m^3$. i.e. $n(A^3) = [n(A)]^3$

In general, $n(A \times A \times$ .... r times$) = n(A)^r$

5.2.4 Definitions of relation, Domain, Co- domain and Range of a Relation:

Relation:

The concept of the term Relation is drawn from the meaning of relation in English language, according to which two objects or quantities are related if there is recognizable link between them.

Members in a family are often related to each other. We describe relation as A is friend of B, F is father of S, P is sister of Q etc. In Mathematics we can have different relations. e.g. among integers we can define the relation as m is factor of n. If this relation is denoted by R, then we write 2R4, 3R6, 5R10.

A well defined relation between elements of A and B is given by aRb, $a \in A$ and $b \in B$. Thus the relation gives ordered pair (a,b) and defines a subset of $A \times B$.

e.g. if $A = \{2, 3, 4, 5, 6)$ and $B = \{6, 7, 8, 10\}$ then the relation aRb if a is factor of b gives the subset $\{(2, 6), (2, 8), (2, 10), (3, 6), (4, 8), (5, 10)\}$ of $A \times B$.

On the other hand, if we consider the subset $S' = \{(2, 7), (3, 10), (4, 7)\}$ of $A \times B$, then there is a unique relation R' given by this subset. i.e. 2R'7, 3R'10, 4R'7. Thus S' has defined R'.

For every relation between A and B, there is a unique subset defined in $A \times B$ and for every subset of $A \times B$, there is a unique relation associated with it. Hence we can say that every relation between A and B is a subset of $A \times B$ and all relationships are ordered pairs in $A \times B$.

When A = B, that is when we have relation between the elements of A, that will be subset of $A \times A$. If a is nonempty, any relation in $A \times A$ is called **binary relation** on A.

Ex. : Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 4, 5\}$

Let R be a relation such that $(x, y) \in R$ implies $x < y$. We list the elements of R.

Solution: Here $A = \{1,2, 3, 4, 5\}$ and $B = \{1, 4, 5\}$

$\therefore R = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5)\}$

Arrow diagram for this relation R is given by

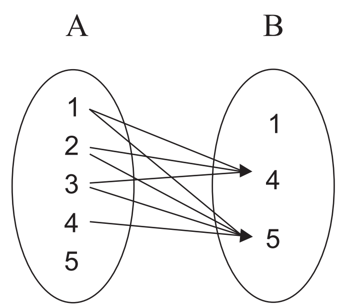


Fig. 5.19

Domain:

The set of all first components of the ordered pairs in a relation R is called the domain of the relation R.

i.e. domain (R) = $\{a/(a, b) \in R\}$

Range:

The set of all second components of all ordered pairs in a relation R is called the range of the relation.

i.e. range (R) = $\{b/(a, b) \in R\}$

Co-domain:

If R is a relation from A to B then set B is called the co-domain of the relation R.

Note : Range is a subset of co-domain (check!).

5.2.5 Binary relation on a set:

Let A be nonempty set then every subset of $A \times A$ is binary relation on A.

Illustrative Examples:

Ex. 1.: Let A = {1, 2, 3} and

R = {(1, 2), (2, 2), (3, 1), (3, 2)}

$A \times A$ = {(1,1), (1,2), (1,3), (2,1), (2,2,), (2,3), (3,1), (3,2), (3,3)}

$R \subset A \times A$ and therefore, R is a binary relation on A.

Ex. 2.: Let N be the set of all natural numbers and R = $\{(a, b) / a, b \in N \text{ and } 2a + b = 10\}$

Since $R \subset N \times N$, R is a binary relation on N. R = {(1, 8), (2, 6), (3, 4), (4, 2)}

We can state domain, range and co-domain of the relation.

They are as follows:

Domain (R) = {1, 2, 3, 4}

Domain (R) = {2, 4, 6, 8}

Co-domain (R) = N

Ex. 3.: If R' is defined in $Z \times Z$ as aR'b if (a+b) is even. Let R be defined $Z \times Z$ as a Rb if (a–b) is even.

Note that R and R' define the same relation and subset related to them is {a, b)/a and b both even or a and b both odd}.

Note : i) Since $\phi \subset A \times A$, ϕ is a relation on A and is called the empty or void relation on A.

ii) Since $A \times A \subset A \times A$, $A \times A$ is a relation on A called the universal relation on A. i.e. $R = A \times A$

Ex. 4 : If A = {2, 4, 6}

then R = $A \times A$ = {(2, 2), (2, 4), (2, 6), (4, 2), (4, 4), (4, 6), (6, 2), (6, 4), (6, 6)} and $R = A \times A$ is the universal relation on A.

Note : The total number of relation that can be defined from a set A to a set B is the number of possible subsets of $A \times B$.

if $n(A) = m_1$ and $n(B) = m_2$

then $n(A \times B) = m_1 m_2$

and the total number of relations is $2^{m_1 m_2}$.

5.2.6 Identity Relation:

If the relation in A is given by aRa for every $a \in A$, then the subset given by this relation is the diagonal subset $\{a,a)/a \in A\}$. Hence, aRb if b = a. This is called identity relation.

5.2.7 Types of relations:

Let A be a non–empty set. Then a binary relation R on A is said to be

(i) Reflexive, if $(a, a) \in R$ for every $a \in A$ i.e. aRa for every $a \in A$

(ii) Symmetric, if $(a, b) \in R$
$\Rightarrow (b, a) \in R$ for all $a, b \in A$
i.e. aRb $\Rightarrow$ bRa for all $a, b \in A$

(iii) Transitive, if $(a, b) \in R$ and $(b, c) \in R$
$\Rightarrow (a, c) \in R$ for all $a, b, c \in A$.

Note: Read the symbol ' $\Rightarrow$' as 'implies'.

Equivalence relation:

A relation which is reflexive, symmetric and transitive is called an equivalence relation.

Illustrative examples:

Ex. 1: Let R be a relation on Q, defined by

$R = \{(a, b)/a, b \in Q \text{ and } a-b \in Z\}$

Show that R is an equivalence relation.

Solution: Given $R = \{(a, b)/a, b \in Q \text{ and } a-b \in Z\}$

i) Let $a \in Q$ then $a - a = 0 \in Z$

$\therefore (a, a) \in R$

So, R is reflexive.

ii) $(a, b) \in R \Rightarrow (a-b) \in Z$

i.e. $(a-b)$ is an integer

$\Rightarrow -(a-b)$ is an integer

$\Rightarrow (b-a)$ is an integer

$\Rightarrow (b, a) \in R$

Thus $(a, b) \in R \Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

iii) $(a, b) \in R$ and $(b, c) \in R$

$\Rightarrow (a-b)$ is an integer and $(b-c)$ is an integer

$\Rightarrow \{(a-b) + (b-c)\}$ is an integer

$\Rightarrow (a-c)$ is an integer

$\Rightarrow (a, c) \in R$

Thus $(a, b) \in R$ and $(b, c) \in R \Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

Thus, R is reflexive, symmetric and transitive.

$\therefore$ R is an equivalence relation.

Ex. 2 : Show that the relation "is congruent to" on the set of all triangles in a plane is an equivalence relation.

Solution: Let S be the set of all triangles in a plane. Then, the congruence relation on S is

i) Reflexive, since $\Delta \cong \Delta$ for every $\Delta \in S$

ii) Symmetric, since $\Delta_1 \cong \Delta_2 \Rightarrow \Delta_2 \cong \Delta_1$ for all $\Delta_1, \Delta_2 \in S$.

iii) Transitive, since $\Delta_1 \cong \Delta_2$ and $\Delta_2 \cong \Delta_3$, then $\Delta_1 \cong \Delta_3$ for all $\Delta_1, \Delta_2, \Delta_3 \in S$. Hence, the given relation is an equivalence relation.

Congruence Modulo:

Let n be any positive integer. $a = b \pmod{m} \Leftrightarrow (a - b)$ is divisible by m. read as 'a is congruent to b modulo m' e.g: $14 \equiv 4 \pmod 5$ since $14 - 4 = 10$ is divisible by 5.

Ex. $R = \{(a,b) : a,b \in Z, a \equiv b \pmod{m}\}$ is an equivalence relation.

Solution: Since $a - a = 0$ is divisible by m $\therefore a \equiv a \pmod{m}$ $\therefore$ R is reflexive.

If $a \equiv b \pmod{m}$ i.e. $a - b$ is divisible by m then $b - a$ is also divisible by m

i.e. $b \equiv a \pmod{m}$. $\therefore$ R is symmetric.

If $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

i.e. $a - b$ is divisible by m and $b - c$ is divisible by m.

Then $a - c = a - b + b - c = (a - b) + (b - c)$ is also divisible by m. i.e. $a \equiv c \pmod{m}$.

$\therefore$ R is transitivie.

So, R is an equivalence relation.

Note:

Since each relation defines a unique subset of $A\times B$.

1) If $n(A) = m$ and $n(B) = n$ then number of relations S from A to B is 2^{mn}.

2) If R is relation on A and $n(A) = m$ then number of relations on A is 2^{n^2}

SOLVED EXAMPLES

Ex. 1: If $(x+1, y-2) = (3, 1)$ find the value of x and y.

Solution: Since the order pairs are equal, the coresponding elements are equal.

$\therefore\ x + 1 = 3$ and $y – 2 = 1$

$\therefore\ x = 2$ and $y = 3$

Ex. 2: If A = {1, 2}, find A×A

Solution: We have A = {1, 2}

$\therefore$ A×A = {(1, 1), (1, 2), (2, 1), (2, 2)}

Ex. 3: If A = {1, 3, 5} and B = {2, 3} find A×B and B×A show that A×B ≠ B×A

Solution: We have A×B = {(1, 2), (1, 3), (3, 2), (3, 3), (5, 2), (5, 3)} and B×A = {(2, 1), (2, 3), (2, 5), (3, 1), (3, 3), (3, 5)}

All elements in A×B, B×A except (3, 3) are different.

$\therefore$ A×B ≠ B×A

Ex. 4 : (Activity) If A×B = {(3, 2), (3, 4), (5, 2), (5, 4)} then find A and B

Solution: Clearly, we have

A = Set of all first components of A×B

$\therefore$ A = {$\boxed{}$, $\boxed{}$}

B = Set of all second components of A×B

$\therefore$ B = {$\boxed{}$, $\boxed{}$}

Ex. 5: A and B are two sets given in such a way that A×B contains 6 elements. If three elements of A×B are (1, 3), (2, 5) and (3, 3), find its remaining elements.

Solution: Since (1, 3), (2, 5) and (3, 3) are in A×B it follows that 1, 2, 3 are elements of A and 3, 5 are elements of B. As A×B contains 6 elements and if we let.

A = {1, 2, 3} and B = {3, 5}, then

A×B = {(1, 3), (1, 5), (2, 3), (2, 5), (3, 3) (3, 5)} giving n (A×B) = 6.

Hence, the remaining elements of A×B are (1, 5), (2, 3) and (3, 5).

Ex. 6: Express $\{(x, y)/x^2 + y^2 = 25$ where $x, y \in$ W} as a set of ordered pairs.

Solution: We have $x^2 + y^2 = 25$

$\therefore\ x = 0, y = 5 \Rightarrow x^2 + y^2 = (0)^2 + (5)^2 = 25$

$x = 4, y = 3 \Rightarrow x^2 + y^2 = (4)^2 + (3)^2 = 25$

$x = 5, y = 0 \Rightarrow x^2 + y^2 = (5)^2 + (0)^2 = 25$

$x = 3, y = 4 \Rightarrow x^2 + y^2 = (3)^2 + (4)^2 = 25$

$\therefore$ The set = {(0, 5), (3, 4), (4, 3), (5, 0)}

Ex. 7: Let A = {1, 2, 3} and B = {2, 4, 6} Show that R = {(1, 2), (1, 4), (3, 2), (3, 4)} is a relation from A to B. Find i) domain (R) ii) Co-domain (R) iii) Range (R).

Solution : Here A = {1, 2, 3} , B = {2, 4, 6}

and R = {(1, 2), (1, 4), (3, 2), (3, 4)}

Since R⊂A×B, R is a relation from A to B

i) Domain (R) = Set of first components of R = {1, 3}

ii) Co-domain (R) = B = {2, 4, 6}

iii) Range (R) = Set of second components of R = {2, 4}

Ex. 8: Let A = {1, 2, 3, 4, 5} and B = {1, 4, 5}. Let R be a relation from A to B. Such that $(x, y) \in$ R if $2x < y$.

i) List the elements of R.

ii) Find the domain, co-domain and range of R.

Solution : A = {1, 2, 3, 4, 5} and B = {1, 4, 5}

i) The elements of R are as follows:

R = {(1, 4), (1, 5), (2, 5),}

ii) Domian (R) = {1, 2}

Range (R) = {4, 5}

Co-domain (R) = {1, 4, 5} = B

Ex. 9: Let A = {1, 2, 3, 4, 5, 6} Define a relation R from A to A by R = {$(x, y) / y = x + 1$}.

Write down the domain, co-domain and range of R.

Solution : A relation R from A to A is given by

R = $(x, y) / y = x + 1$} is given by

R = {(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)

Domian (R) = {1, 2, 3, 4, 5}

Range (R) = {2, 3, 4, 5, 6}

Co-domain (R) = {1, 2, 3, 4, 5, 6}

Note:

1) R represents set of real numbers on number line.
2) R × R represents set of all ordered pairs on cartesian plane.
3) R × R × R represents set of all ordered triplets in three dimensional space.

EXERCISE 5.2

1) If $(x-1, y+4) = (1, 2)$ find the values of x and y.

2) If $(x + \frac{1}{3}, \frac{y}{3} - 1) = (\frac{1}{2}, \frac{3}{2})$, find x and y,

3) If A = {a, b, c}, B = (x, y) find A×B, B×A, A×A, B×B.

4) If P = {1, 2, 3) and Q = {14}, find sets P × Q and Q × P

5) Let A = {1, 2, 3, 4), B = {4, 5, 6}, C = {5, 6}.

Verify, i) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

ii) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

6) Express $\{(x, y) / x^2 + y^2 = 100$, where $x, y \in W\}$ as a set of ordered pairs.

7) Let A = {6, 8} and B = {1, 3, 5}

Show that R_1 = {(a, b)/a∈ A, b∈B, a–b is an even number} is a null relation. R_2 = {(a, b)/a∈ A, b∈B, a+b is odd number} is an universal relation.

8) Write the relation in the Roster form. State its domain and range.

i) $R_1 = \{(a, a^2)$ / a is prime number less than 15}

ii) $R_2 = \{(a, \frac{1}{a}) / 0<a \le 5, a \in N\}$

iii) $R_3 = \{(x, y) / y = 3x, y \in \{3, 6, 9, 12\}, x \in \{1, 2, 3\}$

iv) $R_4 = \{(x, y) / y > x+1, x = 1, 2$ and $y = 2, 4, 6\}$

v) $R_5 = \{(x, y) / x+y = 3, x, y \in \{0, 1, 2, 3\}$

vi) $R_6 = \{(a, b) / a \in N, a < 6$ and $b = 4\}$

vii) $R_7 = \{(a, b) / a, b \in N, a + b = 6\}$

viii) $R_8 = \{(a, b) / b = a + 2, a \in z, 0 < a < 5\}$

9) Identify which of if the following relations are reflexive, symmetric, and transitive. [Activity]

Relation	Reflexive	Symmetrice	Transitive
R = {$(a,b) : a,b \in Z, a-b$ is an integer}			
R = {$(a,b) : a,b \in N, a+b$ is even}	✓	✓	×
R = {$(a,b) : a,b \in N, a$ divides b}			
R = {$(a,b) : a,b \in N, a^2 - 4ab + 3b^2 = 0$}			
R = {(a,b) : a is sister of b and $a,b \in G$ = Set of girls}			
R = {(a,b) : Line a is perpendicular to line b in a place}			
R = {$(a,b) : a,b \in R, a < b$}			
R = {$(a,b) : a,b \in R, a \le b^3$}			

Let's Remember

- $A \cup B = \{x \mid x \in A \text{ OR } x \in B\}$
- $A \cap B = \{x \mid x \in A \text{ AND } x \in B\}$
- $A' = \{x \mid x \in U, x \notin A\}$ where U is universal set.
- $A - B = \{x \mid x \in A, x \notin B\}$
- $A \Delta B = (A - B) \cup (B - A)$
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
- $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$
- $A \times B = \{(x, y) \mid x \in A, y \in B\}$
- Relation R from set A to set B is a subset of $A \times B$.
- Domain of R = $\{x \mid (x, y) \in R, x \in A, y \in B\} \subseteq A$.
- Range of R = $\{y \mid (x, y) \in R, x \in A, y \in B\} \subseteq B$ (Co-domain of R).
- R is a relation on set A and if $(x, x) \in R$ for all $x \in A$ then R is reflexive.
- R is a relation on set A and if $(x, y) \in R$ then $(y, x) \in R$ for all $x, y \in A$ then R is symmetric.
- R is a relation on set A and if $(x, y) \in R$ and $(y, z) \in R$ then $(x, z) \in R$ for all $x, y, z \in A$ then R is transitive.
- If R is reflexive, symmetric and transitive then R is an equivalence relation.

MISCELLANEOUS EXERCISE 5

(I) Select the correct answer from given alternative (Q:1 to Q:10).

1) For the set A = {a, b, c, d, e} the correct statement is

A) $\{a, b\} \in A$ B) $\{a\} \in A$

C) $a \in A$ D) $a \notin A$

2) If $aN = \{ax : x \in N\}$, then set $6N \cap 8N =$

A) 8N B) 48N C) 12N D) 24N

3) If set A is empty set then n[P [P [P (A)]]] is

A) 6 B) 16 C) 2 D) 4

4) In a city 20% of the population travels by car, 50% travels by bus and 10% travels by both car and bus. Then, persons travelling by car or bus are

A) 80% B) 40% C) 60% D) 70%

5) If the two sets A and B are having 43 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ is

A) 43^2 B) 2^{43} C) 43^{43} D) 2^{86}

6) Let R be a relation on the set N be defined by $\{(x, y) / x, y \in N, 2x + y = 41\}$ Then R is

A) Reflexive B) Symmetric

C) Transitive D) None of these

7) The relation ">" in the set of N (Natural number) is

A) Symmetric B) Reflexive

C) Transitive D) Equivalent relation

8) A relation between A and B is

A) only $A \times B$

B) An Universal set of $A \times B$

C) An equivalent set of $A \times B$

D) A subset of $A \times B$

9) If $(x,y) \in R \times R$, then $xy = x^2$ is a relation which is

A) Symmetric B) Reflexive

C) Transitive D) Equivalence

10) If A = {a, b, c} The total no. of distinct relations in $A \times A$ is

A) 3 B) 9 C) 8 D) 2^9

(II) Answer the following.

1) Write down the following sets in set builder form

i) {10, 20, 30, 40, 50},

ii) {a, e, i, o, u)

ii) {Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday}

2) If $U = \{x/x \in N, 1 \le x \le 12\}$

$A = \{1, 4, 7, 10\}$ $B = \{2, 4, 6, 7, 11\}$

$C = \{3, 5, 8\ 9, 12\}$

Write down the sets

i) $A \cup B$ ii) $B \cap C$ iii) A-B

iv) $B \cap C'$ v) $A \cup B \cup C$ vi) $A \cap (B \cup C)$

3) In a survey of 425 students in a school, it was found that 115 drink apple juice, 160 drink orange juice and 80 drink both apple as well as orange juice. How many drink neither apple juice nor orange juice?

4) In a school there are 20 teachers who teach Mathematics or Physics. Of these, 12 teach Mathematics and 4 teach both Physics and Mathematics. How many teachers teach Physics?

5) i) If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$, state the elements of $A \times A$, $A \times B$, $B \times A$, $B \times B$, $(A \times B) \cap (B \times A)$

ii) If $A = \{-1, 1\}$, find $A \times A \times A$

6) If $A = \{1, 2, 3\}$, $B = \{4, 5, 6\}$ check if the following are relations from A to B. Also write its domain and range.

i) $R_1 = \{(1, 4), (1, 5), (1, 6)\}$

ii) $R_2 = \{(1, 5), (2, 4), (3, 6)\}$

iii) $R_3 = \{(1, 4), (1, 5), (3, 6), (2, 6), (3, 4)\}$

ii) $R_4 = \{(4, 2), (2, 6), (5, 1), (2, 4)\}$

7) Determine the domain and range of the following relations.

i) $R = \{(a, b) / a \in N, a < 5, b = 4\}$

ii) $R = \{(a, b) / b = |a-1| a \in z, |a| < 3\}$

8) Find $R : A \rightarrow A$ when $A = \{1,2,3,4\}$ such that

i) $R = (a, b) / a-b = 10\}$

ii) $R = \{(a, b) / |a - b|) \ge 0\}$

9) $R = \{1,2,3\} \rightarrow \{1,2,3\}$ given by $R = \{(1,1)$ $(2,2), (3,3), (1,2)\ (2,3)\}$ Check if R is

a) relflexive b) symmentric

c) transitive

10) Check if $R : Z \rightarrow Z$, $R = \{(a, b) | 2$ divides $a\text{-}b\}$ is equivalence relation.

11) Show that the relation R in the set $A = \{1, 2, 3, 4, 5\}$ Given by $R = \{(a, b) / |a-b|$ is even$\}$ is an equivalence relation

12) Show that following are equivalence relation

a) R in A is set of all books. given by $R = \{ (x, y) / x$ and y have same number of pages$\}$

b) R in $A = \{x \in Z | 0 \le x \le 12\}$ given by $R = \{(a, b) / |a-b|$ is a multiple of 4$\}$

c) R in $A = \{x \in N/x \le 10\}$ given by $R = \{(a, b) | a=b\}$

6 FUNCTIONS

Let's Study

- Function, Domain, Co-domain, Range
- Types of functions
- Representation of function
- Basic types of functions
- Piece-wise defined and special functions

Let's : Learn

6.1 Function

Definition : A function (or mapping) f from a set A to set B (f : A→B) is a relation which associates for each element x in A, a unique (exactly one) element y in B.

Then the element y is expressed as $y = f(x)$.

y is the image of x under f.

f is also called a map or transformation.

If such a function exists, then A is called the **domain** of f and B is called the **co-domain** of f.

Illustration:

Examine the following relations which are given by arrows of line segments joining elements in A and elements in B.

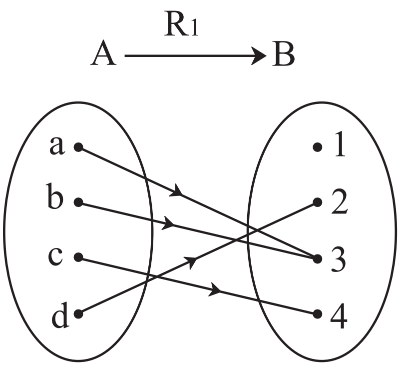


Fig. 6.1

Since, every element from A is associated to exactly one element in B, R_1 is a well defined function.

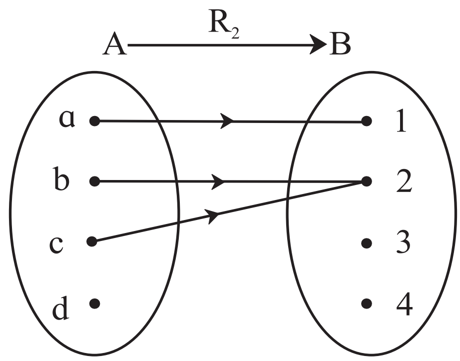


Fig. 6.2

R_2 is not a function because element 'd' in A is not associated to any element in B.

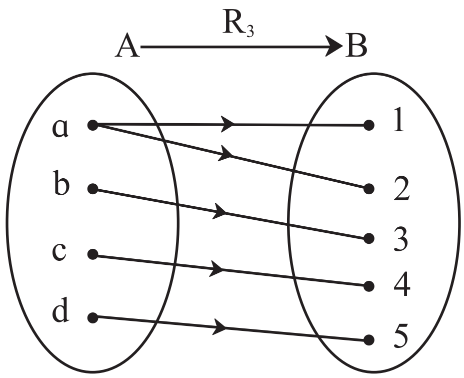


Fig. 6.3

R_3 is not a function because element a in A is associated to two elements in B.

The relation which defines a function f from domain A to co-domain B is often given by an algebraic rule.

For example, A = Z, the set of integers and B = Q the set of rational numbers and the function f is given by $f(n) = \frac{n}{7}$ here $n \in$ Z, f(n) $\in$ Q.

6.1.1 Types of function

One-one or **One to one** or **Injective function**

Definition : A function $f : A \to B$ is said to be one-one if different elements in A have different images in B. The condition is also expressed as

$f(a) = f(b) \Rightarrow a = b$ [As $a \neq b \Rightarrow f(a) \neq f(b)$]

Onto or **Surjective function**

Definition: A function $f : A \to B$ is said to be onto if every element y in B is an image of some x in A (or y in B has preimage x in A)

The image of A can be denoted by $f(A)$.

$f(A) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}$

$f(A)$ is also called the **range** of f.

Note that $f : A \to B$ is onto if $f(A) = f$.

Also range of $f = f(A) \subset$ co-domain of f.

Illustration:

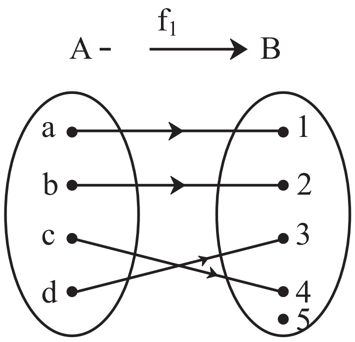


Fig. 6.4

f_1 is one-one, but not onto as element 5 is in B has no pre image in A

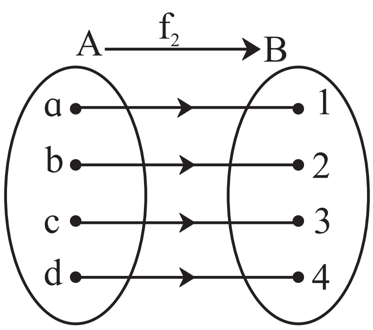


Fig. 6.5

f_2 is one-one, and onto

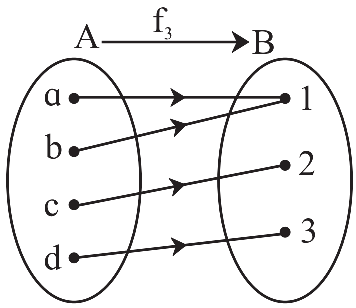


Fig. 6.6

f_3 is onto but not one-one as $f(a) = f(b) = 1$ but $a \neq b$.

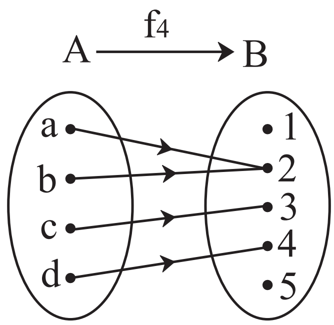
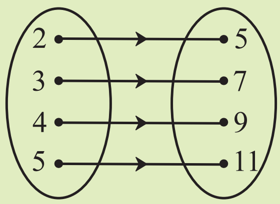


Fig. 6.7

f_4 is neither one-one, nor onto

6.1.2 Representation of Function

Verbal form	Output exceeds twice the input by 1 Domain : Set of inputs Range : Set of outputs
Arrow form on Venn Diagram	 **Fig. 6.8** Domain : Set of pre-images Range: Set of images
Ordered Pair (x, y)	$f = \{(2,5), (3,7), (4,9), (5,11)\}$ Domain : Set of 1st components from each ordered pair = $\{2, 3, 4, 5\}$ Range : Set of 2nd components from each ordered pair = $\{5, 7, 9, 11\}$

<table>
<tr><td>Rule / Formula</td><td>$y = f(x) = 2x + 1$
Where $x \in N$, $1 < x < 6$
$f(x)$ read as 'f of x' or 'function of x'
Domain : Set of values of x for which $f(x)$ is defined
Range : Set of values of y for which $f(x)$ is defined</td></tr>
<tr><td>Tabular Form</td><td>

x	y
2	5
3	7
4	9
5	11

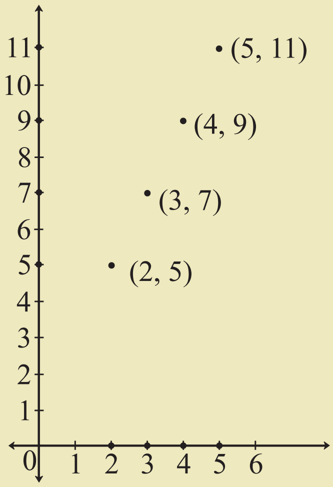
Domain : x values
Range: y values</td></tr>
<tr><td>Graphical form</td><td>

Fig. 6.9
Domain: Projection of graph on x-axis.
Range: Projection of graph on y-axis.</td></tr>
</table>

6.1.3 Graph of a function:

If the domain of function is in R, we can show the function by a graph in xy plane. The graph consists of points (x,y), where $y = f(x)$.

Vertical Line Test

Given a graph, let us find if the graph represents a function of x i.e. $f(x)$

A graph represents function of x, only if no vertical line intersects the curve in more than one point.

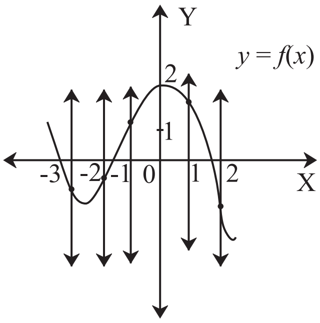


Fig. 6.10

Since every x has a unique associated value of y.

It is a function.

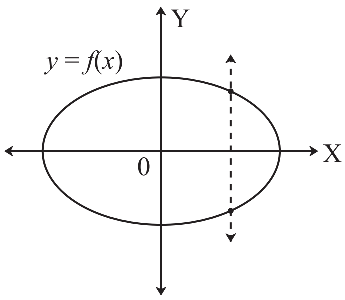


Fig. 6.11

This graph does not represent a function as vertical line intersects at more than one point some x has more than one values of y.

Horizontal Line Test:

If no horizontal line intersects the graph of a function in more than one point, then the function is one-one function.

Illustration:

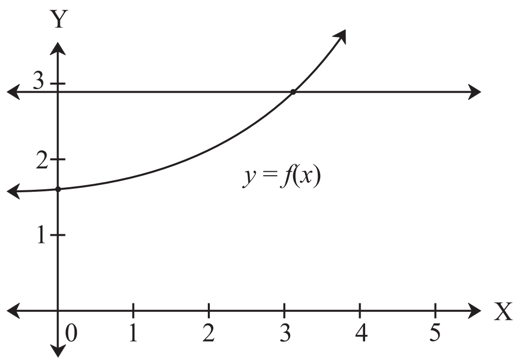


Fig. 6.12

The graph is a one-one function as a horizontal line intersects the graph at only one point.

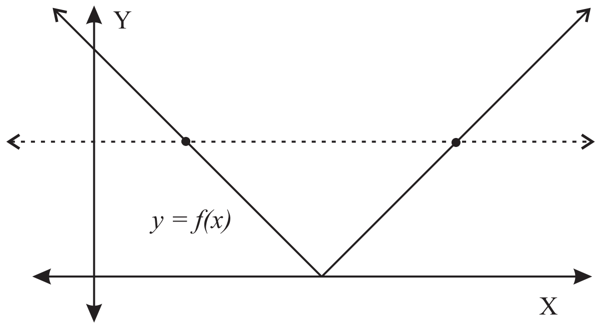


Fig. 6.13

The graph is a one-one function

6.1.4 Value of funcation : f(a) is called the value of funcation $f(x)$ at x = a

Evaluation of function:

Ex. 1) Evaluate $f(x) = 2x^2 - 3x + 4$ at

$x = 7$ & $x = -2t$

Solution : $f(x)$ at $x = 7$ is $f(7)$

$f(7) = 2(7)^2 - 3(7) + 4$

$= 2(49) - 21 + 4$

$= 98 - 21 + 4$

$= 81$

$f(-2t) = 2(-2t)^2 - 3(-2t) + 4$

$= 2(4t^2) + 6t + 4$

$= 8t^2 + 6t + 4$

Ex. 2) Using the graph of $y = g(x)$, find $g(-4)$ and $g(3)$

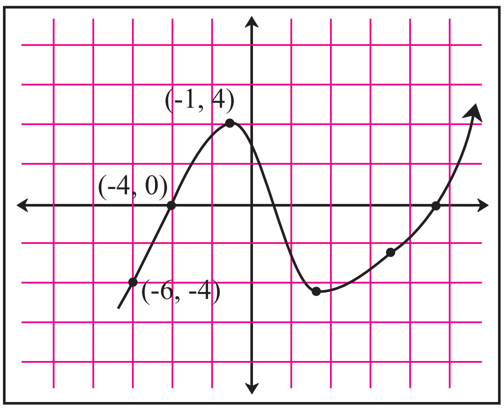


Fig. 6.14

Solution : From graph when $x = -4$, $y = 0$

so $g(-4) = 0$

From graph when $x = 3$, $y = -5$ so $g(3) = -5$

Function Solution:

Ex. 3) If $t(m) = 3m^2 - m$ and $t(m) = 4$, then find m

Solution : As

$t(m) = 4$

$3m^2 - m = 4$

$3m^2 - m - 4 = 0$

$3m^2 - 4m + 3m - 4 = 0$

$m(3m - 4) + 1(3m - 4) = 0$

$(3m - 4)(m + 1) = 0$

Therefore , $m = \frac{4}{3}$ or $m = -1$

Ex. 4) From the graph below find x for which $f(x) = 4$

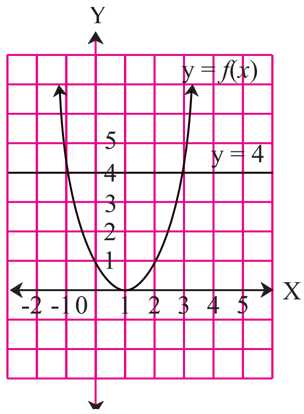


Fig. 6.15

Solution : To solve $f(x) = 4$ i.e. $y = 4$

Find the values of x where graph intersects line $y = 4$

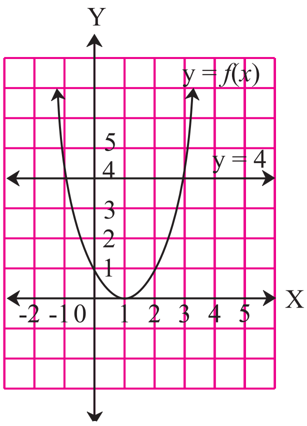


Fig. 6.16

Therefore , $x = -1$ and $x = 3$.

Function from equation:

Ex. 5) (Activity) From the equation $4x + 7y = 1$ express

i) y as a function of x

ii) x as a function of y

Solution : Given equation is $4x + 7y = 1$

i) From the given equation

$7y = \boxed{}$

$y = \boxed{}$ = function of x

So $y = f(x) = \boxed{}$

ii) From the given equation

$4x = \boxed{}$

$x = \boxed{}$ = function of y

So $x = g(y) = \boxed{}$

6.1.5 Some Basic Functions

(Here f : R $\rightarrow$ R Unless stated otherwise)

1) Constant Function

Form : $f(x) = k$, $k \in R$

Example : Graph of $f(x) = 2$

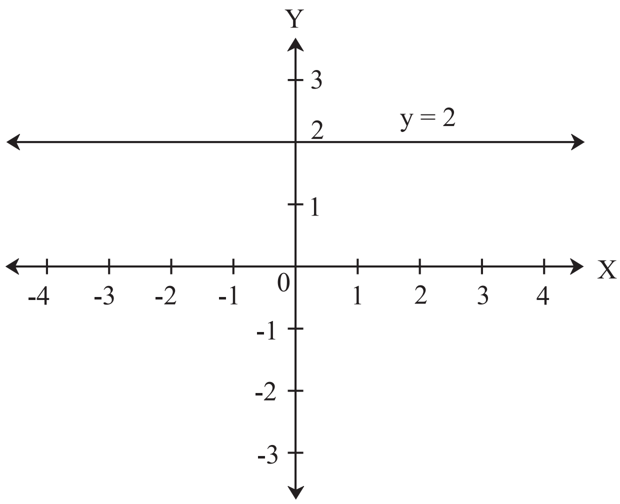


Fig. 6.17

Domain : R or $(-\infty, \infty)$ and **Range :** $\{2\}$

2) Identity function

If f : R $\rightarrow$ R then identity function is defined by $f(x) = x$, for every $x \in R$.

Identity function is given in the graph below.

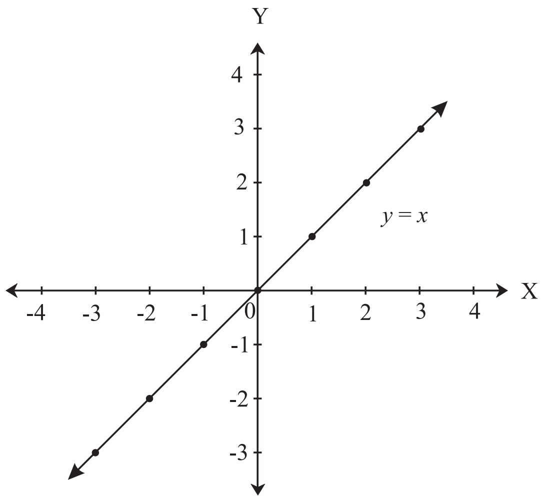


Fig. 6.18

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

[**Note :** Identity function is also given by I $(x) = x$].

3) Power Functions : $f(x) = ax^n$, $n \in N$

(Note that this function is a multiple of n^{th} power of x)

i) Square Function

Example : $f(x) = x^2$

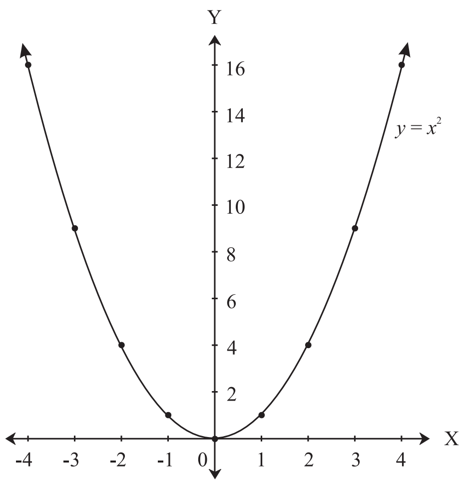


Fig. 6.19

Domain : R or $(-\infty, \infty)$ and **Range :** $[0, \infty)$

Properties:

1) Graph of $f(x) = x^2$ is a parabola opening upwards and with vertex at origin.
2) Graph is symmetric about y - axis .
3) The graph of even powers of x looks similar to square function. (verify !) e.g. x^4, x^6.
4) $(y - k) = (x - h)^2$ represents parabola with vertex at (h, k)
5) If $-2 \le x \le 2$ then $0 \le x^2 \le 4$ (see fig.) and if $-3 \le x \le 2$ then $0 \le x^2 \le 9$ (see fig).

ii) Cube Function

Example : $f(x) = x^3$

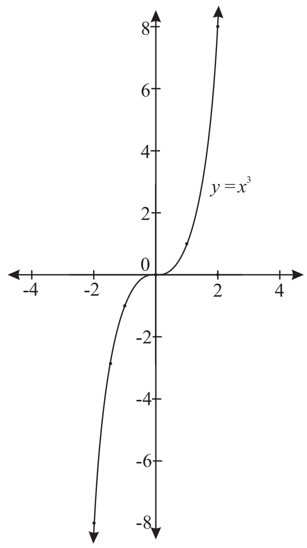


Fig. 6.20

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

Properties:

1) The graph of odd powers of x (more than 1) looks similar to cube function. e.g. x^5, x^7.

4) Polynomial Function

$$f(x) = a_0 x^n + a_1 x^{n-1} + ... + a_{n-1} x + a_n$$

is polynomial function of degree n , if $a_0 \neq 0$, and a_i s are real.

i) Linear Function

Form : $f(x) = ax + b \ (a \neq 0)$

Example : $f(x) = -2x + 3, \ x \in R$

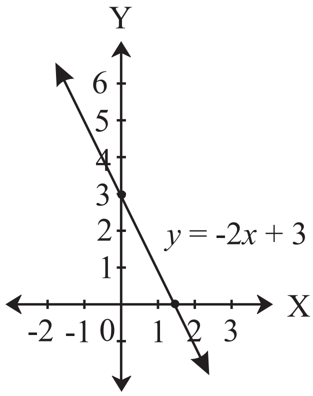


Fig. 6.21

Domain : R or $(-\infty, \infty)$ and **Range :** R or $(-\infty, \infty)$

Properties :

1) Graph of $f(x) = ax + b$ is a line with slope 'a', y-intercept 'b' and x-intercept $\left(-\dfrac{b}{a}\right)$.
2) Function : is increasing when slope is positive and deceasing when slope is negative.

ii) Quadratic Function

Form : $f(x) = ax^2 + bx + c \ (a \neq 0)$

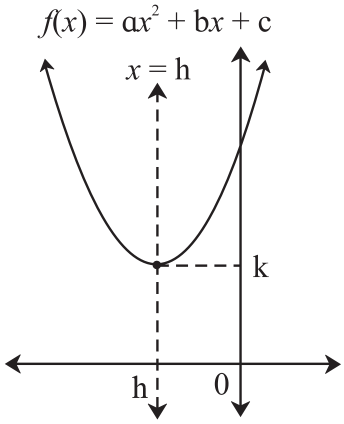


Fig. 6.22

Domain : R or $(-\infty, \infty)$ and **Range :** $[k, \infty)$

Properties :

1) Graph of $f(x) = ax^2 + bx + c$ and where $a > 0$ is a parabola.

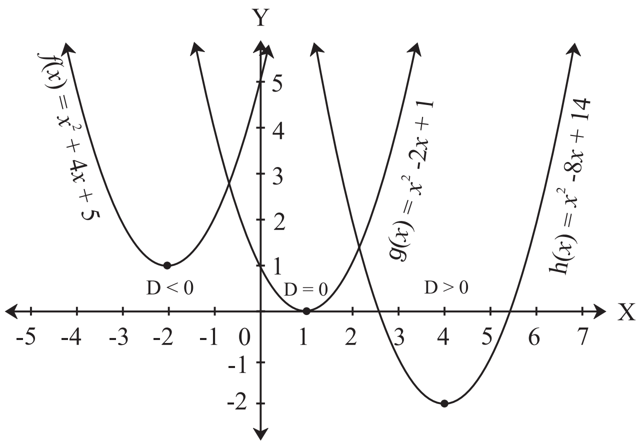


Fig. 6.23

Consider, $y = ax^2 + bx + c$

$$= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$$\left(y + \frac{b^2 - 4ac}{4a}\right) = a\left(x + \frac{b}{2a}\right)^2$$

With change of variable

$$X = x + \frac{b}{2a}, \quad Y = y + \frac{b^2 - 4ac}{4a}$$

this is a parabola $Y = aX^2$

This is a parabola with vertex

$\left(-\frac{b}{2a}, \frac{b^2-4ac}{4a}\right)$ or $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$ where

$D = b^2 - 4ac$ and the parabola is opening upwards.

There are three possibilities.

For a>0,

i) If $D = b^2 - 4ac = 0$, the parabola touches x-axis and $y \geq 0$ for all x. e.g. $g(x) = x^2 - 2x + 1$

ii) If $D = b^2 - 4ac > 0$, then parabola intersects x-axis at 2 distinct points. Here y is negative for values of x between the 2 roots and positive for large or small x.

iii) If $D = b^2 - 4ac < 0$, the parabola lies above x-axis and $y \neq 0$ for any x. Here y is positive for all values of x. e.g. $f(x) = x^2 + 4x + 5$

iii) Cubic Function

Example : $f(x) = ax^3 + bx^2 + cx + d \ (a \neq 0)$

Domain : R or $(-\infty, \infty)$ and

Range : R or $(-\infty, \infty)$

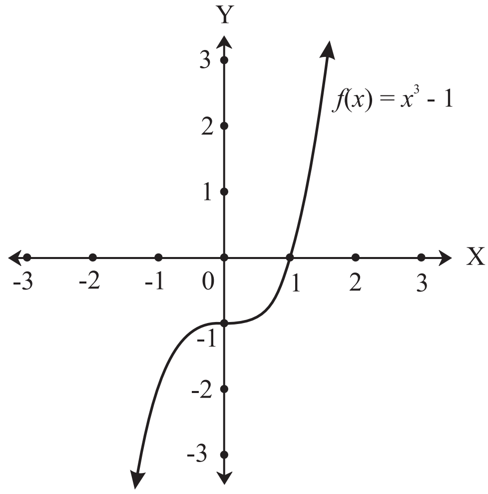


Fig. 6.24

Property:

1) Graph of $f(x) = x^3 - 1$

$f(x) = (x - 1)(x^2 + x + 1)$ cuts x-axis at only one point (1,0) , which means $f(x)$ has one real root & two complex roots.

Note that, any polynomial of odd degree must have at least one real root, because the complex roots appear in conjugate pairs.

5) Radical Function

Ex: $f(x) = \sqrt[n]{x}$, $n \in N$

1. Square root function

$f(x) = \sqrt{x}$, $x \geq 0$

(Since square root of negative number is not a real number, so the domain of $\sqrt{x}$ is restricted to positive values of x).

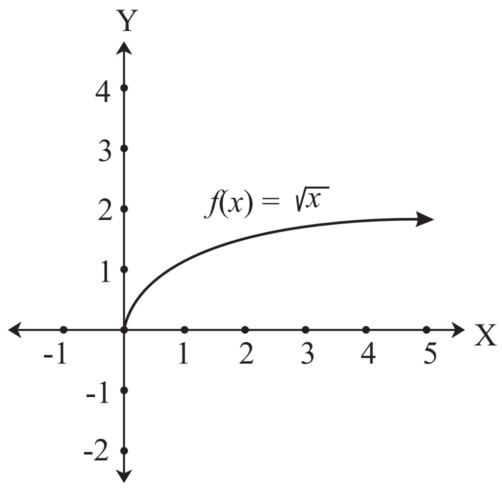


Fig. 6.25

Domain : $[0, \infty)$ and **Range :** $[0, \infty)$

Note :

1) If x is positive, there are two square roots of x. By convention $\sqrt{x}$ is positive root and $-\sqrt{x}$ is the negattive root.

2) If $-4 < x < 9$, as $\sqrt{x}$ is only deifned for $x \geq 0$, so $0 \leq \sqrt{x} < 3$.

Ex. 6 : Find the domain and range of $f(x) = \sqrt{9-x^2}$.

Soln. : $f(x) = \sqrt{9-x^2}$ is defined for

$9 - x^2 \geq 0$, i.e. $x^2 - 9 \leq 0$ i.e. $(x-3)(x+3) \leq 0$

Therefore $[-3, 3]$ is domain of $f(x)$. (Verify !)

To find range, let $\sqrt{9-x^2} = y$

Since square root is always positive, so $y \geq 0$ ...(I)

Also, on squaring we get $9 - x^2 = y^2$

Since, $3 \leq x \leq 3$

i.e. $0 \leq x^2 \leq 9$

i.e. $0 \geq -x^2 \geq -9$

i.e. $9 \geq 9 - x^2 \geq 9 - 9$

i.e. $9 \geq 9 - x^2 \geq 0$

i.e. $3 \geq \sqrt{9-x^2} \geq 0$

$\therefore\ 3 \geq y \geq 0$ ...(II)

From (I) and (II), $y \in [0,3]$ is range of $f(x)$.

2. Cube root function

$f(x) = \sqrt[3]{x}$,

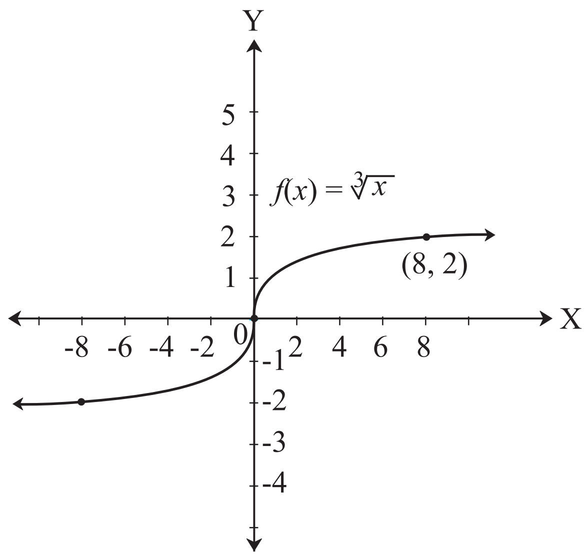


Fig. 6.26

Domain : R and **Range :** R

Note : If $-8 \leq x \leq 1$ then $-2 \leq \sqrt[3]{x} \leq 1$.

Ex. 7 : Find the domain $f(x) = \sqrt{x^3 - 8}$.

Soln. : $f(x)$ is defined for $x^3 - 8 \geq 0$

i.e. $x^3 - 2^3 \geq 0$, $(x-2)(x^2 + 2x + 4) \geq 0$

In $x^2 + 2x + 4$, $a = 1 > 0$ and $D = b^2 - 4ac = 2^2 - 4 \times 1 \times 4 = -12 < 0$

Therefore, $x^2 + 2x + 4$ is a positive quadratic.

i. e. $x^2 + 2x + 4 > 0$ for all x

Therefore $x - 2 \geq 0$, $x \geq 2$ is the domain.

i.e. Domain is $x \in [2, \infty)$

6) Rational Function

Definition: Given polynomials

$p(x)$, $q(x)$ $f(x) = \dfrac{p(x)}{q(x)}$ is defined for x if $q(x) \neq 0$.

Example : $f(x) = \dfrac{1}{x}$, $x \neq 0$

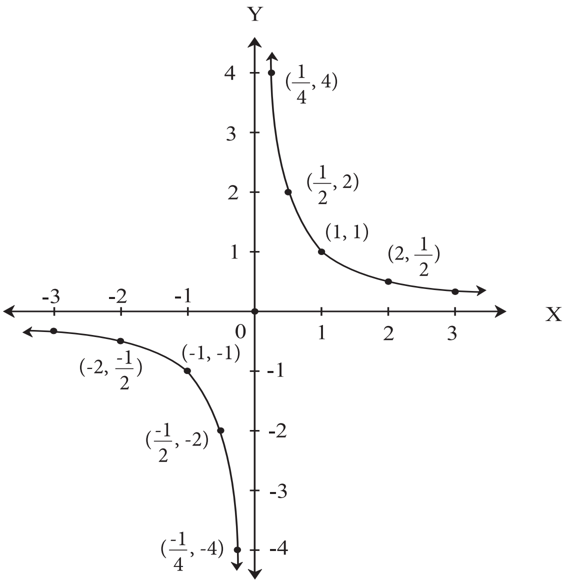


Fig. 6.27

Domain : R-{0} and **Range :** $R-\{0\}$

Properties:

1) As $x \to 0$ i.e. (As x approaches 0) $f(x) \to \infty$ or $f(x) \to -\infty$, so the line $x = 0$ i.e y-axis is called vertical asymptote.(A straight line which does not intersect the curve but as x approaches to ∞ or $-\infty$ the distance between the line and the curve tends to 0, is called an asymptote of the curve.)

2) As As $x \to \infty$ or $x \to -\infty$, $f(x) \to 0$, $y = 0$ the line i.e y-axis is called horizontal asymptote.

3) The domain of rational function $f(x) = \dfrac{p(x)}{q(x)}$ is all the real values of except the zeroes of $q(x)$.

Ex. 8 : Find domain and range of the function $f(x) = \dfrac{6-4x^2}{4x+5}$

Solution : $f(x)$ is defined for all $x \in R$ exccept when denominators is 0.

Since, $4x + 5 = 0 \Rightarrow x = -\dfrac{5}{4}$.

So Domain of $f(x)$ is $R - \left\{-\dfrac{5}{4}\right\}$.

To find the range, let $y = \dfrac{6-4x^2}{4x+5}$

i.e. $y(4x + 5) = 6 - 4x^2$

i.e. $4x^2 + (4y)x + 5y - 6 = 0$.

This is a quadratic equation in x with y as constant.

Since $x \in R - \{-5/4\}$, i.e. x is real, we get

Solution if, $D = b^2 - 4ac \geq 0$

i.e. $(4y)^2 - 4(4)(5y - 6) \geq 0$

$16y^2 - 16(5y - 6) \geq 0$

$y^2 - 5y + 6 \geq 0$

$(y - 2)(y - 3) \geq 0$

Therefore $y \leq 2$ or $y \geq 3$ (Verify!)

Range of $f(x)$ is $(-\infty, 2] \cup [3,\infty)$

7) Exponential Function

Form : $f(x) = a^x$ is an exponential function with base a and exponent (or index) x, $a \neq 0$, $a > 0$ and $x \in R$.

Example : $f(x) = 2^x$ and $f(x) = 2^{-x}$

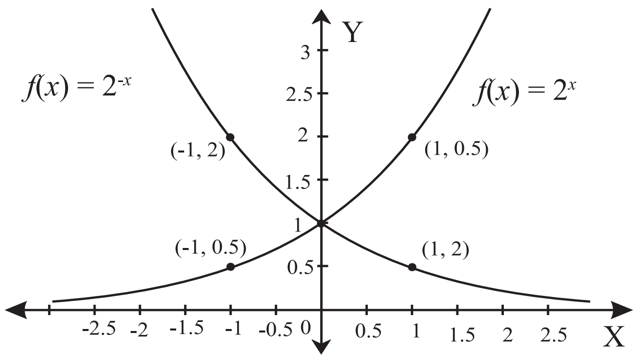


Fig. 6.28

Domain: R and **Range :** $(0, \infty)$

Properties:

1) As $x \to -\infty$, then $f(x) = 2^x \to 0$, so the graph has horizontal asymptote $(y = 0)$

2) By taking the natural base e ($\approx$ 2.718), graph of $f(x) = e^x$ is similar to that of 2^x in appearance

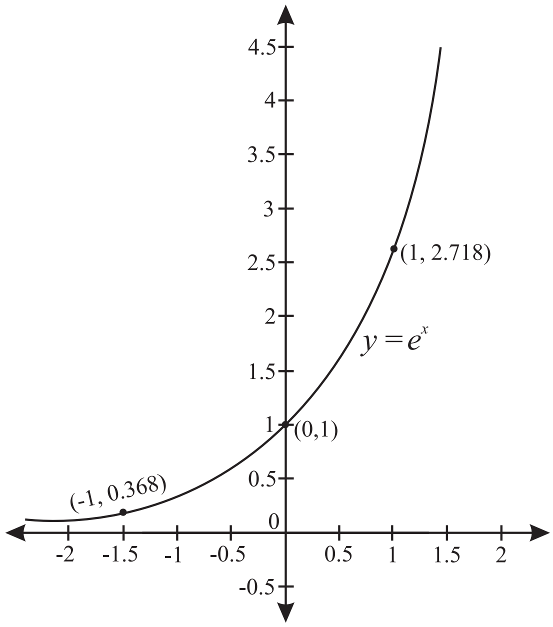


Fig. 6.29

3) For $a > 0$, $a \neq 1$, if $a^x = a^y$ then $x = y$. So a^x is one-one function. (check graph for horizontal line test).

4) $r > 1$, $m > n \Rightarrow r^m > r^n$ and
$r < 1$, $m > n \Rightarrow r^m < r^n$

Ex. 9 : Solve $5^{2x+7} = 125$.

Solution : As $5^{2x+7} = 125$

i.e : $5^{2x+7} = 5^3$, $\therefore$ $2x + 7 = 3$

and $x = \dfrac{3-7}{2} = \dfrac{-4}{2} = -2$

Ex. 10 : Find the domain of $f(x) = \sqrt{6-2^x-2^{3-x}}$

Solution : Since $\sqrt{x}$ is defined for $x \geq 0$

$f(x)$ is defined for $6 - 2^x - 2^{3-x} \geq 0$

i.e. $6 - 2^x - \dfrac{2^3}{2^x} \geq 0$

i.e. $6.2^x - (2^x)^2 - 8 \geq 0$

i.e. $(2^x)^2 - 6.2^x + 8 \leq 0$

i.e. $(2^x - 4)(2^x - 2) \leq 0$

$2^x \geq 2$ and $2^x \leq 4$ (Verify !)

$2^x \geq 2^1$ and $2^x \leq 2^2$

$x \geq 1$ and $x \leq 2$ or $1 \leq x \leq 2$

Doamin is [1,2]

8) Logarithmic Function:

Let, $a > 0$, $a \neq 1$, Then logarithmic function $\log_a x$, $y = \log_a x$ if $x = a^y$.

for $x > 0$, is defined as

$$\underset{\text{log arithmic form}}{y = \log_a x} \Leftrightarrow \underset{\text{exponential form}}{a^y = x}$$

Properties:

1) As $a^0 = 1$, so $\log_a 1 = 0$ and as $a^1 = a$,

so $\log_a a = 1$

2) As $a^x = a^y \Leftrightarrow x = y$ so $\log_a x = \log_a y \Leftrightarrow x = y$

3) Product rule of logarithms.

For a, b, $c > 0$ and $a \neq 1$,

$\log_a bc = \log_a b + \log_a c$ (Verify !)

4) Quotient rule of logarithms.

For a, b, $c > 0$ and $a \neq 1$,

$\log_a \dfrac{b}{c} = \log_a b - \log_a c$ (Verify !)

5) Power/Exponent rule of logarithms.

For a, b, $c > 0$ and $a \neq 1$,

$\log_a b^c = c \log_a b$ (Verify !)

6) For natural base e, $\log_e x = ln\, x$ as Natural Logarithm Function.

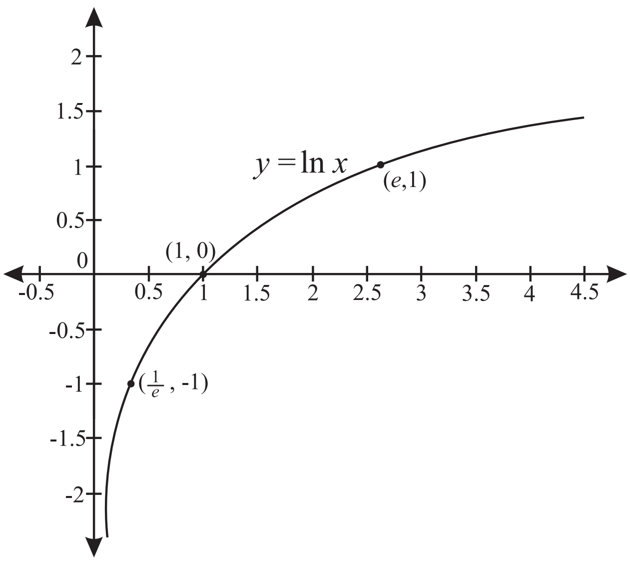


Fig. 6.30

Here domain of $ln\, x$ is $(0, \infty)$ and range is $(-\infty, \infty)$.

8) Logarithmic inequalities:

(i) If $a>1$, $0<m<n$ then $\log_a m < \log_a n$

e.g. $\log_{10} 20 < \log_{10} 30$

(ii) If $0>a<1$, $0<m<n$ then $\log_a m > \log_a n$

e.g. $\log_{0.1} 20 > \log_{0.1} 30$

(iii) For a, $m>0$ if a and m lies on the same side of unity (i.e. 1) then $\log_a m>0$.

e.g. $\log_2 3>0$, $\log_{0.3} 0.5>0$

(iv) For a, $m>0$ if a and m lies on the different sides of unity (i.e. 1) then $\log_a m>0$.

e.g. $\log_{0.2} 3<0$, $\log_3 0.5>0$

Ex. 11 : Write log72 in terms of log2 and log3.

Solution : $\log 72 = \log(2^3.3^2)$

$= \log 2^3 + \log 3^2$ ($\because$ Power rule)

$= 3 \log 2 + 2 \log 3$($\because$ Power rule)

Ex. 12 : Evaluate $\ln e^9 - \ln e^4$.

Solution : $\ln e^9 - \ln e^4 = \log_e e^9 - \log_e e^4$

$= 9 \log_e e - 4 \log_e e$

$= 9(1) - 4(1)$ $(\therefore \ln e = 1)$

$= 5$

Ex. 13 : Expand $\log \left[\dfrac{x^3(x+3)}{2(x-4)^2} \right]$

Solution : Using Quotient rule

$= \log [x^3(x+3)] - \log [2\,(x-4)^2\,]$

Using Product rule

$= [\log x^3 + \log (x+3)] - [\log 2 + \log (x-4)^2]$

Using Power rule

$= [3\log x + \log (x+3)] - [\log 2 - 2\log (x-4)]$

$= 3\log x + \log (x+3) - \log 2 + 2\log (x-4)$

Ex. 14 : Combine

$3\ln\,(p + 1) - \dfrac{1}{2}\ln r + 5\ln(2q + 3)$ into single logarithm.

Solution : Using Power rule,

$= \ln\,(p + 1)^3 - \ln r^{\frac{1}{2}} + \ln(2q + 3)^5$

Using Quotient rule

$= \ln \dfrac{(p+1)^3}{\sqrt{r}} + \ln(2q + 3)^5$

Using Product rule

$= \ln \left[\dfrac{(p+1)^3}{\sqrt{r}} (2q+3)^5 \right]$

Ex. 15 : Find the domain of $ln(x - 5)$.

Solution : As $ln(x - 5)$ is defined for $(x - 5) > 0$

that is $x > 5$ so domain is $(5, \infty)$.

Let's note:

1) $\log(x+y) \neq \log x + \log y$

2) $\log x \log y \neq \log(xy)$

3) $\dfrac{\log x}{\log y} \neq \log\left(\dfrac{x}{y}\right)$

4) $(\log x)^n \neq n \log^n$

9) Change of base formula:

For $a, x, b > 0$ and $a, b \neq 1$, $\log_a x = \dfrac{\log_b x}{\log_b a}$

Note: $\log_a x = \dfrac{\log_x x}{\log_x a} = \dfrac{1}{\log_x a}$ (Verify !)

Ex. 16 : Evaluate $\dfrac{\log_4 81}{\log_4 9}$

Solution : By Change of base law, as the base is same (that is 4)

$$\frac{\log_4 81}{\log_4 9} = \log_9 81 = 2$$

Ex. 17 : Prove that, $2\log_b a^4 . \log_c b^3 . \log_a c^5 = 120$

Solution : L.H.S. $= 2\log_b a^4 . \log_c b^3 . \log_a c^5$

$= 4 \times 2\log_b a \times 3\log_c b \times 5\log_a c$

Using change of base law,

$= 4 \times 2 \dfrac{\log a}{\log b} \times 3 \dfrac{\log b}{\log c} \times 5 \dfrac{\log c}{\log a}$

$= 120$

Ex. 18 : Find the domain of $f(x) = \log_{x+5}(x^2 - 4)$

Solution : SInce $\log_a x$ is definedd for a, $x > 0$ and $a \neq 1$ $f(x)$ is defined for $(x^2 - 4) > 0$, $x + 5 > 0$, $x + 5 \neq 1$.

i.e. $(x-2)(x+2) > 0$, $x > -5$, $x \neq -4$

i.e. $x < -2$ or $x > 2$ and $x > -5$ and $x \neq -4$

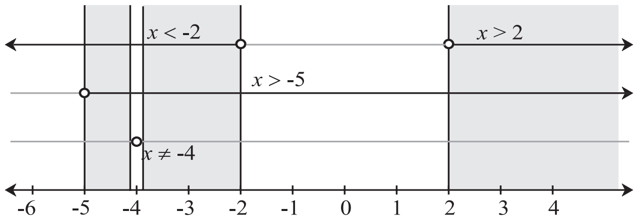


Fig. 6.31

9) Trigonometric function

The graphs of trigonometric functions are discusse in chapter 2 of Mathematics Book I.

$f(x)$	Domain	Range
$\sin x$	R	$[-1,1]$
$\cos x$	R	$[-1,1]$
$\tan x$	$R - \left\{\pm\frac{\pi}{2}, \pm\frac{3\pi}{2} \ldots\right\}$	R

EXERCISE 6.1

1) Check if the following relations are functions.

(a)

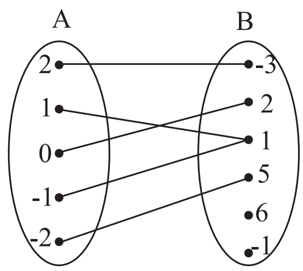


Fig. 6.32

(b)

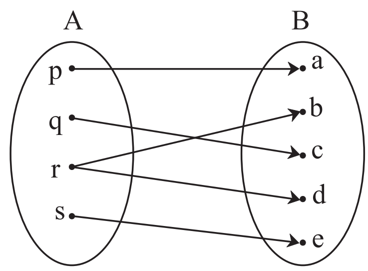


Fig. 6.33

(c)

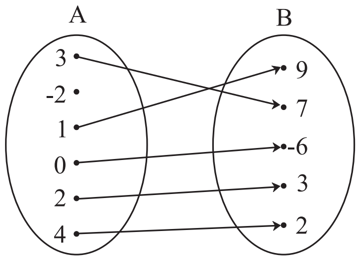


Fig. 6.34

2) Which sets of ordered pairs represent functions from A = {1, 2, 3, 4} to B = {−1, 0, 1, 2, 3}? Justify.

(a) {(1,0), (3,3), (2,−1), (4,1), (2,2)}

(b) {(1,2), (2,−1), (3,1), (4,3)}

(c) {(1,3), (4,1), (2,2)}

(d) {(1,1), (2,1), (3,1), (4,1)}

3) Check if the relation given by the equation represents y as function of x.

(a) $2x + 3y = 12$ (b) $x + y^2 = 9$

(c) $x^2 - y = 25$ (d) $2y + 10 = 0$

(e) $3x - 6 = 21$

4) If $f(m) = m^2 - 3m + 1$, find

(a) $f(0)$ (b) $f(-3)$

(c) $f\left(\frac{1}{2}\right)$ (d) $f(x + 1)$

(e) $f(-x)$

(f) $\left(\dfrac{f(2+h)-f(2)}{h}\right), h \neq 0$.

5) Find x, if $g(x) = 0$ where

(a) $g(x) = \dfrac{5x-6}{7}$ (b) $g(x) = \dfrac{18-2x^2}{7}$

(c) $g(x) = 6x^2 + x - 2$

(d) $g(x) = x^3 - 2x^2 - 5x + 6$

6) Find x, if $f(x) = g(x)$ where

(a) $f(x) = x^4 + 2x^2$, $g(x) = 11x^2$

(b) $f(x) = \sqrt{x} - 3$, $g(x) = 5 - x$

7) If $f(x) = \dfrac{a-x}{b-x}$, $f(2)$ is undefined, and $f(3) = 5$, find a and b.

8) Find the domain and range of the follwoing functions.

(a) $f(x) = 7x^2 + 4x - 1$

(b) $g(x) = \dfrac{x+4}{x-2}$

(c) $h(x) = \dfrac{\sqrt{x+5}}{5+x}$

(d) $f(x) = \sqrt[3]{x+1}$

(e) $f(x) = \sqrt{(x-2)(5-x)}$

(f) $f(x) = \sqrt{\dfrac{x-3}{7-x}}$

(g) $f(x) = \sqrt{16-x^2}$

9) Express the area A of a square as a function of its (a) side s (b) perimeter P.

10) Express the area A of circle as a function of its (a) radius r (b) diameter d (c) circumference C.

11) An open box is made from a square of cardboard of 30 cms side, by cutting squares of length x centimeters from each corner and folding the sides up. Express the volume of the box as a function of x. Also find its domain.

Let f be a subset of Z × Z defined by

12) $f = \{(ab, a+b) : a, b \in Z\}$. Is f a function from Z to Z? Justify.

14) Check the injectivity and surjectivity of the following functions.

(a) $f : N \rightarrow N$ given by $f(x) = x^2$

(b) $f : Z \rightarrow Z$ given by $f(x) = x^2$

(c) $f : R \rightarrow R$ given by $f(x) = x^2$

(d) $f: \mathrm{N} \rightarrow \mathrm{N}$ given by $f(x) = x^3$

(e) $f: \mathrm{R} \rightarrow \mathrm{R}$ given by $f(x) = x^3$

14) Show that if $f: \mathrm{A} \rightarrow \mathrm{B}$ and $g: \mathrm{B} \rightarrow \mathrm{C}$ are one-one, then $g \circ f$ is also one-one.

15) Show that if $f: \mathrm{A} \rightarrow \mathrm{B}$ and $g: \mathrm{B} \rightarrow \mathrm{C}$ are onto, then $g \circ f$ is also onto.

16) If $f(x) = 3(4^{x+1})$ find $f(-3)$.

17) Express the following exponential equations in logarithmic form

(a) $2^5 = 32$ (b) $54^0 = 1$

(c) $23^1 = 23$ (d) $9^{3/2} = 27$

(e) $3^{-4} = \frac{1}{81}$ (f) $10^{-2} = 0.01$

(g) $e^2 = 7.3890$ (h) $e^{1/2} = 1.6487$

(i) $e^{-x} = 6$

18) Express the following logarithmic equations in exponential form

(a) $\log_2 64 = 6$ (b) $\log_5 \frac{1}{25} = -2$

(c) $\log_{10} 0.001 = -3$ (d) $\log_{1/2}(-8) = 3$

(e) $\ln 1 = 0$ (f) $\ln e = 1$

(g) $\ln \frac{1}{2} = -0.693$

19) Find the domain of

(a) $f(x) = \ln(x-5)$

(b) $f(x) = \log_{10}(x^2 - 5x + 6)$

20) Write the following expressions as sum or difference of logarithms

(a) $\log\left(\frac{pq}{rs}\right)$ (b) $\log\left(\sqrt{x}\sqrt[3]{y}\right)$

(c) $\ln\left(\frac{a^3(a-2)^2}{\sqrt{b^2+5}}\right)$

(d) $\ln\left[\frac{\sqrt[3]{x-2}\,(2x+1)^4}{(x+4)\sqrt{2x+4}}\right]^2$

21) Write the following expressions as a single logarithm.

(a) $5\log x + 7\log y - \log z$

(b) $\frac{1}{3}\log(x-1) + \frac{1}{2}\log(x)$

(c) $\ln(x+2) + \ln(x-2) - 3\ln(x+5)$

22) Given that $\log 2 = a$ and $\log 3 = b$,

write $\log \sqrt{96}$ in terms of a and b.

23) Prove that

(a) $b^{\log_b a} = a$ (b) $\log_{b^m} a = \frac{1}{m}\log_b a$

(c) $a^{\log_c b} = b^{\log_c a}$

24) If $f(x) = ax^2 - bx + 6$ and $f(2) = 3$ and $f(4) = 30$, find a and b

25) Solve for x.

(a) $\log 2 + \log(x+3) - \log(3x-5) = \log 3$

(b) $2\log_{10} x = 1 + \log_{10}\left(x + \frac{11}{10}\right)$

(c) $\log_2 x + \log_4 x + \log_{16} x = \frac{21}{4}$

(d) $x + \log_{10}(1+2^x) = x\log_{10} 5 + \log_{10} 6$

26) If $\log\left(\frac{x+y}{3}\right) = \frac{1}{2}\log x + \frac{1}{2}\log y$,

show that $\frac{x}{y} + \frac{y}{x} = 7$.

27) If $\log\left(\frac{x-y}{4}\right) = \log\sqrt{x} + \log\sqrt{y}$,

show that $(x+y)^2 = 20xy$

28) If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$

then prove that $\frac{1}{1+x} + \frac{1}{1+y} + \frac{1}{1+z} = 1$

6.2 Algebra of functions:

Let f and g be functions with domains A and B. Then the functions $f+g, f-g, fg, \frac{f}{g}$ are defined on $A \cap B$ as follows.

Operations
$(f+g)(x) = f(x) + g(x)$
$(f-g)(x) = f(x) - g(x)$
$(f.g)(x) = f(x).g(x)$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ where $g(x) \neq 0$

Ex. 1 : If $f(x) = x^2 + 2$ and $g(x) = 5x - 8$, then find

i) $(f+g)(1)$

ii) $(f-g)(-2)$

ii) $(f \circ g)(3m)$

iv) $\frac{f}{g}(0)$

Solution : i) As $(f+g)(x) = f(x) + g(x)$

$(f+g)(1) = f(1) + g(1)$

$= [(1)^2 + 2] + [5(1) - 8]$

$= 3 + (-3)$

$= 0$

ii) As $(f-g)(x) = f(x) - g(x)$

$(f-g)(-2) = f(-2) - g(-2)$

$= [(-2)^2 + 2] - [5(-2) - 8]$

$= [4 + 2] - [-10-8]$

$= 6 + 18$

$= 24$

iii) As $(fg)(x) = f(x)\,g(x)$

$(f \circ g)(3m) = f(3m)g(3m)$

$= [(3m)^2 + 2][5(3m) - 8]$

$= [9m^2 + 2][15m-8]$

$= 135m^3 - 72m^2 + 30m - 16$

iv) As $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$

$\left(\frac{f}{g}\right)(0) = \frac{f(0)}{g(0)} = \frac{0^2+2}{5(0)-8}$

$= \frac{2}{-8} = -\frac{1}{4}$

Ex. 2 : Given the function $f(x) = 5x^2$ and $g(x) = \sqrt{4-x}$ find the domain of

i) $(f+g)(x)$ ii) $(f \circ g)(x)$ iii) $\frac{f}{g}(x)$

Solution : i) Domain of $f(x) = 5x^2$ is $(-\infty, \infty)$.

To find domain of $g(x) = \sqrt{4-x}$

$4 - x \geq 0$

$x - 4 \leq 0$

Let $x \leq 4$, So domain is $(-\infty, 4]$.

Therefore, domain of $(f+g)(x)$ is

$(-\infty, \infty) \cap (-\infty, 4]$, that is $(-\infty, 4]$

ii) Similarly, domain of $(f \circ g)(x) = 5x^2\sqrt{4-x}$ is $(-\infty, 4]$

iii) And domain of $\left(\frac{f}{g}\right)(x) = \frac{5x^2}{\sqrt{4-x}}$ is $(-\infty, 4)$

As , at $x = 4$ the denominator $g(x) = 0$.

6.2.1 Composition of Functions:

A method of combining the function f : A→B with g : B→C is composition of functions, defined as $(f \circ g)(x) = f[g(x)]$ an read as 'f composed with g'

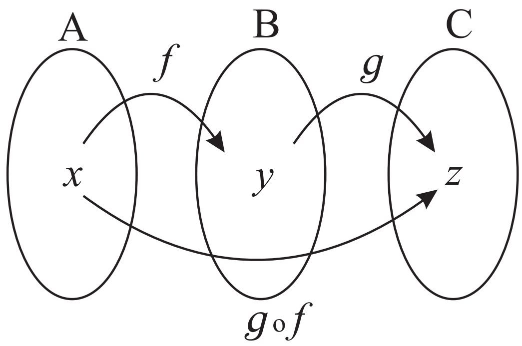


Fig. 6.35

Note:

1) The domain of $g \circ f$ is the set of all x in A such that $f(x)$ is in the B. The range of $g \circ f$ is set of all g $[f(x)]$ in C such that f(x) is in B.

2) Domain of $g \circ f \subseteq$ Domain of f and Range of $g \circ f \subseteq$ Range of g.

Illustration:

A cow produces 4 liters of milk in a day. Then x number of cows produce $4x$ liters of milk in a day. This is given by function $f(x) = 4x =$ 'y'. Price of one liter milk is Rs. 50. Then the price of y liters of the milk is Rs. $50y$. This is given by another function $g(y) = 50y$. Now a function $h(x)$ gives the money earned from x number of cows in a day as a composite function of f and g as $h(x) = (g \circ f)(x) = g[f(x)] = g(4x) = 50(4x) = 200x$.

Ex. 3 : If $f(x) = \dfrac{2}{x+5}$ and $g(x) = x^2 - 1$, then find

i) $(f \circ g)(x)$ ii) $(g \circ f)(3)$

Solution :

i) As $(f \circ g)(x) = f[g(x)]$ and $f(x) = \dfrac{2}{x+5}$

Replace x from $f(x)$ by $g(x)$, to get

$$(f \circ g)(x) = \frac{2}{g(x)+5}$$

$$= \frac{2}{x^2-1+5}$$

$$= \frac{2}{x^2+4}$$

ii) As $(g \circ f)(x) = g[f(x)]$ and $g(x) = x^2 - 1$

Replace x by $f(x)$, to get

$$(g \circ f)(x) = [f(x)]^2 - 1$$

$$= \left(\frac{2}{x+5}\right)^2 - 1$$

Now let $x = 3$

$$(g \circ f)(3) = \left(\frac{2}{3+5}\right)^2 - 1$$

$$= \left(\frac{2}{8}\right)^2 - 1$$

$$= \left(\frac{1}{4}\right)^2 - 1$$

$$= \frac{1-16}{16}$$

$$= -\frac{15}{16}$$

Ex 4 : If $f(x) = x^2$, $g(x) = x + 5$, and $h(x) = \dfrac{1}{x}$, $x \neq 0$, find $(g \circ f \circ h)(x)$

Solution : $(g \circ f \circ h)(x)$

$$= g\{f[h(x)]\}$$

$$= g\left[f\left(\frac{1}{x}\right)\right]$$

$$= g\left[f\left(\frac{1}{x}\right)^2\right]$$

$$= \left(\frac{1}{x}\right)^2 + 5$$

$$= \frac{1}{x^2} + 5$$

Ex. 5 : If $h(x) = (x - 5)^2$, find the functions f and g, such that $h = f \circ g$.

→ In $h(x)$, 5 is subtracted from x first and then squared. Let $g(x) = x - 5$ and $f(x) = x^2$, (verify)

Ex. 6 : Express $m(x) = \dfrac{1}{x^3+7}$ in the form of $f \circ g \circ h$

→ In $m(x)$, x is cubed first then 7 is added and then its reciprocal taken. So,

$h(x) = x^3$, $g(x) = x + 7$ and $f(x) = \dfrac{1}{x}$, (verify)

6.2.2 Inverse functions:

Let $f : A \rightarrow B$ be one-one and onto function and $f(x) = y$ for $x \in A$. The inverse function

$f^{-1} : B \rightarrow A$ is defined as $f^{-1}(y) = x$ if $f(x) = y$

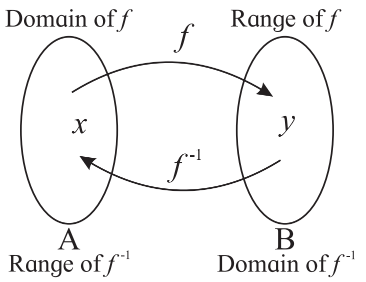


Fig. 6.36

Note:

1) As f is one-one and onto every element $y \in B$ has a unique element $x \in A$ such that $y = f(x)$.

2) If f and g are one-one and onto functions such that $f[g(x)] = x$ for every $x \in$ Domain of g and $g[f(x)] = x$ for every $x \in$ Domain of f, then g is called inverse of function f. Function g is denoted by f^{-1} (read as f inverse).
i.e. $f[g(x)] = g[f(x)] = x$ then $g = f^{-1}$ which Moreover this means $f[f^{-1}(x)] = f^{-1}[f(x)] = x$

3) $f^{-1}(x) \neq [f(x)]^{-1}$, because $[f(x)]^{-1} = \dfrac{1}{f(x)}$
$[f(x)]^{-1}$ is reciprocal of function $f(x)$ where as $f^{-1}(x)$ is the inverse function of $f(x)$.

e.g. If f is one-one onto function with $f(3) = 7$ then $f^{-1}(7) = 3$.

Ex. 7 : If f is one-one onto function with $f(x) = 9 - 5x$, find $f^{-1}(-1)$.

Soln. : $\rightarrow$ Let $f^{-1}(-1) = m$, then $-1 = f(m)$

Therefore,

$$-1 = 9 - 5m$$
$$5m = 9 + 1$$
$$5m = 10$$
$$m = 2$$

That is $f(2) = -1$, so $f^{-1}(-1) = 2$.

Ex. 8 : Verify that $f(x) = \dfrac{x-5}{8}$ and $g(x) = 8x + 5$ are inverse functions of each other.

Solution : As $f(x) = \dfrac{x-5}{8}$, replace x in $f(x)$ with $g(x)$

$$f[g(x)] = \frac{g(x)-5}{8} = \frac{8x+5-5}{8} = \frac{8x}{8} = x$$

and $g(x) = 8x + 5$, replace x in $g(x)$ with $f(x)$

$$g[f(x)] = 8f(x) + 5 = 8\left[\frac{x-5}{8}\right] + 5 = x - 5 + 5$$
$= x$

As $f[g(x)] = x$ and $g[f(x)] = x$, f and g are inverse functions of each other.

Ex. 9 : Determine whether the function

$f(x) = \dfrac{2x+1}{x-3}$ has inverse, if it exists find it.

Solution : f^{-1} exists only if f is one-one and onto.

Consider $f(x_1) = f(x_2)$,

Therefore,

$$\frac{2x_1+1}{x_1-3} = \frac{2x_2+1}{x_2-3}$$
$$(2x_1+1)(x_2-3) = (2x_2+1)(x_1-3)$$
$$2x_1x_2 - 6x_1 + x_2 - 3 = 2x_1x_2 - 6x_2 + x_1 - 3$$
$$-6x_1 + x_2 = -6x_2 + x_1$$
$$6x_1 + x_2 = 6x_2 + x_1$$
$$7x_2 = 7x_1$$
$$x_2 = x_1$$

Hence, f is one-one function.

Let $f(x) = y$, so $y = \dfrac{2x+1}{x-3}$

Express x as function of y , as follows

$$y = \frac{2x+1}{x-3}$$
$$y(x-3) = 2x + 1$$

$$xy - 3y = 2x + 1$$
$$xy - 2x = 3y + 1$$
$$x(y-2) = 3y + 1$$

$\therefore \quad x = \dfrac{3y+1}{y-2}$ for $y \neq 2$.

Thus for any $y \neq 2$,

we have x such that $f(x) = y$

f^{-1} is well defined on R - {2}

If $f(x) = 2$ i.e. $2x + 1 = 2(x - 3)$

i.e. $2x + 1 = 2x - 6$ i.e. $1 = -6$

Which is contradiction.

$2 \notin$ Range of f.

Here the range of $f(x)$ is $R - \{2\}$.

x is defined for all y in the range.

Therefore $f(x)$ is onto function.

As f is one-one and onto, so f^{-1} exists.

As $f(x) = y$, so $f^{-1}(y) = x$

Therefore, $f^{-1}(y) = \dfrac{3y+1}{y-2}$

Replace x by y, to get

$f^{-1}(x) = \dfrac{3x+1}{x-2}$.

6.2.3 Piecewise Defined Functions:

A function defined by two or more equations on different parts of the given domain is called piecewise defind function.

e.g.: If $f(x) = \begin{cases} x+1 & \text{if } x<1 \\ 4-x & \text{if } x\geq 1 \end{cases}$

Here $f(3) = 4 - 3 = 1$ as $3 > 1$,

whereas $f(-2) = -2 + 1 = -1$ as $-2 < 1$ and

$f(1) = 4 - 1 = 3$.

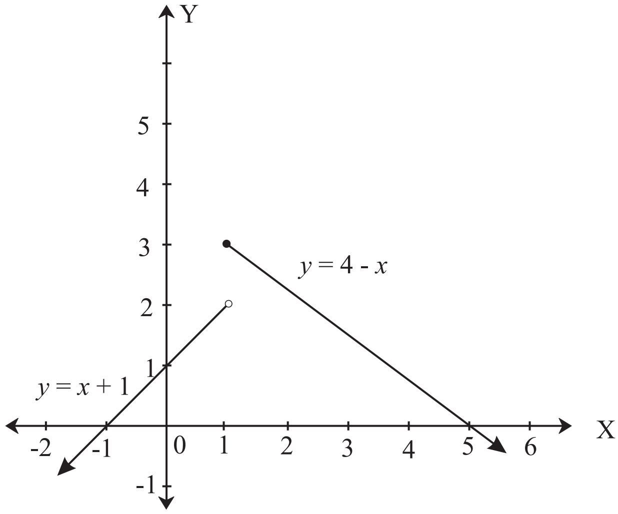


Fig. 6.37

As (1,3) lies on line $y = 4 - x$, so it is shown by small black disc on that line. (1,2) is shown by small white disc on the line $y = x + 1$, because it is not on the line.

1) Signum function :

Definition: $f(x) = \text{sgn}(x)$ is a piecewise function

$$f(x) = \begin{cases} -1 & \text{if } x<0 \\ 0 & \text{if } x=0 \\ 1 & \text{if } x>0 \end{cases}$$

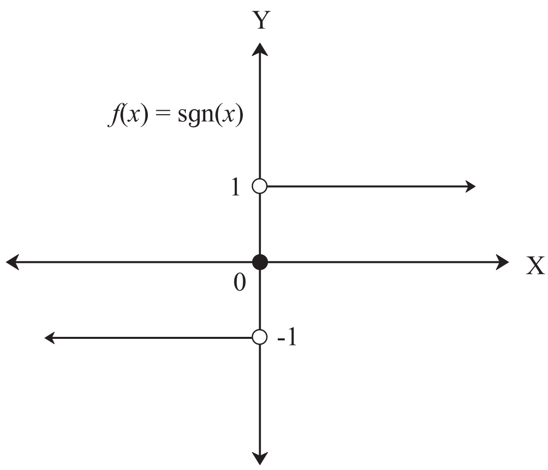


Fig. 6.38

Domain: R and **Range:** {−1, 0, 1}

Properties:

1) For $x > 0$, the graph is line $y = 1$ and for $x < 0$, the graph is line $y = -1$.

2) For $f(0) = 0$, so point (0,0) is shown by black disc, whereas points $(0,-1)$ and $(0,1)$ are shown by white discs.

Absolute value function (Modulus function):

Definition: $f(x) = |x|$, is a piece wise function

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

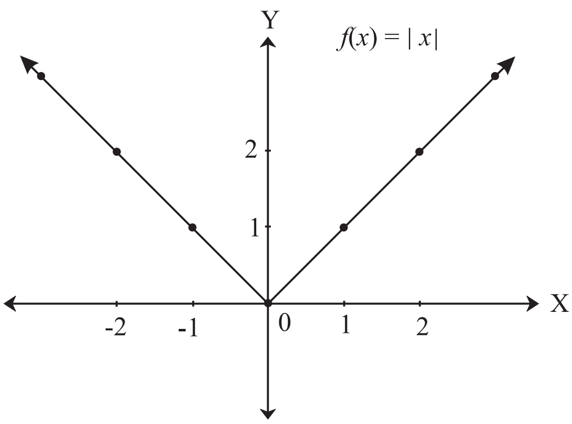


Fig. 6.39

Domain : R or $(-\infty,\infty)$ and **Range :** $[0,\infty)$

Properties:

1) Graph of $f(x) = |x|$ is union of line $y = x$ from quadrant I with the line $y = -x$ from quadrant II. As origin marks the change of directions of the two lines, we call it a critical point.

2) Graph is symmetric about y-axis.

3) Graph of $f(x) = |x-3|$ is the graph of $|x|$ shifted 3 units right and the critical point is (3,0).

4) $f(x) = |x|$, represents the distance of x from origin.

5) If $|x| = m$, then it represents every x whose distance from origin is m, that is $x = +m$ or $x = -m$.

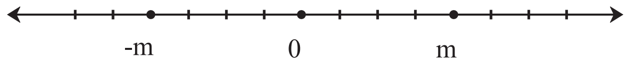


Fig. 6.40

6) If $|x| < m$, then it represents every x whose distance from origin is less than m, $0 \leq x < m$ and $0 \geq x > -m$ That is $-m < x < m$. In interval notation $x \in (-m, m)$

Fig. 6.41

7) If $|x| \geq m$, then it represents every x whose distance from origin is greater than or equal to m, so, $x \geq m$ and $x \leq -m$. In interval notation $x \in (-\infty, m] \cup [m, \infty)$

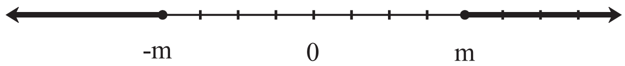


Fig. 6.42

8) If $m < |x| < n$, then it represents all x whose distance from origin is greater than m but less than n. That is $x \in (-n, -m) \cup (m, n)$.

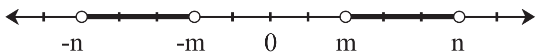


Fig. 6.43

9) Triangle inequality $|x + y| \leq |x| + |y|$.

Verify by taking different values for x and y (positive or negative).

10) $|x|$ can also be defined as $|x| = \sqrt{x^2}$ $= \max\{x, -x\}$.

Ex. 10 : Solve $|4x - 5| \leq 3$.

Solution : If $|x| \leq m$, then $-m \leq x \leq m$

Therefore

$$-3 \leq 4x - 5 \leq 3$$

$$-3 + 5 \leq 4x \leq 3 + 5$$

$$2 \leq 4x \leq 8$$

$$\frac{2}{4} \leq x \leq \frac{8}{4}$$

$$\frac{1}{2} \leq x \leq 2$$

Ex. 11 : Find the domain of $\frac{1}{\sqrt{||x|-1|-3}}$

Solution : As function is defined for $||x|-1|-3>0$

Therefore $||x|-1|>3$

So $|x|-1>3$ or $|x|-1<-3$

That is

$|x|>3+1$ or $|x|<-3+1$

$|x|>4$ or $|x|<-2$

But $|x|<-2$ is not possible as $|x|>0$ always

So $-4<x<4$, $x \in (-4, 4)$.

Ex. 12 : Solve $|x-1|+|x+2|=8$.

Solution : Let $f(x)=|x-1|+|x+2|$

Here the critical points are at $x=1$ and $x=-2$.

They divide number line into 3 parts, as follows.

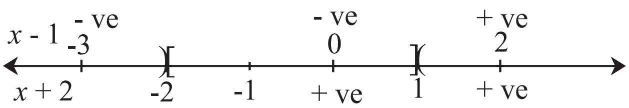


Fig. 6.44

Region	Test Value	Sign	$f(x)$
I $x<-2$	-3	$(x-1)<0$, $(x+2)<0$	$-(x-1)-(x+2)$ $=-2x-1$
II $-2\le x\le 1$	0	$(x-1)<0$, $(x+2)>0$	$-(x-1)+(x+2)$ $=3$
III $x>1$	2	$(x-1)>0$ $(x+2)>0$	$(x-1)+(x+2)$ $=2x+1$

As $f(x)=8$

From I, $-2x-1=8$ $\therefore -2x=9$ $\therefore x=-\frac{9}{2}$.

From II, $3=8$, which is impossible, hence there is no solution in this region.

From III, $2x+1=8$ $\therefore 2x=7$ $\therefore x=\frac{7}{2}$.

Solutions are $x=-\frac{9}{2}$ and $x=\frac{7}{2}$.

3) Greatest Integer Function (Step Function):

Definition: For every real x, $f(x)=[x]=$ The greatest integer less than or equal to x. $[x]$ is also called as floor function and represented by $\lfloor x \rfloor$.

Illustrations:

1) $f(5.7)=[5.7]$ = greatest integer less than or equal to 5.7

Integers less than or equal to 5.7 are 5, 4, 3, 2 of which 5 is the greatest.

2) $f(-6.3)=[-6.3]$ = greatest integer less than or equal to -6.3.

Integers less than or equal to -6.3 are $-10, -9, -8, -7$ of which -7 is the greatest.

$\therefore [-6.3]=-7$

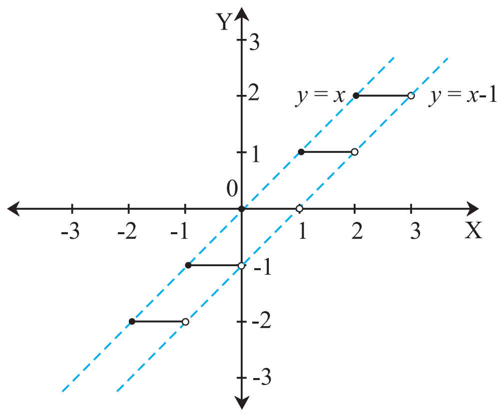
3) $f(2)=[2]$ = greatest integer less than or equal to $2=2$.

4) $[\pi]=3$ 5) $[e]=2$

The function can be defined piece-wise as follows

$f(x)=n$, if $n\le x<n+1$ or $x\in[n, n+1)$, $n\in I$

$$f(x)=\begin{cases} -2 & \text{if } -2\le x<-1 \quad \text{or } x\in[-2,-1) \\ -1 & \text{if } -1\le x<0 \quad \text{or } x\in[-1,0) \\ 0 & \text{if } 0\le x<1 \quad \text{or } x\in[0,1) \\ 1 & \text{if } 1\le x<2 \quad \text{or } x\in[1,2) \\ 2 & \text{if } 2\le x<3 \quad \text{or } x\in[2,3) \end{cases}$$



Graph of $f(x)=[x]$

Fig. 6.45

Domain = R and Range = I (Set of integers)

Properties:

1) If $x \in [2,3)$, $f(x) = 2$ shown by horizontal line. At exactly $x = 2$, $f(2) = 2$, $2 \in [2,3)$ hence shown by black disc, whereas $3 \notin [2,3)$ hence shown by white disc.

2) Graph of $y = [x]$ lies in the region bounded by lines $y = x$ and $y = x-1$. So $x-1 \le [x] < x$

3) $[x] + [-x] = \begin{cases} 0 & \text{if } x \in \text{I} \\ -1 & \text{if } x \notin \text{I} \end{cases}$

Ex. $[3.4] + [-3.4] = 3 + (-4) = -1$ where $3.4 \notin$ I

$[5] + [-5] = 5 + (-5) = 0$ where $5 \in$ I

4) $[x+n] = [x] + n$, where $n \in$ I

Ex. $[4.5 + 7] = [11.5] = 11$ and

$[4.5] + 7 = 4 + 7 = 11$

4) Fractional part function:

Definition: For every real x, $f(x) = \{x\}$ is defined as $\{x\} = x - [x]$

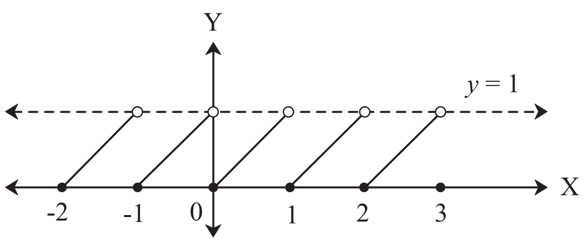
Illustrations:

$f(4.8) = \{4.8\} = 4.8 - [4.8] = 4.8 - 4 = 0.8$

$f(-7.1) = \{-7.1\} = -7.1 - [-7.1]$

$= -7.1 - (-8) = -7.1 + 8 = 0.9$

$f(8) = \{8\} = 8 - [8] = 8 - 8 = 0$



Graph of $f(x) = \{x\}$

Fig. 6.46

Domain = R and Range = [0,1)

Properties:

1) If $x \in [0,1]$, $f(x) = \{x\} \in [0,1)$ shown by slant line $y = x$. At $x = 0$, $f(0) = 0$, $0 \in [0,1)$ hence shown by black disc, whereas at $x = 1$, $f(1) = 1$, $1 \notin [0,1)$ hence shown by white disc.

2) Graph of $y = \{x\}$ lies in the region bounded by $y = 0$ and $y = 1$. So $0 \le \{x\} < 1$

3) $\{x\} + \{-x\} = \begin{cases} 0 & \text{if } x \in \text{I} \\ 1 & \text{if } x \notin \text{I} \end{cases}$

Ex. 13: $\{5.2\} + \{-5.2\} = 0.2 + 0.8 = 1$ where $5.2 \in$ I

$\{7\} + \{-7\} = 0 + (0) = 0$ where $7 \in$ I

4) $\{x \pm n\} = \{x\}$, where $n \in$ I

Ex. 14 : $\{2.8 + 5\} = \{7.8\} = 0.8$ and $\{2.8\} = 0.8$

$\{2.8 - 5\} = \{-2.2\} = -2.2 - (-2.2) = -2.2 - (-3)$

$= 0.8$ ($\because$ $\{x\} = x - [x]$)

Ex. 15 : If $\{x\}$ and $[x]$ are the fractional part function and greatest integer function of x respectively. Solve for x, if $\{x + 1\} + 2x = 4[x + 1] - 6$.

Solution : $\{x + 1\} + 2x = 4\,[x + 1] - 6$

Since $\{x + n\} = \{x\}$ and $[x + n] = [x] + n$, for $n \in I$, also $x = [x] + \{x\}$

$\therefore$ $\{x\} + 2(\{x\} + [x]) = 4([x] + 1) - 6$

$\therefore$ $\{x\} + 2\{x\} + 2[x] = 4[x] + 4 - 6$

$\therefore$ $3\{x\} = 4[x] - 2[x] - 2$

$\therefore$ $3\{x\} = 2[x] - 2$ ... (I)

Since $0 \le \{x\} < 1$

$\therefore$ $0 \le 3\{x\} < 3$

$\therefore$ $0 \le 2\,[x] - 2 < 3$ ($\because$ from I)

$\therefore$ $0 + 2 \le 2\,[x] < 3 + 2$

$\therefore$ $2 \le 2\,[x] < 5$

$\therefore$ $\dfrac{2}{2} \le [x] < \dfrac{5}{2}$

$\therefore$ $1 \le [x] < 2.5$

But as $[x]$ takes only integer values

$[x] = 1, 2$ since $[x] = 1 \Rightarrow 1 \le x < 2$ and $[x] = 2 \Rightarrow 2 \le x < 3$

Therefore $x \in [1,3)$

Note:

1)

Property	$f(x)$
$f(x+y) = f(x) + f(y)$	kx
$f(x+y) = f(x)\, f(y)$	a^{kx}
$f(xy) = f(x)\, f(y)$	x^n
$f(xy) = f(x) + f(y)$	$\log x$

2) If $n(A) = m$ and $n(B) = n$ then

(a) number of functions from A and B is n^m (b) for $m \le n$, number of one-one functions is $\dfrac{n!}{(n-m)!}$

(c) for $m > n$, number of one-one functions is 0

(d) for $m \ge n$, number of onto functions are $n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \ldots + (-1)^{n-1}\, {}^nC_{n-1}$

(e) for $m < n$, number of onto functions are 0.

(f) number of constant fuctions is m.

3) Characteristic & Mantissa of Common Logarithm $\log_{10} x$:

As $x = [x] + \{x\}$

$\log_{10} x = [\log_{10} x] + \{\log_{10} x\}$

Where, integral part $[\log_{10} x]$ is called Characteristic & fractional part $\{\log_{10} x\}$ is called Mantissa.

Illustration : For $\log_{10} 23$,

$\log_{10} 10 < \log_{10} 23 < \log_{10} 100$

$\log_{10} 10 < \log_{10} 23 < \log_{10} 10^2$

$\log_{10} 10 < \log_{10} 23 < 2\log_{10} 10$

$1 < \log_{10} 23 < 2 \qquad (\because \log_{10}^{10} = 1)$

Then $[\log_{10} 23] = 1$, hence Characteristic of $\log_{10} 23$ is 1.

The characteristic of the logarithm of a number N, with 'm' digits in its integral part is 'm−1'.

Ex. 16 : Given that $\log_{10} 2 = 0.3010$, find the number of digits in the number 20^{10}.

Solution : Let $x = 20^{10}$, taking $\log_{10}$ on either sides, we get

$$\begin{aligned} \log_{10} x &= \log_{10}(20^{10}) = 10\log_{10} 20 \\ &= 10\log_{10}(2\times 10) = 10\{\log_{10} 2 + \log_{10} 10\} \\ &= 10\{\log_{10} 2 + 1\} = 10\{0.3010 + 1\} \\ &= 10\,(1.3010) = 13.010 \end{aligned}$$

That is characteristic of x is 13.

So number of digits in x is $13 + 1 = 14$

EXERCISE 6.2

1) If $f(x) = 3x + 5$, $g(x) = 6x - 1$, then find

(a) $(f+g)(x)$ (b) $(f-g)(2)$

(c) $(fg)(3)$ (d) $(f/g)(x)$ and its domain.

2) Let $f:\{2,4,5\} \to \{2,3,6\}$ and $g:\{2,3,6\} \to \{2,4\}$ be given by $f = \{(2,3), (4,6), (5,2)\}$ and $g = \{(2,4), (3,4), (6,2)\}$. Write down $g \circ f$

3) If $f(x) = 2x^2 + 3$, $g(x) = 5x - 2$, then find

(a) $f \circ g$ (b) $g \circ f$

(c) $f \circ f$ (d) $g \circ g$

4) Verify that f and g are inverse functions of each other, where

(a) $f(x) = \dfrac{x-7}{4}$, $g(x) = 4x + 7$

(b) $f(x) = x^3 + 4$, $g(x) = \sqrt[3]{x-4}$

(c) $f(x) = \dfrac{x+3}{x-2}$, $g(x) = \dfrac{2x+3}{x-1}$

5) Check if the following functions have an inverse function. If yes, find the inverse function.

(a) $f(x) = 5x^2$ (b) $f(x) = 8$

(c) $f(x) = \dfrac{6x-7}{3}$ (d) $f(x) = \sqrt{4x+5}$

(e) $f(x) = 9x^3+8$

(f) $f(x) = \begin{cases} x+7 & x<0 \\ 8-x & x \geq 0 \end{cases}$

6) If $f(x) = \begin{cases} x^2+3, & x \leq 2 \\ 5x+7, & x>2 \end{cases}$, then find

(a) $f(3)$ (b) $f(2)$ (c) $f(0)$

7) If $f(x) = \begin{cases} 4x-2, & x \leq -3 \\ 5, & -3<x<3 \\ x^2, & x \geq 3 \end{cases}$, then find

(a) $f(-4)$ (b) $f(-3)$

(c) $f(1)$ (d) $f(5)$

8) If $f(x) = 2|x| + 3x$, then find

(a) $f(2)$ (b) $f(-5)$

9) If $f(x) = 4[x] - 3$, where $[x]$ is greatest integer function of x, then find

(a) $f(7.2)$ (b) $f(0.5)$

(c) $f\left(-\dfrac{5}{2}\right)$ (d) $f(2\pi)$, where $\pi = 3.14$

10) If $f(x) = 2\{x\} + 5x$, where $\{x\}$ is fractional part function of x, then find

(a) $f(-1)$ (b) $f\left(\dfrac{1}{4}\right)$

(c) $f(-1.2)$ (d) $f(-6)$

11) Solve the following for x, where $|x|$ is modulus function, $[x]$ is greatest integer function, $[x]$ is a fractional part function.

(a) $|x+4| \geq 5$ (b) $|x-4| + |x-2| = 3$

(b) $x^2 + 7|x| + 12 = 0$ (d) $|x| \leq 3$

(e) $2|x| = 5$

(f) $[x + [x + [x]]] = 9$ (g) $\{x\} > 4$

(h) $\{x\} = 0$ (i) $\{x\} = 0.5$

(j) $2\{x\} = x + [x]$

Let's Remember

- If f:A $\rightarrow$ B is a function and $f(x) = y$, where $x \in$ A and $y \in$ B, then

  Domain of f is A = Set of Inputs = Set of Pre-images = Set of values of x for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on X-axis.

  Range of f is f (A) = Set of Outputs = Set of Images = Set of values of y for which $y = f(x)$ is defined = Projection of graph of $f(x)$ on Y-axis.

  Co-domain of f is B.

- If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ then f is **one-one** and for every $y \in$ B, if there exists $x \in$ A such that $f(x) = y$ then f is **onto**.

- If f:A $\rightarrow$ B. g:B $\rightarrow$ C then a function $g \circ f$:A $\rightarrow$ C is a **composite function**.

- If f:A $\rightarrow$ B, then f^{-1}:B $\rightarrow$ A is **inverse function** of f.

- If f:R $\rightarrow$ R is a real valued function of real variable, the following table is formed.

Type of f	Form of f	Domain of f	Range of f
Constant function	$f(x) = k$	R	k
Identity function	$f(x) = x$	R	R
Square function	$f(x) = x^2$	R	$[0, \infty)$ or R^+
Cube function	$f(x) = x^3$	R	R
Linear function	$f(x) = ax + b$	R	R
Quadratic function	$f(x) = ax^2 + bx + c$	R	$\left(\frac{4ac - b^2}{4a}, \infty\right)$
Cubic function	$f(x) = ax^3 + bx^2 + cx + d$	R	R
Square root funtion	$f(x) = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$ or R^+
Cube root function	$f(x) = \sqrt[3]{x}$	R	R
Rational function	$f(x) = \frac{p(x)}{q(x)}$	$R - \{x \mid q(x) = 0\}$	depends on $p(x)$ and $q(x)$
Exponential function	$f(x) = a^x$, $a > 1$	R	$(0, \infty)$
Logarithmic function	$f(x) = \log_a x$, $a > 1$	$(0, \infty)$ or R^+	R
Absolute function	$f(x) = \lvert x \rvert$	R	$[0, \infty)$ or R^+
Signum function	$f(x) = \text{sgn}(x)$	R	$\{-1, 0, 1\}$
Greatest Integer function	$f(x) = [x]$	R	I (set of integers)
Fractional Part function	$f(x) = \{x\}$	R	$[0,1)$

MISCELLANEOUS EXERCISE 6

(I) Select the correct answer from given alternatives.

1) If log $(5x - 9)$ – log $(x + 3)$ = log 2 then x = ...

A) 3 B) 5 C) 2 D) 7

2) If $\log_{10}(\log_{10}(\log_{10}x)) = 0$ then x =

A) 1000 B) 10^{10}

C) 10 D) 0

3) Find x, if $2\log_2 x = 4$

A) 4, –4 B) 4

C) –4 D) not defined

4) The equation $\log_{x^2} 16 + \log_{2x} 64 = 3$ has,

A) one irrational solution

B) no prime solution

C) two real solutions

D) one integral solution

5) If $f(x) = \frac{1}{1-x}$, then $f(f\{f(x)\}]$ is

A) $x - 1$ B) $1 - x$ C) x D) $-x$

6) If $f : R \to R$ is defined by $f(x) = x^3$ then f^{-1} (8) is euqal to :

A) {2} B) {–2. 2}

C) {–2} D) (–2. 2)

7) Let the function f be defined by $f(x) = \frac{2x+1}{1-3x}$ then $f^{-1}(x)$ is:

A) $\frac{x-1}{3x+2}$ B) $\frac{x+1}{3x-2}$

C) $\frac{2x+1}{1-3x}$ C) $\frac{3x+2}{x-1}$

8) If $f(x) = 2x^2 + bx + c$ and $f(0) = 3$ and $f(2) = 1$, then $f(1)$ is equal to

A) –2 B) 0 C) 1 D) 2

9) The domain of $\frac{1}{[x]-x}$ where $[x]$ is greatest integer function is

A) R B) Z C) R–Z D) Q - {o}

10) The domain and range of $f(x) = 2 - |x-5|$ is

A) R^+, $(-\infty, 1]$ B) R, $(-\infty, 2]$

C) R, $(-\infty, 2)$ D) R^+, $(-\infty, 2]$

(II) Answer the following.

1) Which of the following relations are functions? If it is a function determine its domain and range.

i) {(2, 1), (4, 2), (6, 3), (8, 4), (10, 5), (12, 6), (14, 7)}

ii) {(0, 0), (1, 1), (1, –1), (4, 2), (4, –2), (9, 3), (9, –3), (16, 4), (16, –4)}

iii) {12, 1), (3, 1), (5, 2)}

2) Find whether following functions are one-one.

i) f:R→R defined by $f(x) = x^2+5$

ii) f: R–{3}→R defined by $f(x) = \frac{5x+7}{x-3}$ for $x \in$ R–{3}

3) Find whether following functions are onto or not.

i) f: Z→Z defined by $f(x) = 6x–7$ for all $x \in Z$

ii) f: R→R defined by $f(x) = x^2+3$ for all $x \in R$

4) Let f: R→R be a function defined by $f(x) = 5x^3–8$ for all $x \in R$, show that f is one-one and onto. Hence find f^{-1}.

5) A function f: R→R defined by $f(x) = \frac{3x}{5} + 2$, $x \in R$. Show that f is one-one and onto. Hence find f^{-1}.

6) A function f is defined as $f(x) = 4x+5$, for $-4 \le x < 0$. Find the values of $f(-1)$, $f(-2)$, $f(0)$, if they exist.

7) A function f is defined as : $f(x) = 5-x$ for $0 \le x \le 4$. Find the value of x such that (i) $f(x) = 3$ (ii) $f(x) = 5$

8) If $f(x) = 3x^4 - 5x^2 + 7$ find $f(x-1)$.

9) If $f(x) = 3x + a$ and $f(1) = 7$ find a and $f(4)$.

10) If $f(x) = ax^2 + bx + 2$ and $f(1) = 3$, $f(4) = 42$, find a and b.

11) Find composite of f and g

i) f = {(1, 3), (2, 4), (3, 5), (4, 6)}

g = {(3, 6), (4, 8), (5, 10), (6, 12)}

ii) f = {(1, 1), (2, 4), (3, 4), (4, 3)}

g = {(1, 1), (3, 27), (4, 64)}

12) Find fog and gof

i) $f(x) = x^2 + 5$, $g(x) = x-8$

ii) $f(x) = 3x - 2$, $g(x) = x^2$

iii) $f(x) = 256x^4$, $g(x) = \sqrt{x}$

13) If $f(x) = \frac{2x-1}{5x-2}$, $x \neq \frac{5}{2}$

Show that $(fof)(x) = x$.

14) If $f(x) = \dfrac{x+3}{4x-5}$, $g(x) = \dfrac{3+5x}{4x-1}$ then show that $(fog)\,(x) = x$.

15) Let f: R – {2}$\to$ R be defined by $f(x) = \dfrac{x^2-4}{x-2}$ and g : R $\to$ R be defined by $g\,(x) = x + 2$. Ex whether $f = g$ or not.

16) Let f: R $\to$ R be given by $f(x) = x + 5$ for all $x \in$R. Draw its graph.

17) Let f: R $\to$ R be given by $f(x) = x^3 + 1$ for all $x \in$R. Draw its graph.

18) For any base show that

$\log(1+2+3) = \log 1 + \log 2 + \log 3$.

19) Find x, if $x = 3^{3\log_3 2}$

20) Show that,

$$\log|\sqrt{x^2+1}+x| + \log|\sqrt{x^2+1}-x| = 0$$

21) Show that, $\log\dfrac{a^2}{bc} + \log\dfrac{b^2}{ca} + \log\dfrac{c^2}{ab} = 0$

22) Simplify, $\log(\log x^4) - \log(\log x)$.

23) Simplify

$$\log_{10}\frac{28}{45} - \log_{10}\frac{35}{324} + \log_{10}\frac{325}{432} - \log_{10}\frac{13}{15}$$

24) If $\log\left(\dfrac{a+b}{2}\right) = \dfrac{1}{2}(\log a + \log b)$, then show that a=b

25) If b^2=ac. prove that, log a + log c = 2log b

26) Solve for x, $\log_x(8x-3) - \log_x 4 = 2$

27) If $a^2 + b^2 = 7ab$, show that,

$$\log\left(\frac{a+b}{2}\right) = \frac{1}{2}\log a + \frac{1}{2}\log b$$

28) If $\log\left(\dfrac{x-y}{5}\right) = \dfrac{1}{2}\log x + \dfrac{1}{2}\log y$, show that $x^2 + y^2 = 27xy$.

29) If $\log_3[\log_2(\log_3 x)] = 1$, show that $x = 6561$.

30) If $f(x) = \log(1-x)$, $0 \le x < 1$ show that

$$f\left(\frac{1}{1+x}\right) = f(1-x) - f(-x)$$

31) Without using log tables, prove that

$$\frac{2}{5} < \log_{10}3 < \frac{1}{2}$$

32) Show that

$$7\log\left(\frac{15}{16}\right) + 6\log\left(\frac{8}{3}\right) + 5\log\left(\frac{2}{5}\right) + \log\left(\frac{32}{25}\right) = \log 3$$

33) Solve : $\sqrt{\log_2 x^4} + 4\log_4\sqrt{\dfrac{2}{x}} = 2$

34) Find value of $\dfrac{3+\log_{10}343}{2+\dfrac{1}{2}\log_{10}\left(\dfrac{49}{4}\right)+\dfrac{1}{2}\log_{10}\left(\dfrac{1}{25}\right)}$

35) If $\dfrac{\log a}{x+y-2z} = \dfrac{\log b}{x+y-2x} = \dfrac{\log c}{x+y-2y}$, show that abc = 1.

36) Show that, $\log_y x^3.\log_z y^4.\log_x z^5 = 60$

37) If $\dfrac{\log_2 a}{4} = \dfrac{\log_2 b}{6} = \dfrac{\log_2 c}{3k}$ and $a^3b^2c = 1$ find the value of k.

38) If $a^2 = b^3 = c^4\,d^5$, show that $\log_a bcd = \dfrac{47}{30}$.

39) Solve the following for x, where $|x|$ is modulus function, $[x]$ is greatest interger function, $\{x\}$ is a fractional part function.

a) $1 < |x-1| < 4$ c) $|x^2 - x - 6| = x + 2$

c) $|x^2 - 9| + |x^2 - 4| = 5$

d) $-2 < [x] \le 7$ e) $2[2x - 5] - 1 = 7$

f) $[x^2] - 5\,[x] + 6 = 0$

g) $[x-2] + [x+2] + \{x\} = 0$

h) $\left[\dfrac{x}{2}\right] + \left[\dfrac{x}{3}\right] = \dfrac{5x}{6}$

40) Find the domain of the following functions.

a) $f(x)=\dfrac{x^2+4x+4}{x^2+x-6}$

b) $f(x)=\sqrt{x-3}+\dfrac{1}{\log(5-x)}$

c) $f(x)=\sqrt{1-\sqrt{1-\sqrt{1-x^2}}}$

d) $f(x)=x!$

e) $f(x)={}^{5-x}P_{x-1}$

f) $f(x)=\sqrt{x-x^2}+\sqrt{5-x}$

g) $f(x)=\sqrt{\log(x^2-6x+6)}$

41) Find the range of the following functions.

a) $f(x)=|x-5|$ b) $f(x)=\dfrac{x}{9+x^2}$

c) $f(x)=\dfrac{1}{1+\sqrt{x}}$ d) $f(x)=[x]-x$

e) $f(x)=1+2^x+4^x$

42) Find $(f\circ g)(x)$ and $(g\circ f)(x)$

a) $f(x)=e^x$, $g(x)=\log x$

b) $f(x)=\dfrac{x}{x+1}$, $g(x)=\dfrac{x}{1-x}$

43) Find $f(x)$ if

a) $g(x) = x^2 + x - 2$ and $(g\circ f)(x) = 4x^2 - 10x + 4$

(b) $g(x) = 1+\sqrt{x}$ and $f[g(x)] = 3+2\sqrt{x}+x$.

44) Find $(f\circ f)(x)$ if

(a) $f(x)=\dfrac{x}{\sqrt{1+x^2}}$

(b) $f(x)=\dfrac{2x+1}{3x-2}$

7 LIMITS

Let's Study

- Meaning – Definition of Limit
- Calculation of various limits
- Limits of Trigonometric Functions
- Limits of Exponential and Logarithmic Functions
- Limit at Infinity and Infinite limit

Introduction:

Calculus is an important branch of mathematics. The concept of limit of a function is a fundamental concept in calculus.

7.1.1 Definition of Limit:

Let's :Learn

7.1.1 LIMIT OF A FUNCTION :

Suppose x is a variable and a is a constant. If x takes values closer and closer to 'a' but not equal to 'a', then we say that x tends to a. Symbolically it is denoted as $x \to a$.

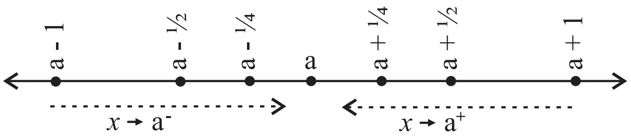


Fig. 7.1

You can observe that the values of x are very near to 'a' but not equal to 'a'.

When $x > a$ and x takes values near a, for example, $x = \mathrm{a} + \frac{1}{8}, x = \mathrm{a} + \frac{1}{4}, x = \mathrm{a} + \frac{1}{2}$ ... etc; we say that $x \to \ a+$ (x tends to a from larger values). When $x < a$ and x takes values near a, for example, $x = \mathrm{a} - \frac{1}{2}, x = \mathrm{a} - \frac{1}{4}, x = \mathrm{a} - \frac{1}{8}$ ... etc. then we say that $x \to a^-$ (x tends to a from smaller values).

We will study functions of x, a real variable, and a, b, c etc will denote constants. $x \to a$ implies that x takes values as near a as possible. So in this case we have to consider x going nearer a from either side. So, $x = a - \frac{1}{4}, a + \frac{1}{4}, a - \frac{1}{2}, a + \frac{1}{2}$ ......

We will illustrate with an example.

Consider the function $f(x) = x + 3$

Take the value of x very close to 3 but not equal to 3.

The following table shows that as x gets nearer to 3, the corresponding values of $f(x)$ also get nearer to 6.

(I)

	x approaches to 3 from left				
x	2.5	2.6	…	2.9	2.99
$f(x)$	5.5	5.6	…	5.9	5.99

(II)

	x approaches to 3 from right				
x	3.6	3.5	…	3.1	3.01
$f(x)$	6.6	6.5	…	6.1	6.01

From the table we see that as $x \to 3$ from either side, $f(x) \to 6$.

This idea can be expressed by saying that the limiting value of $f(x)$ is 6 when x approaches 3, at $x = 3$, which is the limiting value of $f(x)$ as $x \to 3$

Observe that if $P(x)$ is a polynomical in x, then $\lim\limits_{x \to a} P(x) = P(a)$, for any constant '$a$'.

We are going to study the limit of a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ as $x \to$ a.

Here $P(x)$ and $Q(x)$ are polynomials in x.

We get three different cases.

(1) $Q(a) \neq 0$,

(2) $Q(a) = 0$ and $P(a) = 0$

(3) $Q(a) = 0$ and $P(a) \neq 0$

In **case (1)** the limit of $f(x)$ as $x \to a$ is

$$\lim_{x\to a} \frac{P(x)}{Q(x)} = \frac{P(a)}{Q(a)}.$$

In **Case (2)** $(x - a)$ is a factor of P(x) as well as Q(x). So we expresse P(x) and Q(x) as $P(x) = (x - a)P_1(x)$ and $Q(x) = (x - a)Q_1(x)$

$$\text{Now} = \frac{P(x)}{Q(x)} = \frac{(x-a)^r P_1(x)}{(x-a)^s Q_1(x)} \quad (x - a \neq 0)$$

If r = s then $\lim_{x\to a} \dfrac{P(x)}{Q(x)} = \dfrac{P_1(a)}{Q_1(a)}$

If r > s then $\lim_{x\to a} \dfrac{P(x)}{Q(x)} = 0$

If r < s then we proceed to case (3).

In **case (3),** if $Q(a) = 0$ and $P(a) \neq 0$

$\lim_{x\to a} \dfrac{P(x)}{Q(x)}$ does not exist.

7.1.2 Definition of Limit :

We need to confirm that $f(x)$ is very near to l (or as near as expected). This is expressed by $|f(x) - l| < \in$ for any $\in > 0$. Here $\in$ can be arbitrarily small to ensure that f(x) is very near l. If this condition is satisfied for all x near enough, then we can say that $f(x) \to l$ as $x \to a$, the fact that x is near enough a is expressed by $0 < |x - a| < \delta$ where $\delta > 0$. This δ can be very small and depends upon f(x) and $\in$.

Hence the Definition

If given $\in > 0$ there exists $\delta > 0$ such that $|f(x) - l| < \in$ for all $|x - a| < \delta$, then we say that *$f(x) \to l$ as $x \to a$.*

SOLVED EXAMPLES

Strategy : Steps for verifying the $\in$ - δ definition.

Consider $\in > 0$ given, substitute the values of $f(x)$ and l in $|f(x) - l| < \in$ and proceed to find the value of δ. We may have to manipulate the inequalities.

Ex. 1. Consider the example $f(x) = 3x + 1$, take $a = 0$ and $l = 1$

We want to find some $\delta > 0$ such that,

$0 < |x - 0| < \delta$ impies that, $|(3x + 1) - 1| < \in$

if $|3x| < \in$ i.e. if $3|x| < \in$ i.e. if $|x| < \dfrac{\in}{3}$

So, we can choose $\delta = \dfrac{\in}{3}$

(If fact any $\delta \leq \dfrac{\in}{3}$ will do.)

$\therefore\ 0 < |x - 0| < \delta \Rightarrow |f(x) - l| < \in$

$\therefore\ \lim_{x\to 0} (3x + 1) = 1$

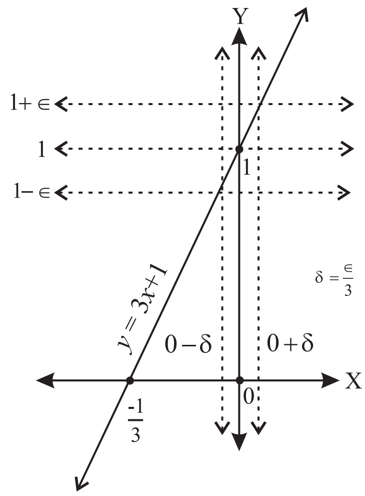


Fig. 7.2

Ex. 2. $f(x) = x^2$

Here take $a = 3$ and $l = 9$

We want to find some $\delta > 0$ such that

$0 < |x - 3| < \delta$ implies $|x^2 - 9| < \in$

i.e. $3 - \delta < x < 3 + \delta \Rightarrow |x^2 - 9| < \in$

δ can be chosen as small as we like and take $\delta < 1$

Then, $3-\delta < x < 3+\delta \Rightarrow 2 < x < 4$ or
$5 < x+3 < 7$

We want $|(x+3)(x-3)| < \in$

But $|(x+3)(x-3)| < 7|x-3|$

So $7|(x-3)| < \in \Rightarrow |x^2-9| < \in$

If $\delta = \frac{\in}{7}$, $|(x-3)| < \delta \Rightarrow |x^2-9| < \in$

So we choose $\delta = \min\{\frac{\in}{7}, 1\}$

then $|(x-3)| < \delta \Rightarrow |f(x)-9| < \in$

* Note that, we want to get rid of factor $|x+3|$ Hence we have to get its lower bound.

Ex. 3. $f(x) = [x]$, $2 < x < 4$ where $[x]$ is a greatest integer function.

We have seen the $f(x) = [x]$, $2 \le x \le 4$

Note that

$[x] = 2$ for $2 \le x < 3$

$= 3$ for $3 \le x < 4$

Let us study the limits of $f(x)$ as $x \to 3$ and $x \to 2.7$

$\lim_{x\to 3^+} f(x) = 3$,

But for $x < 3$, $f(x) = 2$. So, $\lim_{x\to 3^-} f(x) = 2$.

If we take $l = 3$, then for $\in = \frac{1}{2}$ and any $\delta > 0$

$3-\delta < x < 3 \Rightarrow f(x) = 2$ and $|f(x)-l| = 1 \nless \in$

If we take $l = 2$, then $3 < x < 3+\delta \Rightarrow f(x) = 3$,

$|f(x)-2| = 1 \nless \in$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Consider $\lim_{x\to 2.7} f(x)$

Consider $a = 2.7$ We see that for $2 < x < 3$, $f(x) = 2$.

If we choose $\delta = 0.3$,

then $2 < 2.7-\delta < 2.7 < 2.7+\delta < 3$

and $f(x) = 2$ is a constant.

$\therefore \lim_{x\to 2.7} f(x) = 2$

From the above example we notice that the limits of $f(x)$ as $x \to a^+$ or a^- can be different. This induces us to define the following.

7.1.3 One Sided Limit: $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$; if they exist are called one sided limits.

7.1.4 Left hand Limit: *If given $\in > 0$, there exists $\delta > 0$ such that for $|f(x)-l| < \in$ for all x with $a-\delta < x < a$ then*

$\lim_{x\to a^-} f(x) = l$

7.1.5 Right hand Limit : *If given $\in > 0$ there exists $\delta > 0$ such that for $|f(x)-l| < \in$ for all x with $a < x < a+\delta$ then*

$\lim_{x\to a^+} f(x) = l$

7.1.6 Existence of a limit of a function at a point $x = a$

If $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l$, then limit of the function $f(x)$ as $x \to a$ exists and its value is l. And if $\lim_{x\to a^+} f(x) \ne \lim_{x\to a^-} f(x)$ then $\lim_{x\to a} f(x)$ does not exist.

Example:

Find left hand limit and right hand limit for the following function.

$$f(x) = \begin{cases} 3x+1 & \text{if } x < 1 \\ 7x^2-3 & \text{if } x \ge 1 \end{cases}$$

Solution : Right hand limit, $\lim_{x\to 1} f(x)$, for $x > 1$

i.e. $\lim_{x\to 1^+} f(x) = \lim_{x\to 1} (7x^2-3) = 4$

Left hand limit, $\lim_{x\to 1} f(x)$, for $x < 1$:

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1} (3x+1) = 4$

Since left and right-hand limits are equal, the two-sided limit is defined, and $\lim_{x\to 1} f(x) = 4$.

Note : $\lim_{x\to a} f(x)$ means $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x)$

7.1.7 ALGEBRA OF LIMITS:

It is easy to verify the following.

Let $f(x)$ and $g(x)$ be two functions such that $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$, then

1. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
$= l \pm m$

2. $\lim_{x\to a} [f(x) \times g(x)] = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$
$= l \times m$

3. $\lim_{x\to a} [k\, f(x)] = k \times \lim_{x\to a} f(x) = kl$, where 'k' is a constant

4. $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{l}{m}$ where $m \neq 0$

Note:

1) $\lim_{x\to a} k = k$, where k is a constant

2) $\lim_{x\to a} x = a$

3) $\lim_{x\to a} x^n = a^n$

4) $\lim_{x\to a} \sqrt[r]{x} = \sqrt[r]{a}$

5) If p(x) is a polynomial,
then $\lim_{x\to a} p(x) = p(a)$

While evaluating limits, we must always check whether the denominator tends to zero, and if it does, then whether the numerator also tends to zero. In case both tend to zero we have to study the function in detail.

SOLVED EXAMPLES

Ex. 1 : $\lim_{n\to 3}\left(\sum_{r=1}^{n} r^2\right) = \lim_{n\to 3}\left(\dfrac{n(n+1)(2n+1)}{6}\right)$

$$= \frac{(3)\times(3+1)\times(6+1)}{6}$$

$$= \frac{(3)\times(3+1)\times(6+1)}{6}$$

$$= \frac{3\times 4\times 7}{6}$$

$$= 14$$

Ex. 2 : $\lim_{y\to 2} [(y^2-3)(y+2)]$

$= \lim_{y\to 2} [(y^2-3)\,[y+2]$

$= (2^2-3)(2+2) = (8-3)(4) = 5 \times 4 = 20$

Ex. 3 : $\lim_{x\to 3}\left(\dfrac{\sqrt{6+x}-\sqrt{7-x}}{x}\right)$

$$= \frac{\lim_{x\to 3}\left(\sqrt{6+x}\right) - \lim_{x\to 3}\left(\sqrt{7-x}\right)}{\lim_{x\to 3}(x)}$$

$$= \frac{\sqrt{6+3}-\sqrt{7-3}}{3}$$

$$= \frac{\sqrt{9}-\sqrt{4}}{3} = \frac{3-2}{3} = \frac{1}{2}$$

Ex. 4 : $\lim_{x\to 1}\left(\dfrac{\frac{1}{x}-1}{x-1}\right)$

$$= \lim_{x\to 1}\left(\frac{\frac{1-x}{x}}{x-1}\right)$$

$$= \lim_{x\to 1}\left[\frac{(1-x)}{(x-1)\times x}\right]$$

$$= \lim_{x\to 1}\left[\frac{-(x-1)}{(x-1)\times x}\right]$$

$$= \lim_{x\to 1}\left[\frac{-1}{x}\right] \qquad [\text{As } x \to 1,\ x-1 \neq 0]$$

$$= -\lim_{x\to 1}\left[\frac{1}{x}\right] = -\frac{1}{1}$$

$$= -1$$

Ex. 5 : Discuss the limit of the following function as x tends to 3 if

$$f(x) = \begin{cases} x^2+x+1, & 2 \le x \le 3 \\ 2x+1, & 3 < x \le 4 \end{cases}$$

Solution: As $f(x)$ is defined seperately for $x \le 3$ and $x > 3$, we have to find left hand limit (when $x \le 3$) and right hand limit (when $x > 3$) to discuss the existance of limit of $f(x)$ as $x \to 3$.

For the interval $2 \le x \le 3$; $f(x) = x^2 + x + 1$

$$\therefore \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (x^2+x+1) = (3)^2+3+1$$

$$= 9 + 3 + 1 = 13 \qquad \text{-----(I)}$$

Similarly for the interval $3 < x \le 4$; $f(x) = 2x + 1$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (2x+1) = (2 \times 3) + 1$$

$$= 6 + 1 = 7 \qquad \text{-----(II)}$$

From (I) and (II), $\lim_{x\to 3^-} f(x) \ne \lim_{x\to 3^+} f(x)$

$\therefore \lim_{x\to 3} f(x)$ does not exist.

Ex. 6 : For a given $\in > 0$, find $\delta > 0$ such that whenever $|x - a| < \delta$, we have $|f(x) - l| < \in$ so that $\lim_{x\to 1} (4x+3) = 7$

Solution : We want to find δ so that $\lim_{x\to 1} (4x+3) = 7$

Here $a = 1$, $l = 7$ and $f(x) = 4x + 3$

Consider $\in > 0$ and $|f(x) - l| < \in$

$$|(4x+3) - 7| < \in$$

$$\text{if } |(4x-4)| < \in$$

$$\text{i.e. if } |4(x-1)| < \in$$

$$\text{i.e. if } |x-1| < \frac{\varepsilon}{4}$$

We can have $\delta \le \frac{\varepsilon}{4}$ so that $|(x - 1)| < \delta$ $\Rightarrow |f(x) - 7| < \in$

77.1.8 Theorem: Prove that

$$\boxed{\lim_{x\to a} \frac{x^n - a^n}{x - a} = na^{n-1} \text{ where } n \in \mathbb{N},\ a > 0}$$

Proof : We know, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})$ for $n \in \mathbb{N}$.

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) =$$

$$\lim_{x\to a}\left(\frac{(x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots\dots + a^{n-1})}{x-a}\right)$$

$$= \lim_{x\to a} (x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \dots + a^{n-1})$$

for $[x - a \ne 0]$

$$= a^{n-1} + a^{n-2}a + a^{n-3}a^2 + \dots + a^{n-1}$$

$$= a^{n-1} + a^{n-1} + a^{n-1} + \dots + a^{n-1} \ (n \text{ terms})$$

$$= na^{n-1}$$

Note : The above limit can also be found by using the substitution $x - a = h$.

$x - a = h \therefore x = a + h$ and $x \to a \Rightarrow h \to 0$. Use binomial theorem to expand $(a + h)^n$, simplify and apply the limit to get the result

$$\lim_{x\to a}\left[\frac{x^n - a^n}{x-a}\right] = n\,a^{n-1}$$

Verify : If $n < 0$ say $n = -m$ then

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) = \lim_{x\to a}\left(\frac{x^{-m} - a^{-m}}{x-a}\right)$$

$$= -ma^{-m-1}$$

Note : The above theorem can also be verified if n is a fraction say $n = \frac{p}{q}$ where $q \ne 0$. Then

$$\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) = \lim_{x\to a}\left(\frac{x^{\frac{p}{q}} - a^{\frac{p}{q}}}{x-a}\right) = \frac{p}{q}\,a^{\frac{p}{q}-1}$$

SOLVED EXAMPLES

Ex. 1 : Evaluate $\lim_{x\to 5}\left(\dfrac{x^4-625}{x-5}\right)$

Solution : $\lim_{x\to 5}\left(\dfrac{x^4-625}{x-5}\right) = \lim_{x\to 5}\left(\dfrac{x^4-5^4}{x-5}\right)$

$= 4(5)^{4-1}\ \ldots\ldots \left[\because \lim_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}\right]$

$= 4(125) = 500$

Ex. 2 : Evaluate $\lim_{x\to 2}\left(\dfrac{x^7-128}{x^5-32}\right)$

Solution : $\lim_{x\to 2}\left(\dfrac{x^7-128}{x^5-32}\right) = \lim_{x\to 2}\left(\dfrac{x^7-2^7}{x^5-2^5}\right)$

$$= \lim_{x\to 5}\left(\dfrac{\dfrac{x^7-2^7}{x-2}}{\dfrac{x^5-2^5}{x-2}}\right)$$

... [As $x\to 2$, $x-2\neq 0$]

$$= \frac{7(2)^6}{5(2)^4}$$

$\ldots\left[\because \lim_{x\to a}\dfrac{x^n-a^n}{x-a} = na^{n-1}\right]$

$$= \frac{7(2)^2}{5}$$

$$= \frac{28}{5}$$

Ex. 3 : If $\lim_{x\to 4}\left[\dfrac{x^n-4^n}{x-4}\right] = 48$ and $n\in N$, find n.

Solution : Given $\lim_{x\to 4}\left[\dfrac{x^n-4^n}{x-4}\right] = 48$

$\therefore\ n(4)^{n-1} = 48$

$\therefore\ n(4)^{n-1} = 3(4)^{3-1}$

$\therefore\ n = 3$ .... by comparing

Ex. 4 : Evaluate $\lim_{x\to 1}\left[\dfrac{2x-2}{\sqrt[3]{26+x}-3}\right]$

Solution: Put $26 + x = t^3$, $\therefore\ x = t^3 - 26$

As $x\to 1$, $t\to 3$

$$\therefore \lim_{x\to 1}\left[\frac{2x-2}{\sqrt[3]{26+x}-3}\right]$$

$$= \lim_{t\to 3}\left[\frac{2\left(t^3-26\right)-2}{\sqrt[3]{t^3}-3}\right]$$

$$= \lim_{x\to 3}\left[\frac{2\left(t^3-3^3\right)}{t-3}\right]$$

$$= 2\lim_{t\to 3}\left[\frac{t^3-3^3}{t-3}\right]$$

$= 2\times 3(3)^{3-1}$

$= 54$

EXERCISE 7.1

Q.I Evaluate the following limits :

1. $\lim_{z\to -3}\left[\dfrac{\sqrt{z+6}}{z}\right]$

2. $\lim_{y\to -3}\left[\dfrac{y^5+243}{y^3+27}\right]$

3. $\lim_{z\to -5}\left[\dfrac{\left(\dfrac{1}{z}+\dfrac{1}{5}\right)}{z+5}\right]$

Q.II Evaluate the following limits :

1. $= \lim_{x\to 3}\left[\dfrac{\sqrt{2x+6}}{x}\right]$

2. $\lim_{x \to 2}\left[\frac{x^{-3}-2^{-3}}{x-2}\right]$

3. $\lim_{x \to 5}\left[\frac{x^{3}-125}{x^{5}-3125}\right]$

4. If $\lim_{x \to 1}\left[\frac{x^{4}-1}{x-1}\right]=\lim_{x \to a}\left[\frac{x^{3}-a^{3}}{x-a}\right]$,

find all possible values of a.

Q.III Evaluate the following limits :

1. $\lim_{x \to 1}\left[\frac{x+x^{2}+x^{3}+........+x^{n}-n}{x-1}\right]$

2. $\lim_{x \to 7}\left[\frac{\left(\sqrt[3]{x}-\sqrt[3]{7}\right)\left(\sqrt[3]{x}+\sqrt[3]{7}\right)}{x-7}\right]$

3. If $\lim_{x \to 5}\left[\frac{x^{k}-5^{k}}{x-5}\right]=500$, find all possible values of k.

4. $\lim_{x \to 0}\left[\frac{(1-x)^{8}-1}{(1-x)^{2}-1}\right]$

5. $\lim_{x \to 0}\left[\frac{\sqrt[3]{1+x}-\sqrt{1+x}}{x}\right]$

6. $\lim_{y \to 1}\left[\frac{2y-2}{\sqrt[3]{7+y}-2}\right]$

7. $\lim_{z \to a}\left[\frac{(z+2)^{\frac{3}{2}}-(a+2)^{\frac{3}{2}}}{z-a}\right]$

8. $\lim_{x \to 7}\left[\frac{x^{3}-343}{\sqrt{x}-\sqrt{7}}\right]$

9. $\lim_{x \to 1}\left(\frac{x+x^{3}+x^{5}+...+x^{2n-1}-n}{x-1}\right)$

Q.IV In the following examples, given $\epsilon > 0$, find a $\delta > 0$ such that whenever, $|x-a| < \delta$, we must have $|f(x)-l| < \epsilon$

1. $\lim_{x \to 2}(2x+3)=7$
2. $\lim_{x \to -3}(3x+2)=-7$
3. $\lim_{x \to 2}(x^{2}-1)=3$
4. $\lim_{x \to 1}(x^{2}+x+1)=3$

7.2 METHOD OF FACTORIZATION:

$P(x)$ and $Q(x)$ are polynomials in x such that $f(x)=\frac{P(x)}{Q(x)}$. We consider $\lim_{x \to a} f(x)$.

Let's check $\lim_{x \to a} Q(x)$ and $\lim_{x \to a} P(x)$.

1) If $\lim_{x \to a} Q(x) = m \neq 0$

then $\lim_{x \to a} f(x) = \lim_{x \to a}\left[\frac{P(x)}{m}\right]$

2) If $\lim_{x \to a} Q(x)=0$, then $(x-a)$ divides $Q(x)$. In such a case if $(x-a)$ does not divide $P(x)$ then $\lim_{x \to a} f(x)$ does not exist.

(3) Further if $\lim_{x \to a} P(x)$ is also 0, then $(x-a)$ is a factor of both $P(x)$ and Q(x).

So $\lim_{x \to a} f(x) = \lim_{x \to a}\left[\frac{P(x)/(x-a)}{Q(x)/(x-a)}\right]$.

Factorization of polynomials is a useful tool to determine the limits of rational algebraic expressions.

SOLVED EXAMPLES

Ex.1 : Evaluate $\lim_{z \to 3}\left[\frac{z(2z-3)-9}{z^{2}-4z+3}\right]$

Solution: If we substitute $z=3$ in numerator and denominator, we get $z(2z-3) - 9 = 0$ and $z^2 - 4z + 3 = 0$

So $(z - 3)$ is a factor in the numerator and denominator.

$$\lim_{z\to 3}\left[\frac{z(2z-3)-9}{z^2-4z+3}\right] = \lim_{z\to 3}\left[\frac{2z^2-3z-9}{z^2-4z+3}\right]$$

$$= \lim_{z\to 3}\left[\frac{(z-3)(2z+3)}{(z-3)(z-1)}\right]$$

$$= \lim_{z\to 3}\left[\frac{(2z+3)}{(z-1)}\right]$$

...[As $z\to 3$, $z-3\neq 0$]

$$= \frac{2(3)+3}{3-1}$$

$$= \frac{9}{2}$$

Ex.2: Evaluate $\lim_{x\to 4}\left[\frac{(x^3-8x^2+16x)^9}{(x^2-x-12)^{18}}\right]$

Solution : $\lim_{x\to 4}\left[\frac{\left[x(x-4)^2\right]^9}{(x-4)^{18}(x+3)^{18}}\right]$

$$= \lim_{x\to 4}\left[\frac{(x-4)^{18}x^9}{(x-4)^{18}(x+3)^{18}}\right]$$

$$= \lim_{x\to 4}\left[\frac{x^9}{(x+3)^{18}}\right]$$

...[As $x\to 4$, $x-4\neq 0$]

$$= \frac{4^9}{7^{18}}$$

Ex.3: Evaluate $\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{2}{1-x^2}\right]$

Solution : $\lim_{x\to 1}\left[\frac{1}{x-1}+\frac{2}{(1-x)(x+1)}\right]$

$$= \lim_{x\to 1}\left[\frac{1}{x-1}-\frac{2}{(x-1)(x+1)}\right]$$

$$= \lim_{x\to 1}\left[\frac{1+x-2}{(x-1)(x+1)}\right]$$

$$= \lim_{x\to 1}\left[\frac{x-1}{(x-1)(x+1)}\right]$$

$$= \lim_{x\to 1}\left[\frac{1}{(x+1)}\right]$$

...[As $x\to 1$, $x-1\neq 0$]

$$= \frac{1}{2}$$

Ex.4: Evaluate $\lim_{x\to 1}\left[\frac{x^3+x^2-5x+3}{x^2-1}\right]$

Solution : In this case $(x - 1)$ is a factor of the numerator and denominator.

To find another factor we use synthetic division.

Numerator: $x^3 + x^2 - 5x + 3$

1	1	1	–5	3
		1	2	–3
	1	2	–3	0

$\therefore x^3 + x^2 - 5x + 3 = (x-1)(x^2 + 2x - 3)$

Denominator: $x^2 - 1 = (x + 1)(x - 1)$

$$\lim_{x\to 1}\left[\frac{x^3+x^2-5x+3}{x^2-1}\right]$$

$$= \lim_{x\to 1}\left[\frac{(x-1)(x^2+2x-3)}{(x+1)(x-1)}\right]$$

$$= \lim_{x\to 1}\left[\frac{x^2+2x-3}{x+1}\right]$$ ...[As $x\to 1$, $x-1\neq 0$]

$$= \frac{1+2-3}{1+1} = 0$$

Ex.5: Evaluate $\lim_{x\to 0}\left(\frac{1-\sqrt[3]{x^2+1}}{x^2}\right)$

Solution : Put $\sqrt[3]{x^2+1}$ = t, $x^2 + 1 = t^3$

$\therefore x^2 = t^3 - 1$, as $x\to 0$, $t\to 1$

$$\lim_{x\to 0}\left(\frac{1-\sqrt[3]{x^2+1}}{x^2}\right) = \lim_{t\to 1}\left(\frac{(1-t)}{(t^3-1)}\right)$$

$$= \lim_{t\to 1}\left(\frac{-(t-1)}{(t-1)(t^2+t+1)}\right)$$

$$= \lim_{t\to 1}\left(\frac{-1}{t^2+t+1}\right)$$

...[As $t\to 1$, $t - 1 \neq 0$]

$$= \frac{-1}{1+1+1} = -\frac{1}{3}$$

EXERCISE 7.2

Q.I Evaluate the following limits :

1. $\lim_{z\to 2}\left[\frac{z^2-5z+6}{z^2-4}\right]$

2. $\lim_{x\to -3}\left[\frac{x+3}{x^2+4x+3}\right]$

3. $\lim_{y\to 0}\left[\frac{5y^3+8y^2}{3y^4-16y^2}\right]$

4. $\lim_{x\to -2}\left[\frac{-2x-4}{x^3+2x^2}\right]$

5. $\lim_{x\to 3}\left[\frac{x^2+2x-15}{x^2-5x+6}\right]$

Q.II Evaluate the following limits :

1. $\lim_{u\to 1}\left[\frac{u^4-1}{u^3-1}\right]$

2. $\lim_{x\to 3}\left[\frac{1}{x-3}-\frac{9x}{x^3-27}\right]$

3. $\lim_{x\to 2}\left[\frac{x^3-4x^2+4x}{x^2-1}\right]$

4. $\lim_{\Delta x\to 0}\left[\frac{(x+\Delta x)^2-2(x+\Delta x)+1-(x^2-2x+1)}{\Delta x}\right]$

5. $\lim_{x\to \sqrt{2}}\left[\frac{x^2+x\sqrt{2}-4}{x^2-3x\sqrt{2}+4}\right]$

6. $\lim_{x\to 2}\left[\frac{x^3-7x+6}{x^3-7x^2+16x-12}\right]$

Q.III Evaluate the Following limits :

1. $\lim_{y\to \frac{1}{2}}\left[\frac{1-8y^3}{y-4y^3}\right]$

2. $\lim_{x\to 1}\left[\frac{x-2}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$

3. $\lim_{x\to 1}\left[\frac{x^4-3x^2+2}{x^3-5x^2+3x+1}\right]$

4. $\lim_{x\to 1}\left[\frac{x+2}{x^2-5x+4}+\frac{x-4}{3(x^2-3x+2)}\right]$

5. $\lim_{x\to a}\left[\frac{1}{x^2-3ax+2a^2}+\frac{1}{2x^2-3ax+a^2}\right]$

7.3 METHOD OF RATIONALIZATION:

If the function in the limit involves a square root or a trigonometric function, it may be possible to simplify the expression by multiplying and dividing by its rationalizing factor.

SOLVED EXAMPLES

Ex. 1. Evaluate : $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$

Soluton : $\lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\right)$

$$= \lim_{x\to 0}\left(\frac{\sqrt{1+x}-1}{x}\times\frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}\right)$$

$$= \lim_{x\to 0}\left(\frac{1+x-1}{x\left(\sqrt{1+x}+1\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{x}{x\left(\sqrt{1+x}+1\right)}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{\sqrt{1+x}+1}\right) \quad \text{...[As } x\to 0,\ x\neq 0]$$

$$= \frac{1}{\sqrt{1+0}+1} = \frac{1}{2}$$

Ex. 2. Evaluate $\lim_{z\to 0}\left[\frac{(b+z)^{\frac{1}{2}}-(b-z)^{\frac{1}{2}}}{z}\right]$

Solution : $\lim_{z\to 0}\left[\frac{(b+z)^{\frac{1}{2}}-(b-z)^{\frac{1}{2}}}{z}\right]$

$$= \lim_{z\to 0}\left[\frac{\sqrt{b+z}-\sqrt{b-z}}{z}\right]$$

$$= \lim_{z\to 0}\left[\frac{\sqrt{b+z}-\sqrt{b-z}}{z}\times\frac{\sqrt{b+z}+\sqrt{b-z}}{\sqrt{b+z}+\sqrt{b-z}}\right]$$

$$= \lim_{z\to 0}\left[\frac{(b+z)-(b-z)}{z}\times\frac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$$

$$= \lim_{z\to 0}\left[\frac{2z}{z}\times\frac{1}{\sqrt{b+z}+\sqrt{b-z}}\right]$$

$$= \lim_{z\to 0}\left[\frac{2}{\sqrt{b+z}+\sqrt{b-z}}\right] \quad \text{...[As } z\to 0,\ z\neq 0]$$

$$= \frac{2}{\sqrt{b+0}+\sqrt{b-0}}$$

$$= \frac{2}{2\sqrt{b}}$$

$$= \frac{1}{\sqrt{b}}$$

Ex. 3. Evaluate $\lim_{x\to 4}\left(\frac{x^2+x-20}{\sqrt{x^2-7}-\sqrt{25-x^2}}\right)$

Solution : $\lim_{x\to 4}\left(\frac{x^2+x-20}{\sqrt{x^2-7}-\sqrt{25-x^2}}\right)$

$$= \lim_{x\to 4}\left(\frac{x^2+x-20}{\sqrt{x^2-7}-\sqrt{25-x^2}}\times\frac{\sqrt{x^2-7}+\sqrt{25-x^2}}{\sqrt{x^2-7}+\sqrt{25-x^2}}\right)$$

$$= \lim_{x\to 4}\left(\frac{(x-4)(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{x^2-7-25+x^2}\right)$$

$$= \lim_{x\to 4}\left(\frac{(x-4)(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{2(x^2-16)}\right)$$

$$= \lim_{x\to 4}\left(\frac{(x-4)(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{2(x-4)(x+4)}\right)$$

$$= \lim_{x \to 4}\left(\frac{(x+5)\left(\sqrt{x^2-7}+\sqrt{25-x^2}\right)}{2(x+4)}\right)$$

...[As $x \to 4$, $x - 4 \neq 0$]

$$= \frac{(4+5)\left(\sqrt{4^2-7}+\sqrt{25-4^2}\right)}{2(4+4)} = \frac{(9)(3+3)}{2(8)} = \frac{27}{8}$$

EXERCISE 7.3

Q.I Evaluate the following limits :

1. $\lim_{x \to 0}\left[\frac{\sqrt{6+x+x^2}-\sqrt{6}}{x}\right]$

2. $\lim_{x \to 3}\left[\frac{\sqrt{2x+3}-\sqrt{4x-3}}{x^2-9}\right]$

3. $\lim_{y \to 0}\left[\frac{\sqrt{1-y^2}-\sqrt{1+y^2}}{y^2}\right]$

4. $\lim_{x \to 2}\left[\frac{\sqrt{2+x}-\sqrt{6-x}}{\sqrt{x}-\sqrt{2}}\right]$

Q.II Evaluate the following limits :

1. $\lim_{x \to a}\left[\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right]$

2. $\lim_{x \to 2}\left[\frac{x^2-4}{\sqrt{x+2}-\sqrt{3x-2}}\right]$

3. $\lim_{x \to 2}\left[\frac{\sqrt{1+\sqrt{2+x}}-\sqrt{3}}{x-2}\right]$

4. $\lim_{y \to 0}\left[\frac{\sqrt{a+y}-\sqrt{a}}{y\sqrt{a+y}}\right]$

5. $= \lim_{x \to 0}\left(\frac{\sqrt{x^2+9}-\sqrt{2x^2+9}}{\sqrt{3x^2+4}-\sqrt{2x^2+4}}\right)$

Q.III Evaluate the Following limits :

1. $\lim_{x \to 1}\left[\frac{x^2+x\sqrt{x}-2}{x-1}\right]$

2. $\lim_{x \to 0}\left[\frac{\sqrt{1+x^2}-\sqrt{1+x}}{\sqrt{1+x^3}-\sqrt{1+x}}\right]$

3. $\lim_{x \to 4}\left[\frac{x^2+x-20}{\sqrt{3x+4}-4}\right]$

4. $\lim_{z \to 4}\left[\frac{3-\sqrt{5+z}}{1-\sqrt{5-z}}\right]$

5. $\lim_{x \to 0}\left(\frac{3}{x\sqrt{9-x}}-\frac{1}{x}\right)$

7.4 LIMIT OF A TRIGONOMETRIC FUNCTION :

Let's Learn :

To evaluate the limits involving trigonometric functions, we state -

1) $\lim_{x \to a} sinx = sina$

2) $\lim_{x \to a} cosx = cosa$

Using these results and trigonometric identities, we solve some examples.

Evaluation of limits can be done by the method of Factorization, Rationalization or Simplification as the case may be. While solving examples based on trigonometric functions we can use trigonometric identities.

Squeeze theorem (Also known as **Sandwich theorem)**

Suppose $f(x)$, $g(x)$ and $h(x)$ are given functions such that $f(x) \leq g(x) \leq h(x)$ for all x in an open interval about a.

Suppose $\lim\limits_{x\to a} f(x) = L$ and $\lim\limits_{x\to a} h(x) = L$

So, $\lim\limits_{x\to a} f(x) \leq \lim\limits_{x\to a} g(x) \leq \lim\limits_{x\to a} h(x)$

$\Rightarrow L \leq \lim\limits_{x\to a} g(x) \leq L \quad \therefore \lim\limits_{x\to a} g(x) = L$

7.4.2 Theorem : $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$; where θ is measured in radian.

Proof : First consider the case when θ is tending to zero through positive values.

We may take $0 < \theta < \dfrac{\pi}{2}$.

Draw a standard circle with radius r i.e. circles with centre at origin O and radius r.

Let A be the point of intersection of the circle and the X-axis. Take point P on the circle such that $m\angle AOP = \theta$

Draw PM ⊥ OX. Draw a line through A parallel to Y-axis to meet OP extended at B (fig. 7.2)

Area of ΔOAP < Area of sector OAP < Area of ΔOAB

$\therefore \dfrac{1}{2}OA.PM < \dfrac{1}{2}r^2\theta < \dfrac{1}{2}OA.OB \qquad ..... (I)$

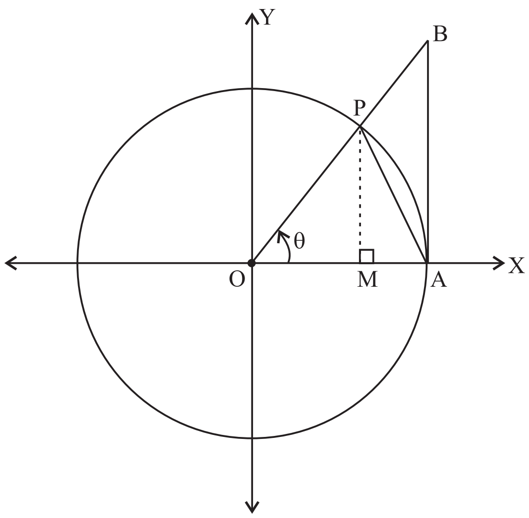


Fig. 7.3

In ΔOAP,

$$\sin\theta = \frac{PM}{OP}$$

$\therefore \quad PM = OP \sin\theta$

$= r \sin\theta$

Also, in ΔOAB

$$\tan\theta = \frac{AB}{OA}$$

$\therefore \quad AB = OA \tan\theta$

$= r \tan\theta$

using these in (I), we get

$$\frac{1}{2} r.r. \sin\theta < \frac{1}{2} r^2 \theta < \frac{1}{2} r.r \tan\theta$$

i.e. $1 < \dfrac{\theta}{\sin\theta} < \dfrac{1}{\cos\theta}$ ... [Divide by $\dfrac{1}{2}r^2 \sin\theta$]

$\therefore \quad 1 > \dfrac{\sin\theta}{\theta} > \cos\theta$

i.e. $\cos\theta < \dfrac{\sin\theta}{\theta} < 1$

Taking limit as $\theta \to 0^+$

$\therefore \quad \lim\limits_{\theta\to 0^+} \cos\theta \leq \lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} \leq \lim\limits_{\theta\to 0^+} 1$

$\therefore \quad 1 \leq \lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} \leq 1$

By using squeeze theorem

$\therefore \quad \lim\limits_{\theta\to 0^+} \dfrac{\sin\theta}{\theta} = 1 \qquad .... (II)$

Now suppose θ→0 through negative values

Let θ = – ϕ where ϕ > 0. Also as θ → 0, ϕ→0

$\therefore \quad \lim\limits_{\theta\to 0^-} \dfrac{\sin\theta}{\theta} = \lim\limits_{\phi\to 0^+} \dfrac{\sin(-\phi)}{-\phi}$

$= \lim\limits_{\phi\to 0^+} \dfrac{-\sin\phi}{-\phi} = \lim\limits_{\phi\to 0^+} \dfrac{\sin\phi}{\phi} = 1$

$\therefore \lim\limits_{\theta \to 0^-} \dfrac{\sin\theta}{\theta} = 1$ .... (III)

$\therefore$ from (II) and (III), $\lim\limits_{\theta \to 0} \dfrac{\sin\theta}{\theta} = 1$

Note:

Corollary 1 : $\lim\limits_{\theta \to 0}\left(\dfrac{\theta}{\sin\theta}\right)=1$

Corollary 2 : $\lim\limits_{\theta \to 0}\left(\dfrac{\tan\theta}{\theta}\right)=1$

Corollary 3 : $\lim\limits_{\theta \to 0}\left(\dfrac{\theta}{\tan\theta}\right)=1$

Corollary 4 : $\lim\limits_{\theta \to 0}\left(\dfrac{\sin p\theta}{p\theta}\right)=1$, ($p$ constant.)

Corollary 5 : $\lim\limits_{\theta \to 0}\left(\dfrac{\tan p\theta}{p\theta}\right)=1$, ($p$ constant.)

Corollary 6 : $\lim\limits_{\theta \to 0}\left(\dfrac{p\theta}{\sin p\theta}\right)=1$, ($p$ constant.)

Corollary 7 : $\lim\limits_{\theta \to 0}\left(\dfrac{p\theta}{\tan p\theta}\right)=1$, ($p$ constant.)

SOLVED EXAMPLES

Ex. 1) If $3x^2 + 2 \le f(x) \le 5x^2 - 6$ for all $x \in R$, then find $\lim\limits_{x \to -2} f(x)$.

Solution : Let $g(x) = 3x^2 + 2$ and $h(x) = 5x^2 - 6$

So, we have $g(x) \le f(x) \le h(x)$

Taking as limit $x \to -2$ throughout we get

$\lim\limits_{x \to -2} g(x) \le \lim\limits_{x \to -2} f(x) \le \lim\limits_{x \to -2} h(x)$

$\lim\limits_{x \to -2} (3x^2+2) \le \lim\limits_{x \to -2} f(x) \le \lim\limits_{x \to -2} (5x^2 - 6)$

$3(-2)^2 + 2 \le \lim\limits_{x \to -2} f(x) \le 5(-2)^2 - 6$

$14 \le \lim\limits_{x \to -2} f(x) \le 14$

$\therefore \lim\limits_{x \to -2} f(x) = 14$ .... [By squeeze theorem]

Ex. 2 : Evaluate : $\lim\limits_{x \to \frac{\pi}{4}}\left(\dfrac{\cot x - 1}{\operatorname{cosec}^2 x - 2}\right)$

Solution : $\lim\limits_{x \to \frac{\pi}{4}}\left(\dfrac{\cot x - 1}{\operatorname{cosec}^2 x - 2}\right)$

$= \lim\limits_{x \to \frac{\pi}{4}}\left(\dfrac{\cot x - 1}{\cot^2 x + 1 - 2}\right)$

$= \lim\limits_{x \to \frac{\pi}{4}}\left(\dfrac{\cot x - 1}{\cot^2 x - 1}\right)$

$= \lim\limits_{x \to \frac{\pi}{4}}\left(\dfrac{\cot x - 1}{(\cot x + 1)(\cot x - 1)}\right)$

$= \lim\limits_{x \to \frac{\pi}{4}}\left(\dfrac{1}{(\cot x + 1)}\right)$

.....[As $x \; \frac{\pi}{4}$ $\cot x - 1 \neq 0$]

$= \dfrac{1}{\cot\left(\frac{\pi}{4}\right) + 1} = \dfrac{1}{1+1}$

$= \dfrac{1}{2}$

Ex. 3) Evaluate : $\lim\limits_{x \to \frac{3\pi}{2}}\left(\dfrac{\sqrt{3 - \sin x} - 2}{\cos^2 x}\right)$

Solution : $\lim\limits_{x \to \frac{3\pi}{2}}\left(\dfrac{\sqrt{3 - \sin x} - 2}{\cos^2 x}\right)$

$= \lim\limits_{x \to \frac{3\pi}{2}}\left(\dfrac{\sqrt{3 - \sin x} - 2}{\cos^2 x} \times \dfrac{\sqrt{3 - \sin x} + 2}{\sqrt{3 - \sin x} + 2}\right)$

$= \lim\limits_{x \to \frac{3\pi}{2}}\left(\dfrac{3 - \sin x - 4}{\cos^2 x\left(\sqrt{3 - \sin x} + 2\right)}\right)$

$= \lim\limits_{x \to \frac{3\pi}{2}}\left(\dfrac{-1 - \sin x}{1 - \sin^2 x\left(\sqrt{3 - \sin x} + 2\right)}\right)$

$= \lim\limits_{x \to \frac{3\pi}{2}}\left(\dfrac{-(1 + \sin x)}{(1 - \sin x)(1 + \sin x)\left(\sqrt{3 - \sin x} + 2\right)}\right)$

.....As $x \to \dfrac{3\pi}{2}$, $\sin x \to -1$ and $1 + \sin x \neq 0$

$$= \left(\frac{-1}{\left(1-\sin\left(\frac{3\pi}{2}\right)\right)\left(\sqrt{3-\sin\left(\frac{3\pi}{2}\right)}+2\right)} \right)$$

$$= \frac{-1}{(1+1)\left(\sqrt{3+1}+2\right)} = -\frac{1}{8}$$

Ex. 4. Evaluate : $\lim\limits_{\theta\to 0}\left[\frac{\sin 7\theta}{\theta}\right]$

Solution : $\lim\limits_{\theta\to 0}\left[\frac{\sin 7\theta}{\theta}\right]$

$$= \lim_{\theta\to 0}\left[\frac{\sin 7\theta}{7\theta}\right] \times 7$$

... as $\theta \to 0,\ 7\theta \to 0$

$$= 1 \times 7 \quad \left[\because \lim_{\theta\to 0}\frac{\sin p\theta}{p\theta} = 1\right]$$

$$= 7$$

Ex. 5. Evaluate : $\lim\limits_{x\to 0}\left[\frac{\sin 8x}{\tan 4x}\right]$

Solution : $\lim\limits_{x\to 0}\left[\frac{\sin 8x}{\tan 4x}\right]$

Divide Numerator and Denominator by x

$$= \lim_{x\to 0}\left[\frac{\frac{\sin 8x}{x}}{\frac{\tan 4x}{x}}\right]$$

$$= \frac{\lim\limits_{x\to 0}\left[\frac{\sin 8x}{x}\right]}{\lim\limits_{x\to 0}\left[\frac{\tan 4x}{x}\right]}$$

$$= \frac{\lim\limits_{x\to 0}\left[\frac{\sin 8x}{8x}\times 8\right]}{\lim\limits_{x\to 0}\left[\frac{\tan 4x}{4x}\times 4\right]}$$

$$= \frac{1\times 8}{1\times 4} = 2 \quad (\text{... as } x\to 0,\ 8x\to 0,\ 4x\to 0)$$

$$\text{....} \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right) = 1 \quad \lim_{\theta\to 0}\left(\frac{\tan p\theta}{p\theta}\right) = 1$$

Ex. 6. Evaluate : $\lim\limits_{x\to 0}\left[\frac{2\sin x - \sin 2x}{x^3}\right]$

Solution : $\lim\limits_{x\to 0}\left[\frac{2\sin x - \sin 2x}{x^3}\right]$

$$= \lim_{x\to 0}\left[\frac{2\sin x - 2\sin x.\cos x}{x^3}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sin x(1-\cos x)}{x.x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{2\sin x}{x}\right]\times\lim_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\right]$$

$$= 2\times\lim_{x\to 0}\left[\frac{\sin x}{x}\right]\times\lim_{x\to 0}\left[\frac{(1-\cos x)}{x^2}\times\frac{(1+\cos x)}{(1+\cos x)}\right]$$

$$= 2\times 1\times\lim_{x\to 0}\left[\frac{(1-\cos^2 x)}{x^2}\times\frac{1}{(1+\cos x)}\right]$$

$$= 2\times\lim_{x\to 0}\left[\frac{(\sin^2 x)}{x^2}\right]\times\lim_{x\to 0}\left[\frac{1}{(1+\cos x)}\right]$$

$$= 2\times\lim_{x\to 0}\left[\frac{\sin x}{x}\right]^2\times\lim_{x\to 0}\left[\frac{1}{(1+\cos 0)}\right]$$

$$= 2(1)^2\times\left[\frac{1}{1+1}\right] \quad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$$= 2\times 1\times\frac{1}{2} = 1$$

Ex. 7. Evaluate : $\lim\limits_{x\to 0}\left[\frac{\sin x^2(1-\cos x^2)}{x^6}\right]$

Solution : $\lim\limits_{x\to 0}\left[\frac{\sin x^2(1-\cos x^2)}{x^6}\right]$

$$= \lim_{x\to 0}\left[\frac{\sin x^2(1-\cos x^2)}{x^6}\times\frac{1+\cos x^2}{1+\cos x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin x^2(1-\cos^2 x^2)}{x^6}\times\frac{1}{1+\cos x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin x^2 . \sin^2 x^2}{x^6}\times\frac{1}{1+\cos x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\sin^3 x^2}{\left(x^2\right)^3}\times\frac{1}{1+\cos x^2}\right]$$

$$= \lim_{x^2\to 0}\left[\frac{\sin x^2}{x^2}\right]^3\times\lim_{x\to 0}\left(\frac{1}{1+\cos x^2}\right)$$

.....[As $x\to 0$, $x^2\to 0$]

$$= (1)^3\ \frac{1}{1+\cos 0} = \frac{1}{1+1} = \frac{1}{2}$$

.....$\left[\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1, \textit{here } \theta = x^2\right]$

Ex. 8. Evaluate : $\lim_{x\to 0}\left(\frac{\cos 5x^o-\cos 3x^o}{x^2}\right)$

Solution : $\lim_{x\to 0}\left(\frac{\cos 5x^o-\cos 3x^o}{x^2}\right)$

$$= \lim_{x\to 0}\left(\frac{-2\sin 4x^o\sin x^o}{x^2}\right)$$

$$= -\ 2\lim_{x\to 0}\left(\frac{\sin 4x^o}{x}\times\frac{\sin x^o}{x}\right)$$

$$= -\ 2\lim_{x\to 0}\left(\frac{\sin 4\left(\frac{\pi x}{180}\right)}{x}\times\frac{\sin\left(\frac{\pi x}{180}\right)}{x}\right)$$

$$= -2\lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi x}{45}\right)}{x}\times\frac{\sin\left(\frac{\pi x}{180}\right)}{x}\right)$$

$$= -2\lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi x}{45}\right)}{\frac{\pi x}{45}}\right)\times\frac{\pi}{45}\times\lim_{x\to 0}\left(\frac{\sin\left(\frac{\pi x}{180}\right)}{\frac{\pi x}{180}}\right)\times\frac{\pi}{180}$$

$$= -2\times(1)\times\frac{\pi}{45}\times(1)\times\frac{\pi}{180}\ ......\left[\because \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right)=1\right]$$

$$= -2\times 4\times\frac{\pi}{180}\times\frac{\pi}{180} = -8\left(\frac{\pi}{180}\right)^2$$

Activity-1 $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$

Solution : $\lim_{x\to 0}\frac{\tan x-\sin x}{\sin^3 x}$

$$= \lim_{x\to 0}\left[\frac{\sin x\times\frac{1}{}-\sin x}{\sin^3 x}\right]$$

$$= \lim_{x\to 0}\frac{\sin x\left(\frac{1}{\cos x}-\square\right)}{\sin^3 x}$$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)}{\cos x}\times\frac{1}{\sin^2 x}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)}{\cos x}\times\frac{1}{1-\square}\right]$$

$$= \lim_{x\to 0}\left[\frac{(1-\cos x)}{\cos x}\times\frac{1}{(\square)(1+\cos x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{\cos x}\times\frac{1}{(1+\cos x)}\right]$$

$$= \left[\frac{1}{\cos\square}\times\frac{1}{(1+\cos\square)}\right]$$

$$= \frac{1}{\square}\times\frac{1}{(1+\square)}$$

$$= \frac{1}{1}\times\frac{1}{\square}$$

$$= \frac{1}{\square}$$

EXERCISE 7.4

Q.I Evaluate the following limits :

1. $\lim_{\theta \to 0}\left[\frac{\sin(m\theta)}{\tan(n\theta)}\right]$

2. $\lim_{\theta \to 0}\left[\frac{1-\cos 2\theta}{\theta^2}\right]$

3. $\lim_{x \to 0}\left[\frac{x.\tan x}{1-\cos x}\right]$

4. $\lim_{x \to 0}\left(\frac{\sec x - 1}{x^2}\right)$

Q.II Evaluate the Following limits :

1. $\lim_{x \to 0}\left[\frac{1-\cos(nx)}{1-\cos(mx)}\right]$

2. $\lim_{x \to \frac{\pi}{6}}\left[\frac{2-\operatorname{cosec} x}{\cot^2 x - 3}\right]$

3. $\lim_{x \to \frac{\pi}{4}}\left[\frac{\cos x - \sin x}{\cos 2x}\right]$

Q.III Evaluate the following limits :

1. $\lim_{x \to 0}\left[\frac{\cos(ax)-\cos(bx)}{\cos(cx)-1}\right]$

2. $\lim_{x \to \pi}\left[\frac{\sqrt{1-\cos x}-\sqrt{2}}{\sin^2 x}\right]$

3. $\lim_{x \to \frac{\pi}{4}}\left[\frac{\tan^2 x - \cot^2 x}{\sec x - \operatorname{cosec} x}\right]$

4. $\lim_{x \to \frac{\pi}{6}}\left[\frac{2\sin^2 x + \sin x - 1}{2\sin^2 x - 3\sin x + 1}\right]$

Activity-1:

$$\lim_{x \to 0} \frac{1-\cos px}{x^2} = \frac{1}{2}\, p^2$$

Consider,

$$\lim_{x \to 0} \frac{1-\cos px}{x^2}$$

$$= \lim_{x \to 0} \frac{2.\boxed{}}{x^2}$$

Use $1-\cos A = 2\sin^2 \frac{A}{2}$

$$= \lim_{x \to 0} 2.\left(\frac{\boxed{}}{x}\right)^2$$

$$= \lim_{x \to 0} 2\left(\frac{\sin\frac{px}{2}}{\frac{px}{2}} \times \square\right)^2$$

$$= \lim_{x \to 0} 2\left(\frac{\sin\frac{px}{2}}{\frac{px}{2}}\right)^2 \left(\frac{p^2}{4}\right)$$

$$= 2\,(1)^2\left(\frac{p^2}{4}\right) \qquad \because \lim_{x \to 0} \frac{\sin\theta}{\theta} = 1$$

$$= \square$$

$$\lim_{x \to 0} \frac{1-\cos px}{x^2} = \frac{1}{2}\, p^2$$

7.5 Substitution Method :

We will consider examples of trigonometric functions in which $x \to a$ where generally a takes the values such as $\pi, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ etc. In such a case we put $x - a = t$ so that as $x \to a$, $t \to o$.

SOLVED EXAMPLE

Ex. 1. Evaluate $\lim\limits_{x\to\frac{\pi}{2}}\left[\dfrac{\cos x}{x-\frac{\pi}{2}}\right]$

Solution : Put $x-\dfrac{\pi}{2}=t \quad \therefore x=\dfrac{\pi}{2}+t$

As $x\to\dfrac{\pi}{2}$; $t\to 0$.

$$\lim_{x\to\frac{\pi}{2}}\left[\frac{\cos x}{x-\frac{\pi}{2}}\right]=\lim_{t\to0}\left[\frac{\cos\left(\frac{\pi}{2}+t\right)}{t}\right]$$

$$=\lim_{t\to0}\left[\frac{-\sin t}{t}\right]$$

$$=-\lim_{t\to0}\left[\frac{\sin t}{t}\right]$$

$$=-1 \qquad \left[\because \lim_{\theta\to0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

Ex. 2. Evaluate $\lim\limits_{x\to a}\left[\dfrac{\cos x-\cos a}{x-a}\right]$

Solution : Put $x-a=t \quad \therefore x=a+t$;

As $x\to a$, $t\to 0$.

$$\lim_{x\to a}\left[\frac{\cos x-\cos a}{x-a}\right]=\lim_{t\to0}\left[\frac{\cos(a+t)-\cos a}{t}\right]$$

$$=\lim_{t\to0}\left[\frac{-2\sin\left(\frac{2a+t}{2}\right)\cdot\sin\frac{t}{2}}{t}\right]$$

$$=2\lim_{t\to0}\left[-\sin\left(a+\frac{t}{2}\right).\frac{\sin(t/2)}{t}\right]$$

$$=-2\lim_{t\to0}\left[\sin\left(a+\frac{t}{2}\right)\right]\lim_{t\to0}\left[\frac{\sin(t/2)}{t/2}\right]\left(\frac{1}{2}\right)$$

$$=-2\sin(a+0).(1)\,\frac{1}{2}\ \ldots\ \left[\because \lim_{\theta\to0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$$=-\sin a$$

Ex. 3. Evaluate $\lim\limits_{x\to1}\left[\dfrac{1+\cos\pi x}{(1-x)^2}\right]$

Solution : Put $1-x=t \quad \therefore x=1-t$;

As $x\to1$, $t\to0$.

$$\lim_{x\to1}\left[\frac{1+\cos\pi x}{(1-x)^2}\right]=\lim_{t\to0}\left[\frac{1+\cos[\pi(1-t)]}{t^2}\right]$$

$$=\lim_{t\to0}\left[\frac{1+\cos(\pi-\pi t)}{(1-x)^2}\right]$$

$$=\lim_{t\to0}\left[\frac{1-\cos\pi t}{t^2}\right]$$

$$=\lim_{t\to0}\left[\frac{2\sin^2\left(\frac{\pi t}{2}\right)}{t^2}\right]$$

$$=2\lim_{t\to0}\left[\frac{\sin\left(\frac{\pi t}{2}\right)}{t}\right]^2$$

$$=2\lim_{t\to0}\left[\frac{\sin\left(\frac{\pi t}{2}\right)}{\frac{\pi t}{2}}\right]^2\left(\frac{\pi}{2}\right)^2$$

$$=2(1)\left(\frac{\pi^2}{4}\right)\ldots\left[\because \lim_{\theta\to0}\left(\frac{\sin P\theta}{P\theta}\right)=1\right]$$

$$=\frac{\pi^2}{2}$$

Ex. 4. Evaluate $\lim\limits_{x\to\frac{\pi}{3}}\left[\dfrac{\sqrt{3}-\tan x}{\pi-3x}\right]$

Solution : Put $\dfrac{\pi}{3}-x=\text{t}$, $\therefore\ x=\dfrac{\pi}{3}-\text{t}$,

$$\text{As } x\to\frac{\pi}{3},\ \text{t}\to 0$$

$$\lim_{x\to\frac{\pi}{3}}\left(\frac{\sqrt{3}-\tan x}{\pi-3x}\right)=\lim_{x\to\frac{\pi}{3}}\left(\frac{\sqrt{3}-\tan}{3\left(\frac{\pi}{3}-x\right)}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\sqrt{3}-\tan\left(\frac{\pi}{3}-t\right)}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\sqrt{3}-\dfrac{\tan(\pi/3)-\tan t}{1+\tan(\pi/3)\tan t}}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\sqrt{3}-\dfrac{\sqrt{3}-\tan t}{1+\sqrt{3}\tan t}}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{\dfrac{\sqrt{3}+3\tan t-\sqrt{3}+\tan t}{1+\sqrt{3}\tan t}}{t}\right)$$

$$=\frac{1}{3}\lim_{t\to 0}\left(\frac{4\tan t}{t(1+\sqrt{3}\tan t)}\right)$$

$$=\frac{4}{3}\lim_{t\to 0}\left(\frac{\tan t}{t}\right)\lim_{t\to 0}\left(\frac{1}{(1+\sqrt{3}\tan t)}\right)$$

$$=\frac{4}{3}(1)\times\frac{1}{\left(1+\sqrt{3}\tan 0\right)}\ \ldots\ldots\left[\because \lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1\right]$$

$$=\frac{4}{3}$$

EXERCISE 7.5

I) Evaluate the following

1) $\lim\limits_{x\to\frac{\pi}{2}}\left[\dfrac{\operatorname{cosec} x-1}{\left(\frac{\pi}{2}-x\right)^2}\right]$

2) $\dfrac{1}{3}\lim\limits_{x\to a}\dfrac{\sin x-\sin a}{\sqrt[5]{x}-\sqrt[5]{a}}$

3) $\lim\limits_{x\to\pi}\left[\dfrac{\sqrt{5+\cos x}-2}{(\pi-x)^2}\right]$

4) $\lim\limits_{x\to\frac{\pi}{6}}\left[\dfrac{\cos x-\sqrt{3}\sin x}{\pi-6x}\right]$

5) $\lim\limits_{x\to 1}\left[\dfrac{1-x^2}{\sin\pi x}\right]$

II) Evaluate the following

1) $\lim\limits_{x\to\frac{\pi}{6}}\left[\dfrac{2\sin x-1}{\pi-6x}\right]$

2) $\lim\limits_{x\to\frac{\pi}{4}}\left[\dfrac{\sqrt{2}-\cos x-\sin x}{(4x-\pi)^2}\right]$

3) $\lim\limits_{x\to\frac{\pi}{6}}\left[\dfrac{2-\sqrt{3}\cos x-\sin x}{(6x-\pi)^2}\right]$

4) $\lim_{x \to a} \left[\frac{\sin(\sqrt{x}) - \sin(\sqrt{a})}{x - a} \right]$

5) $\lim_{x \to \frac{\pi}{2}} \left[\frac{\cos 3x + 3\cos x}{(2x - \pi)^2} \right]$

7.6 LIMITS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS :

Let's :Learn

We use the following results without proof.

1) $\lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) = \log e = 1$

2) $\lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a \;\; (a > 0,\, a \neq 0)$

3) $\lim_{x \to 0} [1 + x]^{\frac{1}{x}} = e$

4) $\lim_{x \to 0} \left(\frac{\log(1 + x)}{x} \right) = 1$

5) $\lim_{x \to 0} \left(\frac{e^{px} - 1}{px} \right) = 1$, ($p$ constant)

6) $\lim_{x \to 0} \left(\frac{a^{px} - 1}{px} \right) = \log a$, (p constant)

7) $\lim_{x \to 0} \left(\frac{\log(1 + px)}{px} \right) = 1$, ($p$ constant)

8) $\lim_{x \to 0} (1 + px)^{\frac{1}{px}} = e$, ($p$ constant)

SOLVED EXAMPLES

Ex. 1. Evaluate : $\lim_{x \to 0} \left[\frac{5^x - 1}{\sin x} \right]$

Solution : $\lim_{x \to 0} \left[\frac{5^x - 1}{\sin x} \right]$

Divide Numerator and Denominator by x

$$= \lim_{x \to 0} \left[\frac{\frac{5^x - 1}{x}}{\frac{\sin x}{x}} \right]$$

$$= \frac{\lim_{x \to 0} \left[\frac{5^x - 1}{x} \right]}{\lim_{x \to 0} \left[\frac{\sin x}{x} \right]}$$

$$= \frac{\log 5}{1}$$

$$\ldots. \left[\because \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right) = 1, \lim_{x \to 0} \left(\frac{a^x - 1}{x} \right) = \log a \right]$$

$$= (\log 5)$$

Ex. 2. Evaluate : $\lim_{x \to 0} \left[\frac{5^x - 3^x}{x} \right]$

Solution : Given $\lim_{x \to 0} \left[\frac{5^x - 3^x}{x} \right]$

$$= \lim_{x \to 0} \left[\frac{5^x - 1 - 3^x + 1}{x} \right]$$

$$= \lim_{x \to 0} \left[\frac{(5^x - 1) - (3^x - 1)}{x} \right]$$

$$= \lim_{x \to 0} \frac{(5^x - 1)}{x} - \lim_{x \to 0} \frac{(3^x - 1)}{x}$$

$$= \log 5 - \log 3 \quad \ldots.. \lim_{x \to a}\left(\frac{a^x - 1}{x}\right) = \log a$$

$$= log\left(\frac{5}{3}\right)$$

Ex. 3. Evaluate : $\lim_{x \to 0}\left[1 + \frac{5x}{6}\right]^{\frac{1}{x}}$

Solution : $\lim_{x \to 0}\left[1 + \frac{5x}{6}\right]^{\frac{1}{x}}$

$$= \lim_{x \to 0}\left[\left(1 + \frac{5x}{6}\right)^{\frac{1}{\frac{6}{5x}}}\right]^{\frac{5}{6}}$$

$$= (e)^{\frac{5}{6}}$$

Ex. 4. Evaluate : $\lim_{x \to 0}\left[\frac{3x+2}{2-5x}\right]^{\frac{1}{3x}}$

Solution: $\lim_{x \to 0}\left[\frac{3x+2}{2-5x}\right]^{\frac{1}{3x}} = \lim_{x \to 0}\left[\frac{2+3x}{2-5x}\right]^{\frac{1}{3x}}$

$$= \lim_{x \to 0}\left[\frac{2\left(1 + \frac{3x}{2}\right)}{2\left(1 - \frac{5x}{2}\right)}\right]^{\frac{1}{3x}}$$

$$= \lim_{x \to 0}\left[\frac{\left(1 + \frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1 - \frac{5x}{2}\right)^{\frac{1}{x}}}\right]^{\frac{1}{3}}$$

$$= \frac{\left[\lim_{x \to 0}\left(1 + \frac{3x}{2}\right)^{\frac{2}{3x}}\right]^{\frac{3}{2}\times\frac{1}{3}}}{\left[\lim_{x \to 0}\left(1 + \frac{-5x}{2}\right)^{\frac{2}{-5x}}\right]^{\frac{-5}{2}\times\frac{1}{3}}}$$

$$= \frac{e^{\frac{3}{6}}}{e^{\frac{-5}{6}}} \ldots.. \left[\because \lim_{x \to 0}(1 + kx)^{\frac{1}{kx}} = e\right]$$

$$= e^{\frac{8}{6}} = e^{\frac{4}{3}}$$

Ex. 5. Evaluate : $\lim_{x \to 0}\left[\frac{\log 4 + \log(0.25 + x)}{x}\right]$

Solution : $\lim_{x \to 0}\left[\frac{\log 4 + \log(0.25 + x)}{x}\right]$

$$= \lim_{x \to 0}\left[\frac{\log[4(0.25 + x)]}{x}\right]$$

$$= \lim_{x \to 0}\left[\frac{\log(1 + 4x)}{x}\right]$$

$$= 4 \times \lim_{x \to 0}\left[\frac{\log(1 + 4x)}{4x}\right]$$

$$= 4(1) \ldots.. \left[\because \lim_{x \to 0}\left[\frac{\log(1 + px)}{px}\right] = 1\right]$$

$$= 4$$

Ex. 6. Evaluate : $\lim_{x \to 0}\left(\frac{e^{2x} + e^{-2x} - 2}{x \sin x}\right)$

Solution :

$$\lim_{x \to 0}\left(\frac{e^{2x} + e^{-2x} - 2}{x \sin x}\right) = \lim_{x \to 0}\left(\frac{e^{2x}\left(e^{2x} + e^{-2x} - 2\right)}{e^{2x} x \sin x}\right)$$

$$= \lim_{x \to 0}\left(\frac{e^{4x} + 1 - 2e^{2x}}{e^{2x} x \sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{(e^{2x})^2 - 2e^{2x} + 1}{e^{2x} x \sin x}\right)$$

$$= \lim_{x\to 0}\left(\frac{(e^{2x}-1)^2}{x\sin x} \times \frac{1}{e^{2x}}\right)$$

$$= \lim_{x\to 0}\left(\frac{\frac{(e^{2x}-1)^2}{x^2}}{\frac{x\sin x}{x^2}} \times \frac{1}{e^{2x}}\right)$$

....[As $x\to 0$, $x \neq 0$, $x^2 \neq 0$]

$$= \frac{\left[\lim_{x\to 0}\left(\frac{e^{2x}-1}{2x}\right)\right]^2 \times 4}{\lim_{x\to 0}\left(\frac{\sin x}{x}\right)} \times \lim_{x\to 0}\left(\frac{1}{e^{2x}}\right)$$

$$= \frac{(1)^2 \times 4}{1} \times \frac{1}{e^o} \;..... \left[\because \lim_{x\to 0}\frac{e^{kx}-1}{kx} = 1, \lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1\right]$$

$= 4$

Ex. 7. Evaluate : $\lim_{x\to 0}\left(\frac{21^x - 7^x - 3^x + 1}{x\log(1+x)}\right)$

Solution : $\lim_{x\to 0}\left(\frac{21^x - 7^x - 3^x + 1}{x\log(1+x)}\right)$

$$= \lim_{x\to 0}\left(\frac{7^x.3^x - 7^x - 3^x + 1}{x\log(1+x)}\right)$$

$$= \lim_{x\to 0}\left(\frac{7^x\left(3^x - 1\right) - \left(3^x - 1\right)}{x\log(1+x)}\right)$$

$$= \lim_{x\to 0}\left[\frac{\left(3^x - 1\right)\left(7^x - 1\right)}{x\log(1+x)}\right]$$

$$= \lim_{x\to 0}\left[\frac{\frac{\left(3^x - 1\right)\left(7^x - 1\right)}{x^2}}{\frac{x\log(1+x)}{x^2}}\right] \text{....}[As\ x \to 0, x^2 \neq 0]$$

$$= \lim_{x\to 0}\left[\frac{\frac{3^x-1}{x} \times \frac{7^x-1}{x}}{\frac{\log(1+x)}{x}}\right]$$

$$= \frac{\log 3.\log 7}{1}$$

$$\text{....}\left[\because \lim_{x\to 0}\frac{a^x-1}{x} = \log a, \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right]$$

$$= \log 3.\log 7$$

Activity-3

Evaluate : $\lim_{x\to 0}\left[\frac{8^x - 4^x - 2^x + 1}{x^2}\right]$

$$= \lim_{x\to 0}\left[\frac{\left(4\times\square\right)^x - 4^x - 2^x + 1}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{\left(4^x\times\square\right)^x - 4^x - 2^x + 1}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{4^x(\square - 1) - (2^x - 1)}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2^x - 1).\quad(4^x - 1)}{x^2}\right]$$

$$= \lim_{x\to 0}\left[\frac{(2^x - 1)}{x}\right] \times \lim_{x\to 0}\left[\frac{(4^x - 1)}{x}\right]$$

$$= \boxed{}\ \boxed{}$$

Activity-4:

Evaluate : $\lim\limits_{x\to 0}\left[\dfrac{e^x-\sin x-1}{x}\right]$

$= \lim\limits_{x\to 0}\left[\dfrac{(e^x-1)-\square}{x}\right]$

$= \lim\limits_{x\to 0}\left[\dfrac{\square}{x}-\dfrac{\sin x}{\square}\right]$

$= \lim\limits_{x\to 0}\left[\dfrac{\square}{x}\right]-\lim\limits_{x\to 0}\left[\dfrac{\sin x}{x}\right]$

$= \boxed{} - 1$

$= 1-1$

$= \boxed{}$

EXERCISE 7.6

Q.I Evaluate the following limits :

1) $\lim\limits_{x\to 0}\left[\dfrac{9^x-5^x}{4^x-1}\right]$

2) $\lim\limits_{x\to 0}\left[\dfrac{5^x+3^x-2^x-1}{x}\right]$

3) $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x-3}{\sin x}\right)$

4) $\lim\limits_{x\to 0}\left(\dfrac{6^x+5^x+4^x-3^{x+1}}{\sin x}\right)$

5) $\lim\limits_{x\to 0}\left(\dfrac{8^{\sin x}-2^{\tan x}}{e^{2x}-1}\right)$

Q.II Evaluate the following limits :

1) $\lim\limits_{x\to 0}\left[\dfrac{3^x+3^{-x}-2}{x.\tan x}\right]$

2) $\lim\limits_{x\to 0}\left[\dfrac{3+x}{3-x}\right]^{\frac{1}{x}}$

3) $\lim\limits_{x\to 0}\left[\dfrac{5x+3}{3-2x}\right]^{\frac{2}{x}}$

4) $\lim\limits_{x\to 0}\left[\dfrac{\log(3-x)-\log(3+x)}{x}\right]$

5) $\lim\limits_{x\to 0}\left[\dfrac{4x+1}{1-4x}\right]^{\frac{1}{x}}$

6) $\lim\limits_{x\to 0}\left[\dfrac{5+7x}{5-3x}\right]^{\frac{1}{3x}}$

Q.III Evaluate the following limits :

1) $\lim\limits_{x\to 0}\left[\dfrac{a^x-b^x}{\sin(4x)-\sin(2x)}\right]$

2) $\lim\limits_{x\to 0}\left[\dfrac{\left(2^x-1\right)^3}{(3^x-1).\sin x.\log(1+x)}\right]$

3) $\lim\limits_{x\to 0}\left[\dfrac{15^x-5^x-3^x+1}{x\ .\ \sin x}\right]$

4) $\lim\limits_{x\to 0}\left[\dfrac{(25)^x-2(5)^x+1}{x\ .\ \sin x}\right]$

5) $\lim\limits_{x\to 0}\left[\dfrac{(49)^x-2(35)^x+(25)^x}{\sin x.\log(1+2x)}\right]$

7.7 LIMIT AT INFINITY : (FUNCTION TENDING TO INFINITY)

Let's :Learn

7.7.1 Limit at infinity :

Let us consider the function $f(x)=\dfrac{1}{x}$

Observe that as x approaches to ∞ or $-\infty$ the value of $f(x)$ is shown below,

i) Observe the following table for $f(x)=\dfrac{1}{x}$

x	1	10	100	1000	10000	100000	...
$f(x)$	1	0.1	0.01	0.001	0.0001	0.00001	...

We see that as x assumes larger and larger values, $\dfrac{1}{x}$ assumes the value nearer and nearer to zero.

$$\therefore \lim_{x\to\infty}\frac{1}{x}=0$$

Definition : A function f is said to tend to limit 'l' as x tends to ∞ if for given $\in > 0$, there exists a positive number M such that $|f(x) - l| < \in$, $\forall$ x in the domain of f for which $x > M$

$$\therefore \lim_{x\to\infty} f(x)=l$$

ii) Observe the following table for $f(x)=\dfrac{1}{x}$

x	-1	-10	-100	-1000	-10000	-100000	...
$f(x)$	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001	...

We see that as x assumes values which tend to $-\infty$, $\dfrac{1}{x}$ assumes the value nearer and nearer to zero.

$$\therefore \lim_{x\to-\infty}\frac{1}{x}=0$$

Definition : A function f is said to tend to limit 'l' as x tends to $-\infty$ if for given $\in > 0$, there exists a positive number M such that $|f(x) - l| < \in$, for all $x > M$

$$\therefore \lim_{x\to-\infty} f(x)=l$$

Note : Whenever expression is of the form $\dfrac{\infty}{\infty}$, then divide, by suitable power of x to get finite limits of numerator as well as denominator.

7.7.2 Infinite Limits :

Let us consider the function $f(x) = \dfrac{1}{x}$. Observe the behavior of $f(x)$ as x approaches zero from right and from left.

i) Observe the following table for $f(x)=\dfrac{1}{x}$

$x =$	1	0.1	0.01	0.001	0.0001	0.00001	...
$f(x)$	1	10	100	1000	10000	100000	...

We see that as x assumes values nearer 0, but greater than 0, $\dfrac{1}{x}$ assumes the values larger and larger.

$$\therefore \lim_{x\to0^+} f(x) = \lim_{x\to0^+}\frac{1}{x}\to\infty$$

ii) Observe the following table for $f(x)=\dfrac{1}{x}$

$x =$	-1	-0.1	-0.01	-0.001	-0.0001	-0.00001	...
$f(x)$	-1	-10	-100	-1000	-10000	-100000	...

We see that as x assumes values nearer to 0, but less than 0, $\dfrac{1}{x}$ assumes the values which tends to $-\infty$

$$\therefore \lim_{x\to0^-} f(x) = \lim_{x\to0^-}\frac{1}{x}\to-\infty$$

SOLVED EXAMPLES

Ex. 1. Evaluate : $\lim_{x\to\infty}\left[\dfrac{ax+b}{cx+d}\right]$

Solution : $\lim_{x\to\infty}\left[\dfrac{ax+b}{cx+d}\right]$

$$= \lim_{x\to\infty}\left[\dfrac{\dfrac{ax+b}{x}}{\dfrac{cx+d}{x}}\right]$$

$$= \dfrac{\lim_{x\to\infty}\left[a+\dfrac{b}{x}\right]}{\lim_{x\to\infty}\left[c+\dfrac{d}{x}\right]}$$

$$= \dfrac{a+0}{c+0} \qquad \text{--- as } x\to\infty,\ \dfrac{1}{x}\to 0$$

$$= \dfrac{a}{c}$$

Ex. 2. Evaluate : $\lim_{x\to\infty}\left[\dfrac{10x^2+5x+3}{5x^2-3x+8}\right]$

Solution : $\lim_{x\to\infty}\left[\dfrac{10x^2+5x+3}{5x^2-3x+8}\right]$

Divide by x^2 to get finite limits of the numerator as well as of the denominator,

$$= \lim_{x\to\infty}\left[\dfrac{\dfrac{10x^2+5x+3}{x^2}}{\dfrac{5x^2-3x+8}{x^2}}\right]$$

$$= \dfrac{\lim_{x\to\infty}\left[10+\dfrac{5}{x}+\dfrac{3}{x^2}\right]}{\lim_{x\to\infty}\left[5-\dfrac{3}{x}+\dfrac{8}{x^2}\right]}$$

$$= \dfrac{10+0+0}{5-0+0} \qquad \left(\because \lim_{x\to\infty}\dfrac{1}{x}=0\right)$$

$$= \dfrac{10}{5}$$

$$= 2$$

Ex. 3. Evaluate : $\lim_{x\to\infty}\left[\sqrt{x^2+3x}-x\right]$

Solution : $\lim_{x\to\infty}\left[\sqrt{x^2+3x}-x\right]$

$$= \lim_{x\to\infty}\left[\dfrac{(\sqrt{x^2+3x}-x)(\sqrt{x^2+3x}+x)}{(\sqrt{x^2+3x}+x)}\right]$$

$$= \lim_{x\to\infty}\left[\dfrac{x^2+3x-x^2}{\sqrt{x^2+3x}+x}\right]$$

$$= \lim_{x\to\infty}\left[\dfrac{3x}{\sqrt{x^2+3x}+x}\right]$$

$$= \lim_{x\to\infty}\left[\dfrac{3x}{x\sqrt{1+\dfrac{3}{x}}+x}\right]$$

$$= \lim_{x\to\infty}\left[\dfrac{3}{\sqrt{1+\dfrac{3}{x}}+1}\right]$$

$$= \dfrac{3}{\sqrt{1+0}+1} \qquad \left(\because \lim_{x\to\infty}\dfrac{1}{x}=0\right)$$

$$= \dfrac{3}{1+1}$$

$$= \dfrac{3}{2}$$

EXERCISE 7.7

I Evaluate the following :

1) $\lim\limits_{x\to\infty}\left[\dfrac{ax^3+bx^2+cx+d}{ex^3+fx^2+gx+h}\right]$

2) $\lim\limits_{x\to\infty}\left[\dfrac{x^3+3x+2}{(x+4)(x-6)(x-3)}\right]$

3) $\lim\limits_{x\to\infty}\left[\dfrac{7x^2+5x-3}{8x^2-2x+7}\right]$

II Evaluate the following :

1) $\lim\limits_{x\to\infty}\left[\dfrac{7x^2+2x-3}{\sqrt{x^4+x+2}}\right]$

2) $\lim\limits_{x\to\infty}\left[\sqrt{x^2+4x+16}-\sqrt{x^2+16}\right]$

3) $\lim\limits_{x\to\infty}\left[\sqrt{x^4+4x^2}-x^2\right]$

III Evaluate the following :

1) $\lim\limits_{x\to\infty}\left[\dfrac{(3x^2+4)(4x^2-6)(5x^2+2)}{4x^6+2x^4-1}\right]$

2) $\lim\limits_{x\to\infty}\left[\dfrac{(3x-4)^3(4x+3)^4}{(3x+2)^7}\right]$

3) $\lim\limits_{x\to\infty}\left[\sqrt{x}\left(\sqrt{x+1}-\sqrt{x}\right)\right]$

4) $\lim\limits_{x\to\infty}\left[\dfrac{(2x-1)^{20}(3x-1)^{30}}{(2x+1)^{50}}\right]$

5) $\lim\limits_{x\to\infty}\left[\dfrac{\sqrt{x^2+5}-\sqrt{x^2-3}}{\sqrt{x^2+3}-\sqrt{x^2+1}}\right]$

Let's Remember

Note : For limits of trigonometric functions, angle is supposed to be in radian.

1) $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

2) $\lim\limits_{x\to a}\sin x=\sin a$

3) $\lim\limits_{x\to 0}\cos x=\cos a$

4) $\lim\limits_{x\to 0}\dfrac{\tan x}{x}=1$

5) $\lim\limits_{x\to 0}\dfrac{\sin kx}{x}=k$

6) $\lim\limits_{x\to 0}\dfrac{\tan kx}{x}=k$

7) $\lim\limits_{x\to 0}x.\sin\left(\dfrac{1}{x}\right)=0$

8) $\lim\limits_{x\to 0}\dfrac{1-\cos p x}{x^2}=\dfrac{p^2}{2}$

9) $\lim\limits_{x\to 0}\dfrac{\cos m x-\cos n x}{x^2}=\dfrac{n^2-m^2}{2}$

10) $\lim\limits_{x\to 0}\dfrac{a^x-1}{x}=\log a,\ \forall\ a>0$

11) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=e$

12) $\lim\limits_{x\to 0}\dfrac{\log(1+x)}{x}=1$

13) $\lim\limits_{x\to a}\left(\dfrac{x^n-a^n}{x-a}\right)=na^{n-1}$ for $a>0$

14) $\lim\limits_{x\to 0}\left(\dfrac{e^x-1}{x}\right)=1$

15) $\lim\limits_{x\to\infty}\left(\dfrac{1}{x}\right)=0$

16) $\lim_{x\to\infty}\left(\frac{k}{x^p}\right) = 0$ for k, p $\in$ R and p > 0

17) As $x \to 0$, $\left(\frac{1}{x}\right) \to \infty$

18) $\lim_{x\to\infty}\left(\frac{a}{b}\right)^x = 0$, if $a < b$

MISCELLANEOUS EXERCISE - 7

I) Select the correct answer from the given alternatives.

1) $\lim_{x\to 2}\left(\frac{x^4-16}{x^2-5x+6}\right) =$

A) 23 B) 32 C) –32 D) –16

2) $\lim_{x\to -2}\left(\frac{x^7+128}{x^3+8}\right) =$

A) $\frac{56}{3}$ B) $\frac{112}{3}$ C) $\frac{121}{3}$ D) $\frac{28}{3}$

3) $\lim_{x\to 3}\left(\frac{1}{x^2-11x+24}+\frac{1}{x^2-x-6}\right) =$

A) $-\frac{2}{25}$ B) $\frac{2}{25}$ C) $\frac{7}{25}$ D) $-\frac{7}{25}$

4) $\lim_{x\to 5}\left(\frac{\sqrt{x+4}-3}{\sqrt{3x-11}-2}\right) =$

A) $\frac{-2}{9}$ B) $\frac{2}{7}$ C) $\frac{5}{9}$ D) $\frac{2}{9}$

5) $\lim_{x\to \frac{\pi}{3}}\left(\frac{\tan^2 x-3}{\sec^3 x-8}\right) =$

A) 1 B) $\frac{1}{2}$ C) $\frac{1}{3}$ D) $\frac{1}{4}$

6) $\lim_{x\to 0}\left(\frac{5\sin x - x\cos x}{2\tan x-3x^2}\right) =$

A) 0 B) 1 C) 2 D) 3

7) $\lim_{x\to \frac{\pi}{2}}\left[\frac{3\cos x+\cos 3x}{(2x-\pi)^3}\right] =$

A) $\frac{3}{2}$ B) $\frac{1}{2}$ C) $-\frac{1}{2}$ D) $\frac{1}{4}$

8) $\lim_{x\to 0}\left(\frac{15^x-3^x-5^x+1}{\sin^2 x}\right) =$

A) log 15 B) log 3 + log 5

C) log 3.log 5 D) 3 log 5

9) $\lim_{x\to 0}\left(\frac{3+5x}{3-4x}\right)^{\frac{1}{x}} =$

A) e^3 B) e^6 C) e^9 D) e^{-3}

10) $\lim_{x\to 0}\left[\frac{\log(5+x)-\log(5-x)}{\sin x}\right] =$

A) $\frac{3}{2}$ B) $-\frac{5}{2}$ C) $-\frac{1}{2}$ D) $\frac{2}{5}$

11) $\lim_{x\to \frac{\pi}{2}}\left(\frac{3^{\cos x}-1}{\frac{\pi}{2}-x}\right) =$

A) 1 B) log 3 C) $3^{\frac{\pi}{2}}$ D) 3 log 3

12) $\lim_{x\to 0}\left[\frac{x.\log(1+3x)}{(e^{3x}-1)^2}\right] =$

A) $\frac{1}{e^9}$ B) $\frac{1}{e^3}$ C) $\frac{1}{9}$ D) $\frac{1}{3}$

13) $\lim_{x\to 0}\left[\frac{(3^{\sin x}-1)^3}{(3^x-1)\cdot\tan x\cdot\log(1+x)}\right] =$

A) 3 log 3 B) 2 log 3

C) $(\log 3)^2$ D) $(\log 3)^3$

14) $\lim_{x\to 3}\left[\frac{5^{x-3}-4^{x-3}}{\sin(x-3)}\right] =$

A) log 5 – 4 B) $\log\frac{5}{4}$

C) $\frac{\log 5}{\log 4}$ D) $\frac{\log 5}{4}$

15) $\lim_{x\to\infty}\left[\frac{(2x+3)^7\,(x-5)^3}{(2x-5)^{10}}\right] =$

A) $\frac{3}{8}$ B) $\frac{1}{8}$ C) $\frac{1}{6}$ D) $\frac{1}{4}$

II) Evaluate the following.

1) $\lim_{x\to0}\left[\frac{(1-x)^5-1}{(1-x)^3-1}\right]$

2) $\lim_{x\to0}[x]$ ([*] is a greatest integer function.)

3) If $f(r) = \pi r^2$ then find $\lim_{h\to0}\left[\frac{f(r+h)-f(r)}{h}\right]$

4) $\lim_{x\to0}\left[\frac{x}{|x|+x^2}\right]$

5) Find the limit of the function, if it exists, at $x = 1$

$$f(x) = \begin{cases} 7-4x & for \quad x<1 \\ x^2+2 & for \quad x\ge1 \end{cases}$$

6) Given that $7x \le f(x) \le 3x^2 - 6$ for all x. Determine the value of $\lim_{x\to3} f(x)$

7) $\lim_{x\to0}\left[\frac{secx^2-1}{x^4}\right]$

8) $\lim_{x\to0}\left[\frac{e^x+e^{-x}-2}{x.tanx}\right]$

9) $\lim_{x\to0}\left[\frac{x(6^x-3^x)}{\cos(6x)-\cos(4x)}\right]$

10) $\lim_{x\to0}\left[\frac{a^{3x}-a^{2x}-a^x+1}{x.tanx}\right]$

11) $\lim_{x\to a}\left[\frac{sinx-sina}{x-a}\right]$

12) $\lim_{x\to2}\left[\frac{\log x-\log 2}{x-2}\right]$

13) $\lim_{x\to1}\left[\frac{ab^x-a^xb}{x^2-1}\right]$

14) $\lim_{x\to0}\left[\frac{(5^x-1)^2}{(2^x-1)\log(1+x)}\right]$

15) $\lim_{x\to\infty}\left[\frac{(2x+1)^2(7x-3)^3}{(5x+2)^5}\right]$

16) $\lim_{x\to a}\left[\frac{x\cos a-a\cos x}{x-a}\right]$

17) $\lim_{x\to\frac{\pi}{4}}\left[\frac{(\sin x-\cos x)^2}{\sqrt{2}-\sin x-\cos x}\right]$

18) $\lim_{x\to1}\left[\frac{2^{2x-2}-2^x+1}{\sin^2(x-1)}\right]$

19) $\lim_{x\to1}\left[\frac{4^{x-1}-2^x+1}{(x-1)^2}\right]$

20) $\lim_{x\to1}\left[\frac{\sqrt{x}-1}{\log x}\right]$

21) $\lim_{x\to0}\left(\frac{\sqrt{1-\cos x}}{x}\right)$

22) $\lim_{x\to1}\left(\frac{x+3x^2+5x^3+\cdots\cdots+(2n-1)x^n-n^2}{x-1}\right)$

23) $\lim_{x\to0}$

$$\left\{\frac{1}{x^{12}}\left[1-\cos\left(\frac{x^2}{2}\right)-\cos\left(\frac{x^4}{4}\right)+\cos\left(\frac{x^2}{2}\right)\cos\left(\frac{x^4}{4}\right)\right]\right\}$$

24) $\lim_{x\to\infty}\left(\frac{8x^2+5x+3}{2x^2-7x-5}\right)^{\frac{4x+3}{8x-1}}$

8 CONTINUITY

Let's Study

- Continuity of a function at a point.
- Continuity of a function over an interval.
- Intermediate value theorem.

Let's Recall

- Different types of functions.
- Limits of Algebraic, Trigonometric, Exponential and Logarithmic functions.
- Left hand and Right hand limits of functions.

8.1 CONTINUOUS AND DISCONTINUOUS FUNCTIONS

The dictionary meaning of the word continuity **is the unbroken and consistent existence over a period of time.** The intuitive idea of continuity is manifested in the following examples.

(i) An unbroken road between two cities.

(ii) Flow of river water.

(iii) Railway tracks.

(iv) The changing temperature of a city during a day.

In winter the temperature of Pune rises from 14^0C at night to 29^0C in the afternoon. This change in the temp is continuous and all the values between 14 and 29 are taken during 12 hours. An activity that takes place gradually, without interruption or abrupt change is called a continuous process. There are no jumps, breaks, gaps or holes in the graph of the function.

8.1.1 CONTINUITY OF A FUNCTION AT A POINT

We are going to study continuity of functions of real variable so the domain will be an interval in R. Before we consider a formal definition of a function to be continuous at a point, let's consider various functions that fail to meet our notion of continuity. The functions are indicated by graphs where $y = f(x)$

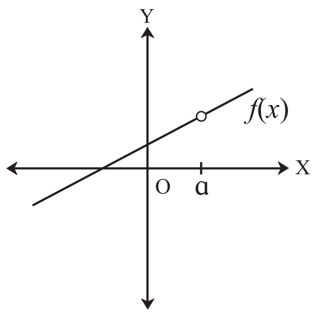


Fig. 8.1

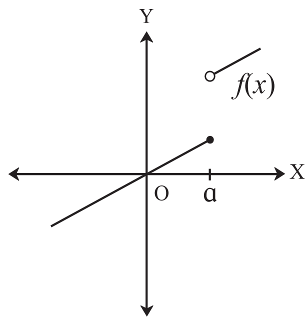


Fig. 8.2

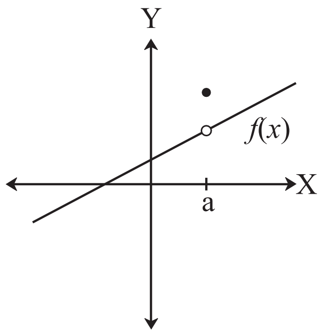


Fig. 8.3

The function in figure 8.1 has a hole at $x = a$. In fact $f(x)$ is not defined at $x = a$.

The function in figure 8.2 has a break at $x = a$.

For the function in figure 8.3, $f(a)$ is not in the continuous line.

8.1.2 DEFINITION OF CONTINUITY

A function $f(x)$ is said to be continuous at a point $x = a$, if the following three conditions are satisfied:

i. f is defined at every point on an open interval containing a.

ii. $\lim_{x \to a} f(x)$ exists

iii. $\lim_{x \to a} f(x) = f(a)$.

Among the three graphs given above, decide which conditions of continuity are not satisfied.

The condition (iii) can be reformulated and the continuity of $f(x)$ at $x = a$, can be restated as follows :

A function $f(x)$ is said to be continuous at a point $x = a$ if it is defined in some neighborhood of 'a' and if

$$\lim_{h \to 0} [f(a+h) - f(a)] = 0.$$

Illustration 1. Let $f(x) = |x|$ be defined on R.

$$f(x) = -x \quad , \text{ for } x < 0$$
$$= x \quad , \text{ for } x \geq 0$$

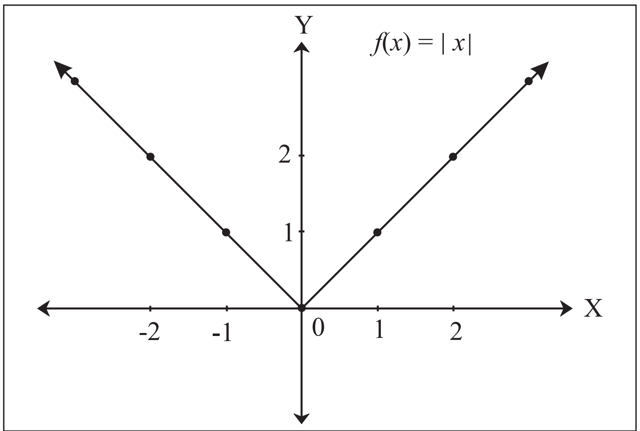


Fig. 8.4

Consider, $\lim_{x \to 0^-} f(x) = \lim_{x \to 0} (-x) = 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} (x) = 0$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0) = 0$$

Hence $f(x)$ is continuous at $x = 0$.

Illustration 2 : Consider $f(x) = x^2$ and let us discuss the continuity of f at $x = 2$.

$$f(x) = x^2$$

$$\therefore \ f(2) = 2^2 = 4$$

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} (x^2) = 2^2 = 4$$

$$\therefore \lim_{x \to 2} f(x) = f(2) = 4$$

$\therefore$ The function $f(x)$ is continuous at $x = 2$.

Observe that $f(x) = x^3, x^4$, ... etc. are continuous at every point. It follows that all polynomials are continuous functions of x.

There are some functions, which are defined in two different ways on either side of a point. In such cases we have to consider the limits of function from left as well as right of that point.

8.1.3 CONTINUITY FROM THE RIGHT AND FROM THE LEFT

A function $f(x)$ is said to be continuous from the right at $x = a$ if $\lim_{x \to a^+} f(x) = f(a)$.

A function $f(x)$ is said to be continuous from the left at $x = a$ if $\lim_{x \to a^-} f(x) = f(a)$.

If a function is continuous on the right and also on the left of a then it is continuous at a because

$$\lim_{x \to a^+} f(x) = f(a) = \lim_{x \to a^-} f(x) .$$

Illustration 3: Consider the function $f(x) = \lfloor x \rfloor$ in the interval [2, 4).

Note : $\lfloor x \rfloor$ is the greatest integer function or floor function.

Solution :

$$f(x) = \lfloor x \rfloor, \quad \text{for } x \in [2, 4)$$

that is $f(x) = 2 \quad , \quad \text{for } x \in [2, 3)$

$$= 3 \quad , \quad \text{for } x \in [3, 4)$$

The graph of which is as shown in figure 8.5

Test of continuity at $x = 3$.

For $x = 3$, $f(3) = 3$

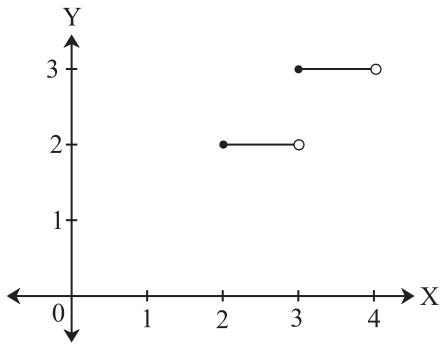


Fig. 8.5

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} \lfloor x \rfloor = \lim_{x\to 3^-} (2) = 2, \text{ and}$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \lfloor x \rfloor = \lim_{x\to 3^+} (3) = 3$$

$$\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$$

$\therefore$ $f(x)$ is discontinuous at $x = 3$.

Illustration 4 :

Consider $f(x) = x^2 + \dfrac{3}{2} \quad \text{for } 0 \leq x \leq 3$

$$= 5x - 4.5 \quad \text{for} \quad 3 < x \leq 5;$$

For $x = 3$, $f(3) = 3^2 + \dfrac{3}{2} = 10.5$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3} (x^2 + \frac{3}{2}) = 3^2 + \frac{3}{2} = 10.5$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3} (5x - 4.5) = 15 - 4.5 = 10.5$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = 10.5$$

$$\therefore \lim_{x\to 3} f(x) = f(3)$$

$\therefore$ $f(x)$ is continuous at $x = 3$.

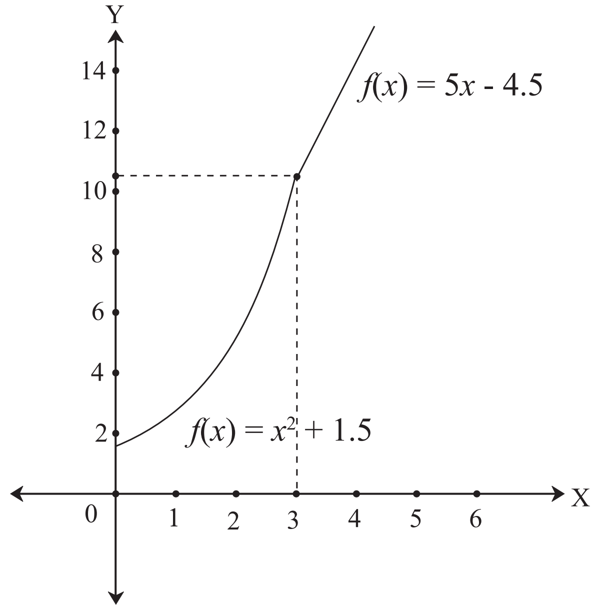


Fig. 8.6

8.1.4 Examples of Continuous Functions.

(1) Constant function, that is $f(x) = k$, is continuous at every point on R.

(2) Power functions, that is $f(x) = x^n$, with positive integral exponents are continuous at every point on R.

(3) Polynomial functions,

$P(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots\ldots + a_{n-1}x + a_n$ are continuous at every point on R

(4) The trigonometric functions $\sin x$ and $\cos x$ are continuous at every point on R.

(5) The exponential function a^x $(a > 0)$ and logarithmic function $\log_b x$ (for $x, > 0$, and b, $b \neq 1$) are continuous on R.

(6) Rational functions are of the form $\dfrac{P(x)}{Q(x)}$, $Q(x) \neq 0$. They are continuous at every point a if $Q(a) \neq 0$.

8.1.5 PROPERTIES OF CONTINUOUS FUNCTIONS:

If the functions f and g are continuous at $x = a$, then,

1. their sum, that is $(f + g)$ is continuous at $x = a$.
2. their difference, that is $(f - g)$ or $(g - f)$ is continuous at $x = a$.
3. the constant multiple of $f(x)$, that is $k.f$, for any $k \in R$, is continuous at $x = a$.
4. their product, that is $(f.g)$ is continuous at $x = a$.
5. their quotient, that is $\dfrac{f}{g}$, if $g(a) \neq 0$, is continuous at $x = a$.
6. their composite function, $f[g(x)]$ or $g[f(x)]$, that is $fog(x)$ or $gof(x)$, is continuous at $x = a$.

8.1.6 TYPES OF DISCONTINUITIES

We have seen that discontinuities have several different types. Let us classify the types of discontinuities.

8.1.7 JUMP DISCONTINUITY

As in figure 8.2, for a function, both left-hand limit and right-hand limits may exist but they are different. So the graph "jumps" at $x = a$. The function is said to have a jump discontinuity.

A function $f(x)$ has a Jump Discontinuity at $x = a$ if the left hand and right-hand limits both exist but are different, that is

$$\lim_{x\to a^+} f(x) \neq \lim_{x\to a^-} f(x)$$

Illustration 5: Consider

$$f(x) = x^2 - x - 5, \text{ for } -4 \leq x < -2.$$
$$= x^3 - 4x - 3, \text{ for } -2 \leq x \leq 1.$$

For $x = -2$, $f(-2) = (-2)^3 - 4(-2) - 3 = -3$

$\lim\limits_{x\to -2^-} f(x) = \lim\limits_{x\to -2} (x^2 - x - 5) = 4 + 2 - 5 = 1$ and

$\lim\limits_{x\to -2^+} f(x) = \lim\limits_{x\to -2} (x^3 - 4x - 3) = -8 + 8 - 3 = -3$

$\therefore \lim\limits_{x\to -2^-} f(x) \neq \lim\limits_{x\to -2^+} f(x)$

Hence $\lim\limits_{x\to -2^-} f(x)$ does not exist.

$\therefore$ the function $f(x)$ has a jump discontinuity.

8.1.8 REMOVABLE DISCONTINUITY

Some functions have a discontinuity at some point, but it is possible to define or redefine the function at that point to make it continuous. These types of functions are said to have a **removable discontinuity**. Let us look at the function $f(x)$ represented by the graph in Figure 8.1 or Figure 8.3. The function has a limit. However, there is a hole or gap at $x = a$. $f(x)$ is not defined at $x = a$. That can be corrected by defining $f(x)$ at $x = a$.

A function $f(x)$ has a discontinuity at $x = a$, and $\lim\limits_{x\to a} f(x)$ exists, but either $f(a)$ is not defined or $\lim\limits_{x\to a} f(x) \neq f(a)$. In such case we define or redefine $f(a)$ as $\lim\limits_{x\to a} f(x)$. Then with new definition, the function $f(x)$ becomes continuous at $x = a$. Such a discontinuity is called a Removable discontinuity.

If the original function is not defined at a and the new definition of f makes it continuous at a, then the new definition is called the extension of the original function.

Illustration 6:

Consider $f(x) = \dfrac{x^2+3x-10}{x^3-8}$, for $x \neq 2$.

Here $f(2)$ is not defined.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}\left(\frac{x^2+3x-10}{x^3-8}\right)$$

$$= \lim_{x\to 2}\left(\frac{(x-2)(x+5)}{(x-2)(x^2+2x+4)}\right)$$

$$= \lim_{x\to 2}\left(\frac{x+5}{x^2+2x+4}\right) = \frac{2+5}{4+4+4} = \frac{7}{12}$$

$$\therefore \lim_{x\to 2}\left(\frac{x^2+3x-10}{x^3-8}\right) = \frac{7}{12}$$

Here $f(2)$ is not defined but $\lim\limits_{x\to 2} f(x)$ exists.

Hence $f(x)$ has a removable discontinuity. The extension of the original function is

$$f(x) = \frac{x^2+3x-10}{x^3-8} \quad \text{for } x \neq 2$$

$$= \frac{7}{12} \quad \text{for } x = 2$$

This is coninuous at $x = 2$.

8.1.9 INFINITE DISCONTINUITY

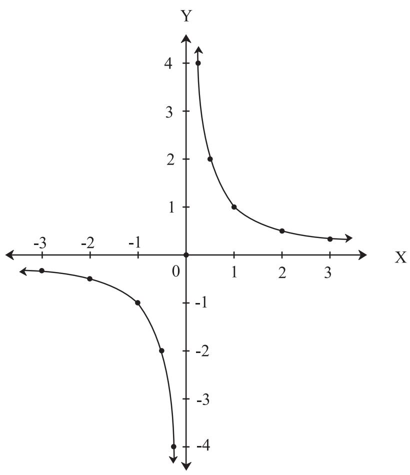


Fig. 8.7

Observe the graph of $xy = 1$. $y = f(x) = \dfrac{1}{x}$ is the function to be considered. It is easy to see that $f(x) \to \infty$ as $x \to 0^+$ and $f(x) \to -\infty$ as $x \to 0^-$. $f(0)$ is not defined. Of course, this function is discontinuous at $x = 0$.

A function $f(x)$ is said to have an infinite discontinuity at $x = a$,

if $\lim\limits_{x\to a^-} f(x) = \pm\infty$ or $\lim\limits_{x\to a^+} f(x) = \pm\infty$

Fig. 8.7 says, $f(x)$ has an infinite discontinuity.

8.1.10 CONTINUITY OVER AN INTERVAL

So far we have explored the concept of continuity of a function at a point. Now we will extend the idea of continuity on an interval.

Let (a, b) be an open interval. **If for every $x \in (a, b)$, f is continuous at x then we say that f is continuous on (a, b) .**

Consider f defined on $[a, b)$. If f is continuous on (a, b) and f is continuous to the right of a, $\lim\limits_{x\to a^+} f(x) = f(a)$ then f is continuous on $[a, b)$

Consider f defined on $(a, b]$. If f is continuous on (a, b) and f is continuous to the left of b $\lim\limits_{x\to a^+} f(x) = f(b)$, then f is continuous on $(a, b]$

Consider a function f continuous on the open interval (a, b). If $\lim\limits_{x\to a^+} f(x)$ and $\lim\limits_{x\to b^-} f(x)$ exists, then we can extend the function to [a, b] so that it is continuous on [a, b].

SOLVED EXAMPLES

Ex. 1. : Discuss the continuity of the function $f(x) = |x - 3|$ at $x = 3$.

Solution : By definition of a modulus function, the given function can be rewritten as

$$f(x) = -(x-3) \quad \text{if } x < 3$$
$$= x - 3 \quad \text{if } x \geq 3$$

Now, for $x = 3$, $f(3) = 3 - 3 = 0$.

$\lim\limits_{x \to 3^-} f(x) = -(3-3) = 0$ and

$\lim\limits_{x \to 3^+} f(x) = 3 - 3 = 0$

so, $\lim\limits_{x \to 3^-} f(x) = \lim\limits_{x \to 3^+} f(x) = 0 \Rightarrow \lim\limits_{x \to 3} f(x) = 0$

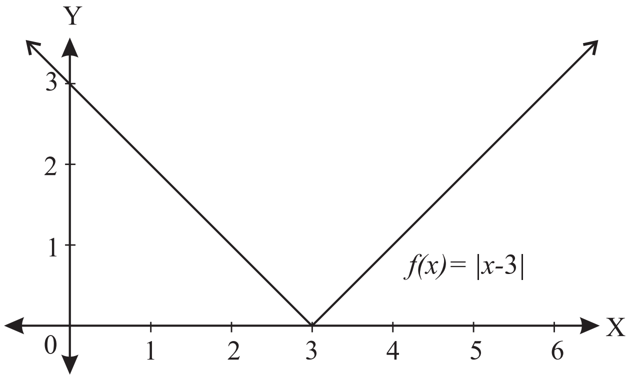


Fig. 8.8

and $\lim\limits_{x \to 3} f(x) = f(3) = 0$

Therefore the function $f(x)$ is continuous at $x = 3$.

Ex. 2 : Determine whether the function f is continuous on the set of real numbers

where $f(x) = 3x + 1, \quad \text{for } x < 2$

$= 7, \quad \text{for } 2 \leq x < 4$

$= x^2 - 8 \quad \text{for } x \geq 4$.

If it is discontinuous, state the type of discontinuity.

Solution : The function is defined in three parts, by polynomial functions, and all polynomial functions are continuous on their respective domains. Any discontinuity, if at all it exists, would be at the points where the definition changes. That is at $x = 2$ and $x = 4$.

Let us check at $x = 2$.

$f(2) = 7$ (Given)

$\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^-} (3x + 1) = 3(2) + 1 = 7$.

and $\lim\limits_{x \to 2^+} f(x) = 7$,

So $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x) = 7 \Rightarrow \lim\limits_{x \to 2} f(x) = 7$

Also, $\lim\limits_{x \to 2} f(x) = f(2) = 7$

$\therefore f(x)$ is continuous at $x = 2$.

Let us check the continuity at $x = 4$.

$f(4) = (4^2 - 8) = 8$

$\lim\limits_{x \to 4^-} f(x) = \lim\limits_{x \to 4} (7) = 7$ and

$\lim\limits_{x \to 4^+} f(x) = \lim\limits_{x \to 4} (x^2 - 8) = 4^2 - 8 = 8$

$\lim\limits_{x \to 4^-} f(x) \neq \lim\limits_{x \to 4^+} f(x)$

so $\lim\limits_{x \to 4} f(x)$ does not exist.

Since one of the three conditions does not hold at $x = 4$, the function. Hence $f(x)$ is discontinuous at $x = 4$. Therefore the function $f(x)$ is continuous on it's domain , except at $x = 4$. There exists a jump discontinuity at $x = 4$.

$\therefore f$ is discontinuous at $x = 4$.

Ex. 3 : Test whether the function $f(x)$ is continuous at $x = -4$, where

$$f(x) = \frac{x^2 + 16x + 48}{x + 4}, \quad \text{for } x \neq -4$$
$$= 8, \quad \text{for } x = -4.$$

Solution : $f(-4) = 8$ (defined)

$$\lim_{x \to -4} f(x) = \lim_{x \to -4} \left(\frac{x^2 + 16x + 48}{x + 4} \right)$$
$$= \lim_{x \to -4} \left(\frac{(x+4)(x+12)}{x+4} \right)$$
$$= \lim_{x \to -4} (x + 12) \; .... [\because x + 4 \neq 0]$$
$$= -4 + 12 = 8$$

$\therefore \lim\limits_{x \to -4} f(x) = f(-4) = 8$

$\therefore$ by definition, the function $f(x)$ is continuous at $x = -4$.

Ex. 4 : Discuss the continuity of $f(x) = \sqrt{9-a^2}$, on the interval $[-3, 3]$.

Solution : The domain of f is $[-3, 3]$.

[Note, $f(x)$ is defined if $9 - x^2 \geq 0$]

Let $x = a$ be any point in the interval $(-3, 3)$ that is $a \in (-3, 3)$.

$$\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{9-x^2}$$

$$= \sqrt{9-a^2} = f(a)$$

$\therefore$ for $a = 3$, $f(3) = 0$ and for $a = -3$, $f(-3) = 0$

Now, $\lim_{x\to 3^-} f(x) = f(3) = 0$

and $\lim_{x\to -3^+} f(x) = f(-3) = 0$

Thus $f(x)$ is continuous at every point on $(-3, 3)$ and also continuous to the right at $x = -3$ and to the left at $x = 3$.

Hence, $f(x)$ is continuous on $[-3, 3]$.

Ex. 5 : Show that the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0, 1$ in the interval $[-1, 2)$

Solution : $f(x) = \lfloor x \rfloor$ for $x \in [-1, 3)$

that is $f(x) = -1$ for $x \in [-1, 0)$

$f(x) = 0$ for $x \in [0, 1)$

$f(x) = 1$ for $x \in [1, 2)$

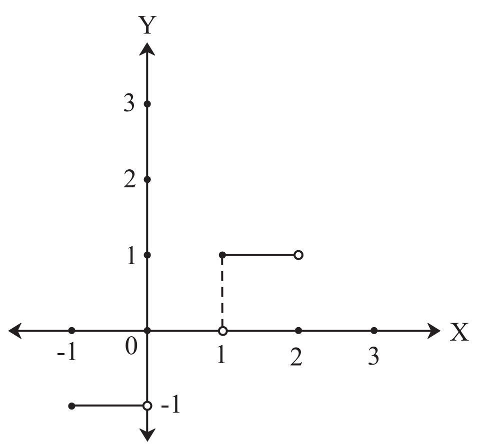


Fig. 8.9

Test of continuity of f at $x = 0$.

For $x = 0$, $f(0) = 0$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \lfloor x \rfloor = \lim_{x\to 0^-} (-1) = -1 \text{ and}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \lfloor x \rfloor = \lim_{x\to 0^+} (0) = 0$$

$$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$$

Therefore $f(x)$ is discontinuous at $x = 0$.

Test of continuity of f at $x = 1$.

For $x = 1$, $f(1) = 1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \lfloor x \rfloor = \lim_{x\to 1^-} (0) = 0 \text{ and}$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \lfloor x \rfloor = \lim_{x\to 1^+} (1) = 1$$

$$\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$$

Therefore $f(x)$ is discontinuous at $x = 1$.

Hence the function $f(x) = \lfloor x \rfloor$ is not continuous at $x = 0, 1$ in the interval $[-1, 2)$.

Ex. 6 : Discuss the continuity of the following function at $x = 0$, where

$$f(x) = x^2 \sin\left(\frac{1}{x}\right), \text{for } x \neq 0$$

$$= 0, \text{ for } x = 0.$$

Solution : The function $f(x)$ is defined for all $x \in \mathbb{R}$.

Let's check the continuity of $f(x)$ at $x = 0$.

Given, for $x = 0$, $f(0) = 0$.

we know that, $-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$ for any $x \neq 0$

Multiplying throughout by x^2 we get

$$-x^2 \leq x^2 \sin\left(\frac{1}{x}\right) \leq x^2$$

Taking limit as $x \to 0$ throughout we get,

$$\lim_{x\to 0}(-x^2) \le \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] \le \lim_{x\to 0}(x^2)$$

$$0 \le \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] \le 0$$

$\therefore$ by squeeze theorem we get,

$$\lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] = 0$$

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\left[x^2\sin\left(\frac{1}{x}\right)\right] = 0$$

$$\lim_{x\to 0} f(x) = f(0) = 0$$

$\therefore f(x)$ is continuous at $x = 0$

Ex. 7 : Find k if $f(x)$ is continuous at $x = 0$, where

$$f(x) = \frac{xe^x + \tan x}{\sin 3x}, \quad \text{for } x \ne 0$$

$$= k, \quad \text{for } x = 0$$

Solution : Given that $f(x)$ is continuous at $x = 0$,

$$\therefore f(0) = \lim_{x\to 0} f(x)$$

$$k = \lim_{x\to 0}\left(\frac{xe^x + \tan x}{\sin 3x}\right)$$

$$= \lim_{x\to 0}\left(\frac{e^x + \frac{\tan x}{x}}{\frac{\sin 3x}{x}}\right)$$

$$= \frac{\lim_{x\to 0}(e^x) + \lim_{x\to 0}\left(\frac{\tan x}{x}\right)}{\lim_{x\to 0}\left(\frac{\sin 3x}{3x}\right)\times 3}$$

$$= \frac{1+1}{1} \times \frac{1}{3} = \frac{2}{3} \quad \text{as } x \to 0,\ 3x \to 0$$

$$\left[\text{since } \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1 = \lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right)\right]$$

$$\therefore k = \frac{2}{3}$$

Ex. 8 : If f is continuous at $x = 1$, where

$$f(x) = \frac{\sin(\pi x)}{x-1} + a, \quad \text{for } x < 1$$

$$= 2\pi, \quad \text{for } x = 1$$

$$= \frac{1+\cos(\pi x)}{\pi(1-x)^2} + b, \quad \text{for } x > 1,$$

then find the values of a and b.

Solution : Given that $f(x)$ is continuous at $x = 1$

$$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) \quad \text{............ (1)}$$

$$\text{Now, } \lim_{x\to 1^-} f(x) = f(1)$$

$$\lim_{x\to 1^-}\left(\frac{\sin(\pi x)}{x-1} + a\right) = 2\pi$$

Put $x - 1 = t$, $x = 1 + t$ as $x \to 1$, $t \to 0$

$$\lim_{t\to 0}\left(\frac{\sin\pi(1+t)}{t} + a\right) = 2\pi$$

$$\lim_{t\to 0}\left(\frac{\sin(\pi + \pi t)}{t} + a\right) = 2\pi$$

$$\lim_{t\to 0}\left(\frac{-\sin\pi t}{t} + a\right) = 2\pi$$

$$-\lim_{t\to 0}\left(\frac{\sin\pi t}{\pi t}\right) \times \pi + \lim_{t\to 0}(a) = 2\pi$$

$$-(1)\pi + a = 2\pi \Rightarrow a = 3\pi$$

$$\left[\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1\right]$$

$$\text{From (1), } \lim_{x\to 1^+} f(x) = f(1)$$

$$\lim_{x\to 1^+}\left(\frac{1+\cos(\pi x)}{\pi(1-x)^2}+b\right) = 2\pi$$

Put $1-x = \theta \therefore x = 1-\theta$, as $x \to 1, \theta \to 0$

$$\therefore \lim_{\theta\to 0}\left(\frac{1+\cos(\pi(1-\theta))}{\pi\theta^2}+b\right) = 2\pi$$

$$\therefore \lim_{\theta\to 0}\left(\frac{1+\cos(\pi-\pi\theta)}{\pi\theta^2}+b\right) = 2\pi$$

$$\therefore \lim_{\theta\to 0}\left(\frac{1-\cos\pi\theta}{\pi\theta^2}+b\right) = 2\pi$$

$$\therefore \lim_{\theta\to 0}\left(\frac{2\sin^2\left(\frac{\pi\theta}{2}\right)}{\pi\theta^2}+b\right) = 2\pi$$

$$\frac{2}{\pi}\lim_{\theta\to 0}\left[\frac{\sin\left(\frac{\pi\theta}{2}\right)}{\frac{\pi\theta}{2}}\right]^2\left(\frac{\pi}{2}\right)^2 + \lim_{\theta\to 0}(b) = 2\pi$$

$$\frac{2}{\pi}(1)\times\frac{\pi^2}{4}+b = 2\pi \qquad \left[\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right) = 1\right]$$

$$\therefore \frac{\pi}{2}+b = 2\pi \qquad \therefore b = \frac{3\pi}{2}$$

$$\therefore a = 3\pi\ ,\ b = \frac{3\pi}{2}$$

Ex. 9 : Identify discontinuities for the following functions as either a jump or a removable discontinuity on R.

(1) $f(x) = \dfrac{x^2-3x-18}{x-6}$,

(2) $g(x) = 3x+1$, for $x < 3$
$\quad = 2-3x$, for $x \geq 3$

(3) $h(x) = 13-x^2$, for $x < 5$
$\quad = 13-5x$, for $x > 5$

Solution :

(1) $f(x) = \dfrac{x^2-3x-18}{x-6}$

Here $f(x)$ is a rational function, which is continuous for all real values of x, except for $x = 6$. Therefore $f(6)$ is not defined.

Now, $\displaystyle\lim_{x\to 6} f(x) = \lim_{x\to 6}\left(\frac{x^2-3x-18}{x-6}\right)$

$$= \lim_{x\to 6}\left(\frac{(x-6)(x+3)}{x-6}\right)$$

$$= \lim_{x\to 6}(x+3) \quad [\because (x-6)\neq 0]$$

$$\therefore \lim_{x\to 6} f(x) = 9$$

Here $f(6)$ is not defined but $\lim_{x\to 6} f(x)$ exists.

Hence $f(x)$ has a removable discontinuity.

(2) $g(x) = 3x+1$, for $x < 3$
$\quad = 2-3x$, for $x \geq 3$

This function is defined by different polynomials on two intervals. So they are continuous on the open intervals $(-\infty, 3)$ and $(3, \infty)$.

We examine continuity at $x = 3$.

For $x = 3$, $g(3) = 2-3(3) = -7$

$$\lim_{x\to 3^-} g(x) = \lim_{x\to 3^-}(3x+1) = 10 \text{ and}$$

$$\lim_{x\to 3^+} g(x) = \lim_{x\to 3^+}(2-3x) = -7$$

$$\lim_{x\to 3^-} g(x) \neq \lim_{x\to 3^+} g(x) \therefore \lim_{x\to 3} g(x) \text{ does not exist.}$$

Hence g is not continuous at $x = 3$.

The function $g(x)$ has a jump discontinuity at $x = 3$.

(3) $h(x) = 13-x^2$, for $x < 5$,
$\quad = 13-5x$, for $x > 5$,

but $h(5)$ is not defined.

$h(x)$ is continuous at any $x < 5$ and $x > 5$

$$\lim_{x\to5^-} h(x) = \lim_{x\to5^-} (13 - x^2) = 13 - 25 = -12$$

and $\lim_{x\to5^+} h(x) = \lim_{x\to5^+} (13 - 5x) = 13 - 25 = -12$

So, $\lim_{x\to5^-} h(x) = \lim_{x\to5^+} h(x) \therefore \lim_{x\to5} h(x) = -12$

But for $x = 5$, $f(x)$ is not defined.

So the function $h(x)$ has a removable discontinuity.

Note :

We have proved that $\lim_{x\to0} \frac{\sin x}{x} = 1$.

Some standard limits are stated without proof.

$$\lim_{x\to0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x\to0} \frac{a^x - 1}{x} = \log a$$

$$\lim_{x\to0} (1+t)^{\frac{1}{t}} = e$$

$$\lim_{x\to0} (1-t)^{\frac{1}{t}} = e^{-1} = \frac{1}{e}$$

$$\lim_{x\to0} \frac{\log(1+x)}{x} = 1,$$

$$\lim_{x\to0} \frac{\log(1-x)}{x} = -1$$

These can be proved using L' Hospital's rule, or expressions in power series which will be studied at advanced stage.

Ex. 10 : Show that the function

$$f(x) = \frac{5^{\cos x} - e^{\left(\frac{\pi}{2}-x\right)}}{\cot x}, \quad \text{for } x \neq \frac{\pi}{2}$$

$$= \log 5 - e, \quad \text{for } x = \frac{\pi}{2}$$

has a removable discontinuity at $x = \frac{\pi}{2}$.

Redefine the function so that it becomes continuous at $x = \frac{\pi}{2}$.

Solution : $f(\pi/2) = \log 5 - e$

Now, $\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}} \left(\frac{5^{\cos x} - e^{\left(\frac{\pi}{2}-x\right)}}{\cot x}\right)$

Let $\frac{\pi}{2} - x = t$, $x = \frac{\pi}{2} - t$ as $x \to \frac{\pi}{2}$, $t \to 0$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{t\to0} \left(\frac{5^{\cos\left(\frac{\pi}{2}-t\right)} - e^t}{\cot\left(\frac{\pi}{2}-t\right)}\right)$$

$$= \lim_{t\to0} \left(\frac{5^{\sin t} - e^t}{\tan t}\right)$$

$$= \lim_{t\to0} \left(\frac{5^{\sin t} - 1 - e^t + 1}{\tan t}\right)$$

$$= \lim_{t\to0} \left(\frac{(5^{\sin t} - 1) - (e^t - 1)}{\tan t}\right)$$

$$= \lim_{t\to0} \left(\frac{\frac{5^{\sin t} - 1}{t} - \frac{e^t - 1}{t}}{\frac{\tan t}{t}}\right)$$

[As t→0, t ≠ 0]

$$= \lim_{t\to0} \left(\frac{\frac{\sin t}{t} \times \frac{5^{\sin t} - 1}{\sin t} - \frac{e^t - 1}{t}}{\frac{\tan t}{t}}\right)$$

[sin t ≠ 0]

$$= \frac{(1).(\log 5) - 1}{1} \quad \left[\begin{array}{l} \lim_{\theta\to0}\left(\frac{\sin\theta}{\theta}\right) = 1 = \lim_{\theta\to0}\left(\frac{\tan\theta}{\theta}\right) \\ \lim_{x\to0}\left(\frac{e^x - 1}{x}\right) = 1, \lim_{x\to0}\left(\frac{a-1}{x}\right) = \log a \end{array}\right]$$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \log 5 - 1$$

$f(\pi/2)$ is defined and $\lim_{x\to\frac{\pi}{2}} f(x)$ exists

But $\lim_{x\to\frac{\pi}{2}} f(x) \neq f\left(\frac{\pi}{2}\right)$

$\therefore$ the function $f(x)$ has removable discontinuity. This discontinuity can be removed by redefining $f(\pi/2) = \log 5 - 1$.

So the function can be redefined as follows

$$f(x) = \left(\frac{5^{\cos x} - e^{\left(\frac{\pi}{2}-x\right)}}{\cot x}\right), \text{ for } x \neq \frac{\pi}{2}$$

$$= \log 5 - 1, \text{ for } x = \frac{\pi}{2}$$

Ex. 11 : If $f(x) = \left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$, for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$

Solution : Given that $f(x)$ is continuous at $x = 0$

$$\therefore f(0) = \lim_{x\to 0}\left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(\frac{2\left(1+\frac{3x}{2}\right)}{2\left(1-\frac{5x}{2}\right)}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to 0}\left(\frac{\left(1+\frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1-\frac{5x}{2}\right)^{\frac{1}{x}}}\right)$$

$$= \frac{\left[\lim_{x\to 0}\left(1+\frac{3x}{2}\right)^{\frac{2}{3x}}\right]^{\frac{3}{2}}}{\left[\lim_{x\to 0}\left(1-\frac{5x}{2}\right)^{\frac{-2}{5x}}\right]^{\frac{-5}{2}}}$$

$$= \frac{e^{\frac{3}{2}}}{e^{\frac{-5}{2}}} \qquad \left[\lim_{x\to 0}(1+kx)^{\frac{1}{kx}} = e\right]$$

$f(0) = e^4$

Ex. 12 : If $f(x)$ is defined on R, discuss the continuity of f at $x = \frac{\pi}{2}$, where

$$f(x) = \frac{5^{\cos x} + 5^{-\cos x} - 2}{(3\cot x).\log\left(\frac{2+\pi-2x}{2}\right)}, \text{ for } x \neq \frac{\pi}{2}$$

$$= \frac{2\log 5}{3}, \text{ for } x = \frac{\pi}{2}.$$

Solution : Given that, for $x = \frac{\pi}{2}$,

$$f\left(\frac{\pi}{2}\right) = \frac{2\log 5}{3},$$

$$\lim_{x\to\frac{\pi}{2}} f(x) = \lim_{x\to\frac{\pi}{2}}\left[\frac{5^{\cos x} + 5^{-\cos x} - 2}{3(\cot x)\log\left(\frac{2+\pi-2x}{2}\right)}\right]$$

$$= \lim_{x\to\frac{\pi}{2}}\left[\frac{5^{\cos x} + 5^{-\cos x} - 2}{3(\cot x)\log\left(1+\frac{\pi-2x}{2}\right)}\right]$$

Let $\frac{\pi-2x}{2} = t \Rightarrow x = \frac{\pi}{2} - t$ as $x \to \frac{\pi}{2}$, $t \to 0$

$$\therefore \lim_{x\to\frac{\pi}{2}} f(x)$$

$$= \lim_{t\to 0}\left(\frac{5^{\cos\left(\frac{\pi}{2}-t\right)} + 5^{-\cos\left(\frac{\pi}{2}-t\right)} - 2}{3\cot\left(\frac{\pi}{2}-t\right).\log(1+t)}\right)$$

$$= \lim_{t\to 0}\left(\frac{5^{\sin t} + 5^{-\sin t} - 2}{3(\tan t).\log(1+t)}\right)$$

$$= \lim_{t\to 0}\left(\frac{5^{\sin t}\left(5^{\sin t} + 5^{-\sin t} - 2\right)}{3(\tan t)\log(1+t).5^{\sin t}}\right)$$

.........[Multiply and divide by $5^{\sin t}$]

$$= \lim_{t\to 0}\left(\frac{(5^{\sin t})^2+1-2\times 5^{\sin t}}{3(\tan t)\log(1+t)(5^{\sin t})}\right)$$

$$= \lim_{t\to 0}\left(\frac{(5^{\sin t}-1)^2}{3(\tan t).\log(1+t).(5^{\sin t})}\right)$$

$$= \lim_{t\to 0}\left[\frac{\frac{(5^{\sin t}-1)^2}{\sin^2 t}\times\sin^2 t}{3(\tan t).\log(1+t).(5^{\sin t})}\right]$$

t is small but $t \neq 0$. Hence $\sin t \neq 0$

So we can multiply and divide the numerator by $\sin^2 t$

$$= \lim_{t\to 0}\left(\frac{\left(\frac{5^{\sin t}-1}{\sin t}\right)^2\times\left(\frac{\sin t}{t}\right)^2}{3\left(\frac{\tan t}{t}\right).\frac{\log(1+t)}{t}.5^{\sin t}}\right)$$

[Dividing Numerator and Denominator by t^2 as $t \neq 0$]

$$= \frac{\left[\lim_{t\to 0}\left(\frac{5^{\sin t}-1}{\sin t}\right)\right]^2\times\lim_{t\to 0}\left(\frac{\sin t}{t}\right)^2}{3\lim_{t\to 0}\left(\frac{\tan t}{t}\right).\lim_{t\to 0}\left(\frac{\log(1+t)}{t}\right)}\times\lim_{t\to 0}\left(\frac{1}{5^{\sin t}}\right)$$

$$= \frac{(\log 5)^2\times(1)}{3(1)(1)}\times\frac{1}{5^0}\quad\left[\begin{array}{l}\lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1,\ \lim_{\theta\to 0}\left(\frac{\tan\theta}{\theta}\right)=1\\ \lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)=1,\ \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\end{array}\right]$$

$$\therefore\ \lim_{x\to\frac{\pi}{2}} f(x) = \frac{(\log 5)^2}{3}$$

$$\therefore\ \lim_{x\to\frac{\pi}{2}} f(x) \neq f\left(\frac{\pi}{2}\right)$$

$\therefore$ $f(x)$ is discontinuous at $x = \left(\frac{\pi}{2}\right)$

Activity 1 :

Discuss the continuity of $f(x)$

where $f(x) = \dfrac{\log x - \log 5}{x-5}$ for $x \neq 5$

$= \dfrac{1}{5}$ for $x = 5$

Solution. : Given that $f(5) = \boxed{}$ .... (I)

$$\therefore\ \lim_{x\to 5} f(x) = \lim_{x\to 5}\left[\frac{\log x-\log 5}{x-5}\right]$$

put $x - 5 = t$ $\therefore$ $x = 5 + t$. As $x \to 5$, $t \to 0$

$$= \lim_{t\to\square}\left[\frac{\log(\square)-\log 5}{t}\right]$$

$$= \lim_{t\to\square}\left[\frac{\log\left[\frac{\square}{5}\right]^{t}}{t}\right]$$

$$= \lim_{t\to\square}\left[\frac{\log\left(1+\frac{t}{5}\right)}{t}\right]$$

$$= \lim_{t\to\square}\left[\frac{\log\left(1+\frac{t}{5}\right)}{\frac{t}{\square}}\right]\times\frac{1}{\square}$$

$$= 1\times\frac{1}{\square}\quad\left[\lim_{x\to 0}\left(\frac{\log(1+px)}{px}\right)=1\right]$$

$$\therefore \lim_{x\to 0} f(x) = \frac{1}{\square}\quad\text{........... (II)}$$

$\therefore$ from (I) and (II)

$$\lim_{x\to 5} f(x) = f(5)$$

$\therefore$ The function $f(x)$ is continuous at $x = 5$.

8.1.11 THE INTERMEDIATE VALUE THEOREM FOR CONTINUOUS FUNCTIONS

Theorem : If f is a continuous function on a closed interval [a, b], and if y_0 is any value between f(a) and f(b) then $y_0 = f$(c) for some c in [a, b].

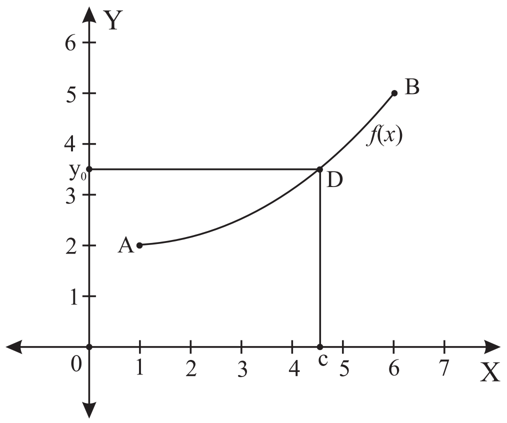


Fig. 8.10

Geometrically, the Intermediate Value Theorem says that any horizontal line $y = y_0$ crossing the Y-axis between the numbers f(a) and f(b) will cross the curve $y = f(x)$ at least once over the interval [a, b]. The proof of the Intermediate Value Theorem depends on the completeness property of the real number system and can be found in more advanced texts. The continuity of f on the interval is essential. If f is discontinuous at even one point of the interval, the conclusion of the theorem may fail.

Illustration 1 : Show that there is a root for the equation $x^3 - x - 1 = 0$ between 1 and 2

Solution : Let $f(x) = x^3 - x - 1$. $f(x)$ is a polynomial function so, it is continuous everywhere. We say that root of $f(x)$ exists if

$f(x) = 0$ for some value of x.

For $x = 1$, $f(1) = 1^3 - 1 - 1 = -1 < 0$

so, $f(1) < 0$

For $x = 2$, $f(2) = 2^3 - 2 - 1 = 5 > 0$ so,

$f(2) > 0$

So by intermediate value theorem there has to be a point c between 1 and 2 with f(c) = 0.

Hence there is a root for the equation

$x^3 - x - 1 = 0$ between 1 and 2.

EXERCISE 8.1

1) Examine the continuity of

(i) $f(x) = x^3 + 2x^2 - x - 2$ at $x = -2$.

(ii) $f(x) = \sin x$, for $x \leq \dfrac{\pi}{4}$

$\quad = \cos x$, for $x > \dfrac{\pi}{4}$, at $x = \dfrac{\pi}{4}$

(iii) $f(x) = \dfrac{x^2 - 9}{x - 3}$, for $x \neq 3$

$\quad = 8$ for $x = 3$

2) Examine whether the function is continuous at the points indicated against them.

(i) $f(x) = x^3 - 2x + 1$, if $x \leq 2$

$\quad = 3x - 2$, if $x > 2$, at $x = 2$.

(ii) $f(x) = \dfrac{x^2 + 18x - 19}{x - 1}$, for $x \neq 1$

$\quad = 20$ for $x = 1$, at $x = 1$

(iii) $f(x) = \dfrac{x}{\tan 3x} + 2$, for $x < 0$

$\quad = \dfrac{7}{3}$, for $x \geq 0$, at $x = 0$.

3) Find all the points of discontinuities of $f(x) = \lfloor x \rfloor$ on the interval (–3, 2).

4) Discuss the continuity of the function $f(x) = |2x + 3|$, at $x = -3/2$

5) Test the continuity of the following functions at the points or interval indicated against them.

(i) $f(x) = \dfrac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$, for $x \neq 2$

$= \dfrac{1}{5}$, for $x = 2$

at $x = 2$

(ii) $f(x) = \dfrac{x^3-8}{\sqrt{x+2}-\sqrt{3x-2}}$ for $x \neq 2$

$= -24$ for $x = 2$ at $x = 2$

(iii) $f(x) = 4x + 1$, for $x \leq 8/3$.

$= \dfrac{59-9x}{3}$, for $x > 8/3$, at $x = 8/3$.

(iv) $f(x) = \dfrac{(27-2x)^{\frac{1}{3}}-3}{9-3(243+5x)^{\frac{1}{5}}}$, for $x \neq 0$

$= 2$ for $x = 0$, at $x = 0$

(v) $f(x) = \dfrac{x^2+8x-20}{2x^2-9x+10}$ for $0 < x < 3; x \neq 2$

$= 12$, for $x = 2$

$= \dfrac{2-2x-x^2}{x-4}$ for $3 \leq x < 4$

at $x = 2$

6) Identify discontinuities for the following functions as either a jump or a removable discontinuity.

(i) $f(x) = \dfrac{x^2-10x+21}{x-7}$.

(ii) $f(x) = x^2 + 3x - 2$, for $x \leq 4$

$= 5x + 3$, for $x > 4$.

(iii) $f(x) = x^2 - 3x - 2$, for $x < -3$

$= 3 + 8x$, for $x > -3$.

(iv) $f(x) = 4 + \sin x$, for $x < \pi$

$= 3 - \cos x$ for $x > \pi$

7) Show that following functions have continuous extension to the point where $f(x)$ is not defined. Also find the extension

(i) $f(x) = \dfrac{1-\cos 2x}{\sin x}$, for $x \neq 0$.

(ii) $f(x) = \dfrac{3\sin^2 x + 2\cos x(1-\cos 2x)}{2\left(1-\cos^2 x\right)}$, for $x \neq 0$.

(iii) $f(x) = \dfrac{x^2-1}{x^3+1}$ for $x \neq -1$.

8) Discuss the continuity of the following functions at the points indicated against them.

(i) $f(x) = \dfrac{\sqrt{3}-\tan x}{\pi-3x}$, $x \neq \dfrac{\pi}{3}$

$= \dfrac{3}{4}$, for $x = \dfrac{\pi}{3}$, at $x = \dfrac{\pi}{3}$.

(ii) $f(x) = \dfrac{e^{1/x}-1}{e^{1/x}+1}$, for $x \neq 0$

$= 1$, for $x = 0$, at $x = 0$.

(iii) $f(x) = \dfrac{4^x - 2^{x+1}+1}{1-\cos 2x}$, for $x \neq 0$

$= \dfrac{(\log 2)^2}{2}$, for $x = 0$, at $x = 0$.

9) Which of the following functions has a removable discontinuity? If it has a removable discontinuity, redefine the function so that it becomes continuous.

(i) $f(x) = \dfrac{e^{5\sin x}-e^{2x}}{5\tan x - 3x}$, for $x \neq 0$

$= 3/4$, for $x = 0$, at $x = 0$.

(ii) $f(x) = \log_{(1+3x)}(1+5x)$ for $x > 0$

$= \dfrac{32^x - 1}{8^x - 1}$, for $x < 0$, at $x = 0$.

(iii) $f(x) = \left(\dfrac{3-8x}{3-2x}\right)^{\frac{1}{x}}$, for $x \neq 0$.

(iv) $f(x) = 3x + 2$, for $-4 \le x \le -2$

$= 2x - 3$, for $-2 < x \le 6$.

(v) $f(x) = \dfrac{x^3 - 8}{x^2 - 4}$, for $x > 2$

$= 3$, for $x = 2$

$= \dfrac{e^{3(x-2)^2} - 1}{2(x-2)^2}$, for $x < 2$

10) (i) If $f(x) = \dfrac{\sqrt{2+\sin x} - \sqrt{3}}{\cos^2 x}$, for $x \neq \dfrac{\pi}{2}$,

is continuous at $x = \dfrac{\pi}{2}$ then find $f\left(\dfrac{\pi}{2}\right)$.

(ii) If $f(x) = \dfrac{\cos^2 x - \sin^2 x - 1}{\sqrt{3x^2+1} - 1}$ for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$.

(iii) If $f(x) = \dfrac{4^{x-\pi} + 4^{\pi-x} - 2}{(x-\pi)^2}$ for $x \neq \pi$,

is continuous at $x = \pi$, then find $f(\pi)$.

11) (i) If $f(x) = \dfrac{24^x - 8^x - 3^x + 1}{12^x - 4^x - 3^x + 1}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, find k.

(ii) If $f(x) = \dfrac{5^x + 5^{-x} - 2}{x^2}$, for $x \neq 0$

$= k$ for $x = 0$

is continuous at $x = 0$, find k.

(iii) If $f(x) = \dfrac{\sin 2x}{5x} - a$, for $x > 0$

$= 4$ for $x = 0$

$= x^2 + b - 3$, for $x < 0$

is continuous at $x = 0$, find a and b.

(iv) For what values of a and b is the function

$f(x) = ax + 2b + 18$, for $x \le 0$

$= x^2 + 3a - b$, for $0 < x \le 2$

$= 8x - 2$, for $x > 2$,

continuous for every x ?

(v) For what values of a and b is the function

$f(x) = \dfrac{x^2 - 4}{x - 2}$, for $x < 2$

$= ax^2 - bx + 3$, for $2 \le x < 3$

$= 2x - a + b$, for $x \ge 3$

continuous for every x on R?

12) Discuss the continuity of f on its domain, where

$f(x) = |x+1|$, for $-3 \le x \le 2$

$= |x-5|$, for $2 < x \le 7$.

13) Discuss the continuity of $f(x)$ at $x = \dfrac{\pi}{4}$ where,

$f(x) = \dfrac{(\sin x + \cos x)^3 - 2\sqrt{2}}{\sin 2x - 1}$, for $x \neq \dfrac{\pi}{4}$

$= \dfrac{3}{\sqrt{2}}$, for $x = \dfrac{\pi}{4}$.

14) Determine the values of p and q such that the following function is continuous on the entire real number line.

$f(x) = x + 1$, for $1 < x < 3$

$= x^2 + px + q$, for $|x-2| \ge 1$.

15) Show that there is a root for the equation $2x^3 - x - 16 = 0$ between 2 and 3.

16) Show that there is a root for the equation $x^3 - 3x = 0$ between 1and 2.

17) Activity : Let $f(x) = ax + b$ (where a and b are unknown)

$$= x^2 + 5 \qquad \text{for } x \in R$$

Find the values of a and b, so that $f(x)$ is continuous at $x = 1$. (Fig. 8.11)

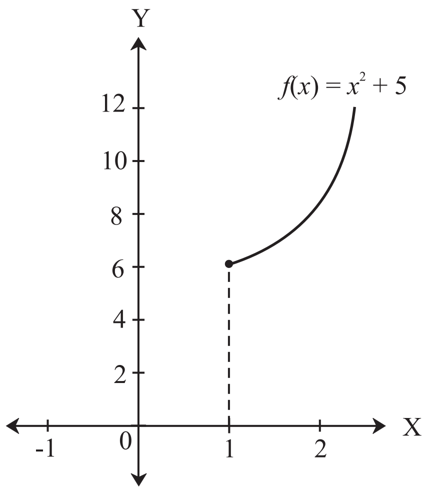


Fig. 8.11

18) Activity : Suppose $f(x) = px + 3$ for $a \le x \le$ b

$$= 5x^2 - \text{q} \quad \text{for } b < x \le \text{c}$$

Find the condition on p, q, so that $f(x)$ is continuous on [a,c], by filling in the boxes.

$f(b) = \boxed{}$

$\lim\limits_{x \to b^+} f(x) = \boxed{}$

$\therefore\ pb + 3 = \boxed{} - q$

$\therefore\ p = \dfrac{\boxed{}}{b}$ is the required condition.

Let's Remember

Continuity at a point

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisfied:

(1) $f(a)$ is defined, (2) $\lim\limits_{x \to a} f(x)$ exists, and

(3) $\lim\limits_{x \to a} f(x) = f(a)$

Continuity from right : A function is continuous from right at a if $\lim\limits_{x \to a^+} f(x) = f(a)$

Continuity from left : A function is continuous from left at b if $\lim\limits_{x \to b^-} f(x) = f(b)$

Continuity over an interval :

Open Interval : A function is continuous over an open interval if it is continuous at every point in the interval.

Closed Interval : A function $f(x)$ is continuous over a closed interval [a,b] if it is continuous at every point in (a,b),and it is continuous from right at a and from left at b.

Discontinuity at a point :

A function is discontinuous at a point or has a pont of discontinuity if it is not continuous at that point

Infinite discontinuity :

An infinite discontinuity occurs at a point a if

$$\lim_{x \to a-} f(x) = \pm\infty \text{ or } \lim_{x \to a^+} f(x) = \pm\infty$$

Jump discontinuity :

A jump discontinuity occurs at a point 'a' if

$\lim\limits_{x \to a-} f(x)$ and $\lim\limits_{x \to a^+} f(x)$ both exist, but

$$\lim_{x \to a-} f(x) \ne \lim_{x \to a^+} f(x)$$

Removable discontinuity :

A removable discontinuity occurs at a point a if $f(x)$ is discontinuous at a, but $\lim_{x \to a} f(x)$ exists and $f(a)$ may or may not be defined.

Intermediate Value Theorem :

Let f be continuous over a closed bounded interval [a,b]. If z is any real number between $f(a)$ and $f(b)$, then there is a number c in $[a,b]$ satisfying $f(c) = z$.

MISCELLANEOUS EXERCISE-8

(I) Select the correct answer from the given alternatives.

(1) $f(x) = \dfrac{2^{\cot x} - 1}{\pi - 2x}$, for $x \neq \dfrac{\pi}{2}$

$= \log \sqrt{2}$, for $x = \dfrac{\pi}{2}$

(A) f is continuous at $x = \dfrac{\pi}{2}$

(B) f has a jump discontinuity at $x = \dfrac{\pi}{2}$

(C) f has a removable discontinuity

(D) $\lim_{x \to \frac{\pi}{2}} f(x) = 2\log 3$

(2) If $f(x) = \dfrac{1 - \sqrt{2}\sin x}{\pi - 4x}$, for $x \neq \dfrac{\pi}{4}$

is continuous at $x = \dfrac{\pi}{4}$, then $f\left(\dfrac{\pi}{4}\right) =$

(A) $\dfrac{1}{\sqrt{2}}$ (B) $-\dfrac{1}{\sqrt{2}}$ (C) $-\dfrac{1}{4}$ (D) $\dfrac{1}{4}$

(3) If $f(x) = \dfrac{(\sin 2x)\tan 5x}{(e^{2x} - 1)^2}$, for $x \neq 0$

is continuous at $x = 0$, then $f(0)$ is

(A) $\dfrac{10}{e^2}$ (B) $\dfrac{10}{e^4}$ (C) $\dfrac{5}{4}$ (D) $\dfrac{5}{2}$

(4) $f(x) = \dfrac{x^2 - 7x + 10}{x^2 + 2x - 8}$, for $x \in [-6, -3]$

(A) f is discontinuous at $x = 2$.

(B) f is discontinuous at $x = -4$.

(C) f is discontinuous at $x = 0$.

(D) f is discontinuous at $x = 2$ and $x = -4$.

(5) If $f(x) = ax^2 + bx + 1$, for $|x - 1| \geq 3$ and

$= 4x + 5$, for $-2 < x < 4$

is continuous everywhere then,

(A) $a = -\dfrac{1}{2}$, $b = 5$ (B) $a = -\dfrac{1}{2}$, $b = -5$

(C) $a = \dfrac{1}{2}$, $b = -5$ (D) $a = \dfrac{1}{2}$, $b = 5$

(6) $f(x) = \dfrac{(16^x - 1)(9^x - 1)}{(27^x - 1)(32^x - 1)}$, for $x \neq 0$

$= k$, for $x = 0$

is continuous at $x = 0$, then 'k' =

(A) $\dfrac{8}{3}$ (B) $\dfrac{8}{15}$ (C) $-\dfrac{8}{15}$ (D) $\dfrac{20}{3}$

(7) $f(x) = \dfrac{32^x - 8^x - 4^x + 1}{4^x - 2^{x+1} + 1}$, for $x \neq 0$

$= k$, for $x = 0$,

is continuous at $x = 0$, then value of 'k' is

(A) 6 (B) 4 (C) (log2)(log4) (D) 3log4

(8) If $f(x) = \dfrac{12^x - 4^x - 3^x + 1}{1 - \cos 2x}$, for $x \neq 0$ is continuous at $x = 0$ then the value of $f(0)$ is

(A) $\dfrac{\log 12}{2}$ (B) log2.log3

(C) $\dfrac{\log 2 . \log 3}{2}$ (D) None of these.

(9) If $f(x)=\left(\frac{4+5x}{4-7x}\right)^{\frac{4}{x}}$, for $x \neq 0$ and $f(0) = k$, is continuous at $x = 0$, then k is

(A) e^7 (B) e^3 (C) e^{12} (D) $e^{\frac{3}{4}}$

(10) If $f(x) = \lfloor x \rfloor$ for $x \in (-1,2)$ then f is discontinuous at

(A) $x = -1, 0, 1, 2,$ (B) $x = -1, 0, 1$

(C) $x = 0, 1$ (D) $x = 2$

(II) Discuss the continuity of the following functions at the point(s) or on the interval indicated against them.

(1) $f(x)=\frac{x^2-3x-10}{x-5}$, for $3 \leq x \leq 6, x \neq 5$

$= 10$, for $x = 5$

$=\frac{x^2-3x-10}{x-5}$, for $6 < x \leq 9$

(2) $f(x) = 2x^2 - 2x + 5$, for $0 \leq x \leq 2$

$=\frac{1-3x-x^2}{1-x}$, for $2 < x < 4$

$=\frac{x^2-25}{x-5}$, for $4 \leq x \leq 7$ and $x \neq 5$

$= 7$ for $x = 5$

(3) $f(x)=\frac{\cos 4x-\cos 9x}{1-\cos x}$, for $x \neq 0$

$f(0)=\frac{68}{15}$, at $x = 0$ on $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(4) $f(x)=\frac{\sin^2 \pi x}{3(1-x)^2}$, for $x \neq 1$

$=\frac{\pi^2 \sin^2\left(\frac{\pi x}{2}\right)}{3+4\cos^2\left(\frac{\pi x}{2}\right)}$ for $x = 1$, at $x = 1$.

(5) $f(x)=\frac{|x+1|}{2x^2+x-1}$, for $x \neq -1$

$= 0$ for $x = -1$ at $x = -1$.

(6) $f(x) = [x+1]$ for $x \in [-2, 2)$

Where [*] is greatest integer function.

(7) $f(x) = 2x^2 + x + 1$, for $|x - 3| \geq 2$

$= x^2+3$, for $1 < x < 5$

(III) Identify discontinuities if any for the following functions as either a jump or a removable discontinuity on their respective domains.

(1) $f(x) = x^2 + x - 3$, for $x \in [-5, -2)$

$= x^2 - 5$, for $x \in (-2, 5]$

(2) $f(x) = x^2 + 5x + 1$, for $0 \leq x \leq 3$

$= x^3 + x + 5$, for $3 < x \leq 6$

(3) $f(x)=\frac{x^2+x+1}{x+1}$, for $x \in [0, 3)$

$=\frac{3x+4}{x^2-5}$, for $x \in [3, 6]$.

(IV) Discuss the continuity of the following functions at the point or on the interval indicated against them. If the function is discontinuous, identify the type of discontinuity and state whether the discontinuity is removable. If it has a removable discontinuity, redefine the function so that it becomes continuous.

(1) $f(x) = \frac{(x+3)(x^2-6x+8)}{x^2-x-12}$

(2) $f(x) = x^2 + 2x + 5$, for $x \leq 3$

$= x^3 - 2x^2 - 5$, for $x > 3$

(V) Find k if following functions are continuous at the points indicated against them.

(1) $f(x) = \left(\dfrac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$, for $x \neq 2$

$= k$, for $x = 2$ at $x = 2$.

(2) $f(x) = \dfrac{45^x - 9^x - 5^x + 1}{(k^x - 1)(3^x - 1)}$, for $x \neq 0$

$= \dfrac{2}{3}$, for $x = 0$, at $x = 0$

(VI) Find a and b if following functions are continuous at the points or on the interval indicated against them.

(1) $f(x) = \dfrac{4\tan x + 5\sin x}{a^x - 1}$, for $x < 0$

$= \dfrac{9}{\log 2}$, for $x = 0$

$= \dfrac{11x + 7x.\cos x}{b^x - 1}$, for $x > 0$.

(2) $f(x) = ax^2 + bx + 1$, for $|2x - 3| \geq 2$

$= 3x + 2$, for $\dfrac{1}{2} < x < \dfrac{5}{2}$.

(VII) Find $f(a)$, if f is continuous at $x = a$ where,

(1) $f(x) = \dfrac{1+\cos(\pi x)}{\pi(1-x)^2}$, for $x \neq 1$ and at $a = 1$.

(2) $f(x) = \dfrac{1-\cos[7(x-\pi)]}{5(x-\pi)^2}$, for $x \neq \pi$ at $a = \pi$.

(VIII) Solve using intermediate value theorem.

(1) Show that $5^x - 6x = 0$ has a root in $[1, 2]$

(2) Show that $x^3 - 5x^2 + 3x + 6 = 0$ has at least two real roots between $x = 1$ and $x = 5$.

9 DIFFERENTIATION

Let's :Learn

- The meaning of rate of change.
- Definition of derivative and the formula associated with it.
- Derivatives of some standard functions.
- Relationship between Continuity and Differentiability.

Let's Recall

- Real valued functions on R.
- Limits of functions.
- Continuity of a function at a point and over an interval.

9.1.1 INTRODUCTION :

Suppose we are travelling in a car from Mumbai to Pune. We are displacing ourselves from the origin (Mumbai) from time to time. We know that the average speed of the car

$$= \frac{\text{Total distance travelled}}{\text{Time taken to travel that distance}}$$

But at different times the speed of the car can be different. It is the ratio of a very small distance travelled, with the small time interval required to travel that distance. The limit of this ratio as the time interval tends to zero is the speed of the car at that time. This process of obtaining the speed is given by the differentiation of the distance function with respect to time. This is an example of derivative or differentiation. This measures how quickly the car moves with time. Speed is the rate of change of distance with time.

When we speak of velocity, it is the speed with the direction of movements. In problems with no change in direction, words speed and velocity may be interchanged.

The rate of change in a function at a point with respect to the variable is called the derivative of the function at that point. The process of finding a derivative is called differentiation

9.1.2 DEFINITION OF DERIVATIVE AND DIFFERENTIABILITY

Let $f(x)$ be a function defined on an open interval containing the point 'a'. If

$\lim\limits_{\delta x \to 0}\left(\dfrac{f(a+\delta x)-f(a)}{\delta x}\right)$ exists, then f is said to

be differentiable at $x = a$ and this limit is said to be the derivative of f at a and is denoted by $f'(a)$.

We can calculate derivative of 'f' at any point x in the domain of f.

Let $y = f(x)$ be a function. Let there be a small increment in the value of 'x', say δx, then correspondingly there will be a small increment in the value of y say δy.

$$\therefore \quad y + \delta y = f(x+\delta x)$$

$$\therefore \quad \delta y = f(x+\delta x) - y$$

$$\delta y = f(x+\delta x) - f(x) \qquad .... [\because \ y = f(x)]$$

As δx is a small increment and $\delta x \neq 0$, so dividing

throughout by δx, we get $\dfrac{\delta y}{\delta x} = \dfrac{f(x+\delta x)-f(x)}{\delta x}$

Now, taking the limit as $\delta x \to 0$ we get

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{f(x+\delta x)-f(x)}{\delta x}\right)$$

If the above limit exists, then that limiting value is called the derivative of the function and it is symbolically represented as, $\dfrac{dy}{dx}$

so $\dfrac{dy}{dx} = f'(x)$

We can consider the graph of $f(x)$ i.e. $\{(x, y) / y = f(x)\}$ and write the differentiation in terms of y and x

NOTE : (1) If $y = f(x)$ is a differentiable function of x then $\lim\limits_{\delta x \to 0}\left(\dfrac{\delta y}{\delta x}\right) = \dfrac{dy}{dx}$ and

$$\lim_{\delta x \to 0}\left(\frac{f(x+\delta x) - f(x)}{\delta x}\right) = f'(x)$$

(2) Let $\delta x = h$, Suppose that

$\lim\limits_{h \to 0}\left(\dfrac{f(a+h) - f(a)}{h}\right)$ exists. It implies that

$$\lim_{h \to 0^-}\left(\frac{f(a+h) - f(a)}{h}\right) = \lim_{h \to 0^+}\left(\frac{f(a+h) - f(a)}{h}\right)$$

$\lim\limits_{h \to 0^-}\left(\dfrac{f(a+h) - f(a)}{h}\right)$ is called the Left Hand Derivative (LHD)at $x = a$

$\lim\limits_{h \to 0^+}\left(\dfrac{f(a+h) - f(a)}{h}\right)$ is called the Right Hand Derivative (RHD)at $x = a$

Generally LHD at $x = a$ is represented as $f'(a^-)$ or L$f'(a)$, and RHD at $x = a$ is represented as $f'(a^+)$ or R$f'(a)$

9.1.3 DERIVATIVE BY METHOD OF FIRST PRINCIPLE.

The process of finding the derivative of a function from the definition of derivative is known as derivatives from first principle. Just for the sake of convenience δx can be replaced by h.

If $f(x)$ is a given function on an open interval, then the derivative of $f(x)$ with respect to x by method of first principle is given by

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right) = \frac{dy}{dx}$$

The derivative of $y = f(x)$ with respect to x at $x = a$ by method of first principle is given by

$$f'(a) = \lim_{h \to 0}\left(\frac{f(a+h) - f(a)}{h}\right) = \left(\frac{dy}{dx}\right)_{x\,=\,a}$$

9.1.4 DERIVATIVES OF SOME STANDARD FUNCTIONS

(1) Find the derivative of x^n w. r. t. x. for n$\in$N

Solution :

Let $f(x) = x^n$

$f(x+h) = (x+h)^n$

By method of first principle,

$$f'(x) = \lim_{h \to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

$$f'(x) = \lim_{h \to 0}\left(\frac{(x+h)^n - x^n}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{x^n + {}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + \ldots\ldots + h^n - x^n}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{{}^nC_1 x^{n-1}h + {}^nC_2 x^{n-2}h^2 + {}^nC_3 x^{n-3}h^3 + \ldots\ldots + h^n}{h}\right)$$

$$= \lim_{h \to 0}\left(\frac{h({}^nC_1 x^{n-1} + {}^nC_2 x^{n-2}h + {}^nC_3 x^{n-3}h^2 + \ldots\ldots + h^{n-1})}{h}\right)$$

$$= \lim_{h \to 0}\left(nx^{n-1} + {}^nC_2 x^{n-2}h + {}^nC_3 x^{n-3}h^2 + \ldots\ldots + h^{n-1}\right)$$

... (as h $\to 0$, h $\neq 0$)

$\therefore$ if $f(x) = x^n$, $f'(x) = nx^{n-1}$

(2) Find derivative of sin x w. r. t. x.

Solution :

Let $f(x)=\sin x$

$f(x+h)=\sin(x+h)$

By method of first principle,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\sin(x+h)-\sin x}{h}\right)$$

$$\lim_{h\to 0}\left(\frac{2\cos\left(\frac{2x+h}{2}\right).\sin\left(\frac{h}{2}\right)}{h}\right)$$

$$=\quad 2\lim_{h\to 0}\cos\left(\frac{2x+h}{2}\right)\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\frac{1}{2}\right)$$

$$=\quad 2\lim_{h\to 0}\cos\left(x+\frac{h}{2}\right)\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\frac{1}{2}\right)$$

$$=\quad 2\cos x.(1)\left(\frac{1}{2}\right)\text{.......}\left[\because \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right)=1\right]$$

$\therefore$ if $f(x)=\sin x, f'(x)=\cos x$

(3) Find the derivative of tan x w. r. t. x.

Solution:

Let $f(x)=\tan x$

$f(x+h)=\tan(x+h)$

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\tan(x+h)-\tan x}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\frac{\sin(x+h)}{\cos(x+h)}-\frac{\sin x}{\cos x}}{h}\right)$$

$$=\quad\lim_{h\to 0}\left(\frac{\frac{\sin(x+h).\cos x-\cos(x+h)\sin x}{\cos(x+h).\cos x}}{h}\right)$$

$$=\quad\lim_{h\to 0}\left(\frac{\sin h}{h.\cos(x+h).\cos x}\right)$$

$$=\quad\lim_{h\to 0}\left(\frac{\sin h}{h}\right).\lim_{h\to 0}\left(\frac{1}{\cos(x+h).\cos x}\right)$$

$$=\quad(1).\left(\frac{1}{\cos^2 x}\right)\text{........}\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$\therefore$ if $f(x)=\tan x, f'(x)=\sec^2 x$

(4) Find the derivative of sec x w. r. t. x.

Solution:

Let $f(x)=\sec x$

$f(x+h)=\sec(x+h)$

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\sec(x+h)-\sec x}{h}\right)$$

$$\lim_{h\to 0}\left(\frac{\frac{1}{\cos(x+h)}-\frac{1}{\cos x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\frac{\cos x - \cos(x+h)}{\cos(x+h).\cos x}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-2\sin\left(\frac{2x+h}{2}\right).\sin\left(\frac{-h}{2}\right)}{h.\cos(x+h).\cos x}\right)$$

$$= 2\lim_{h\to 0}\left(\frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h).\cos x}\right)\lim_{h\to 0}\left(\frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}\right)\left(\frac{1}{2}\right)$$

$$= \frac{2\sin x}{\cos^2 x}.(1).\left(\frac{1}{2}\right)\text{.......}\left[\because \lim_{\theta\to 0}\left(\frac{\sin p\theta}{p\theta}\right)=1\right]$$

$\therefore$ if $f(x) = \sec x, f'(x) = \sec x , \tan x$

(5) Find the derivative of $\log x$ w. r. t. $x.(x > 0)$

Solution:

Let $f(x) = \log x$

$f(x+h) = \log(x+h)$

From the definition,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{\log(x+h)-\log(x)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(\frac{x+h}{x}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{x}\right)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}\right)\times\left(\frac{1}{x}\right)$$

$$= 1.\left(\frac{1}{x}\right) \qquad \left[\because \lim_{x\to 0}\left(\frac{\log(1+x)}{x}\right)=1\right]$$

$\therefore$ if $f(x) = \log x, f'(x) = \dfrac{1}{x}$

(6) Find then derivative of a^x w. r. t. x. $(a > 0)$

Solution:

Let $f(x) = a^x$

$f(x+h) = a^{x+h}$

From the definition,

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x) = \lim_{h\to 0}\left(\frac{a^{(x+h)}-a^x}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{a^x(a^h-1)}{h}\right)$$

$$= a^x \lim_{h\to 0}\left(\frac{a^h-1}{h}\right)$$

$$f'(x) = a^x \log a\text{.........}\left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right]$$

$\therefore$ if $f(x) = a^x$, $f'(x) = a^x.\log a$

Try the following

(1) If $f(x)=\frac{1}{x^n}$, for $x \neq 0$, n∈N, then prove that

$$f'(x)=-\frac{n}{x^{n+1}}$$

(2) If f(x) = cos x, then prove that
$f'(x)=-\sin x$

(3) If f(x) = cot x, then prove that
$f'(x)=-\cos ec^2 x$

(4) If f(x) = cosec x, then prove that
$f'(x)=-\cos ecx.\cot x$

(5) If $f(x)=e^x$, then prove that $f'(x)=e^x$

SOLVED EXAMPLES

Ex. 1. Find the derivatives of the following from the definition,

(i) $\sqrt{x}$ (ii) cos (2x+3) (iii) 4^x (iv) log(3x–2)

Solution :

(i) Let $f(x)=\sqrt{x}$

$f(x+h)=\sqrt{x+h}$

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\times\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right)$$

$$=\lim_{h\to 0}\left(\frac{x+h-x}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$=\lim_{h\to 0}\left(\frac{h}{h(\sqrt{x+h}+\sqrt{x})}\right)$$

$$=\lim_{h\to 0}\left(\frac{1}{\sqrt{x+h}+\sqrt{x}}\right) \text{...[As } h\to 0,\ h\neq 0]$$

$$=\frac{1}{2\sqrt{x}}$$

(ii) Let $f(x)$ = cos (2x + 3)

$f(x+h)$ = cos (2(x+h)+3) =cos ((2x+3)+2h)

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\cos[(2x+3)+2h]-\cos(2x+3)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-2\sin\left(\frac{2(2x+3)+2h}{2}\right)\sin(h)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-2\sin(2x+3+h)\sin(h)}{h}\right)$$

$$=\lim_{h\to 0}[-2\sin(2x+3+h)]\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$=[-2\sin(2x+3)](1)\text{.......}\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

$$=-2\sin(2x+3)$$

(iii) Let $f(x)=4^x$

$f(x+h)=4^{x+h}$

From the definition

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{4^{x+h}-4^x}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{4^x(4^h-1)}{h}\right)$$

$$=4^x\lim_{h\to 0}\left(\frac{4^h-1}{h}\right)$$

$$=4^x\log 4 \ldots\ldots \left[\because \lim_{x\to 0}\left(\frac{a^x-1}{x}\right)=\log a\right]$$

(iv) Let $f(x)=\log(3x-2)$

$$f(x+h)=\log[3(x+h)-2]=\log[(3x-2)+3h]$$

From the definition,

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{\log[(3x-2)+3h]-\log(3x-2)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\log\left(\frac{(3x-2)+3h}{3x-2}\right)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\log\left(1+\frac{3h}{3x-2}\right)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{\log\left(1+\frac{3h}{3x-2}\right)}{\frac{3h}{3x-2}}\right)\times\left(\frac{3}{3x-2}\right)$$

$$=(1)\times\left(\frac{3}{3x-2}\right)\quad\left[\because \lim_{x\to 0}\left(\frac{\log(1+px)}{px}\right)=1\right]$$

$$=\frac{3}{3x-2}$$

Ex. 2. Find the derivative of $f(x) = \sin x$, at $x=\pi$

Solution:

$$f(x)=\sin x$$

$$f(\pi)=\sin\pi=0$$

$$f(\pi+h)=\sin(\pi+h)=-\sin h$$

From the definition,

$$f'(a)=\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

$$f'(\pi)=\lim_{h\to 0}\left(\frac{f(\pi+h)-f(\pi)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{-\sin h-0}{h}\right)=\lim_{h\to 0}\left(-\frac{\sin h}{h}\right)$$

$$=-\lim_{h\to 0}\left(\frac{\sin h}{h}\right)$$

$$=-1\quad\left[\because \lim_{\theta\to 0}\left(\frac{\sin\theta}{\theta}\right)=1\right]$$

Ex. 3. Find the derivative of x^2+x+2, at $x=-3$

Solution :

Let $f(x)=x^2+x+2$

For $x = -3$, $f(-3)=(-3)^2-3=9-3+2=8$

$$f(-3+h)=(-3+h)^2+(-3+h)+2$$
$$=h^2-6h+9-3+h+2=h^2-5h+8$$

From the definition,

$$f'(a)=\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

$$f'(-3)=\lim_{h\to 0}\left(\frac{f(-3+h)-f(-3)}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{h^2-5h+8-8}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{h^2-5h}{h}\right)$$

$$=\lim_{h\to 0}\left(\frac{h(h-5)}{h}\right)$$

$$=\lim_{h\to 0}(h-5)......[h\to 0, h\neq 0]$$

$$=-5$$

9.1.5 RELATIONSHIP BETWEEN DIFFERENTIABILITY AND CONTINUITY

Theorem : Every differentiable function is continuous.

Proof : Let $f(x)$ be differentiable at $x = a$.

Then, $f'(a)=\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$ .... (1)

we have to prove that $f(x)$ is continuous at $x = a$.

i.e. we have to prove that $\lim_{x\to a} f(x)=f(a)$

Let $x = a + h$, $x\to a$, $h\to 0$

We need to show that

$$\lim_{h\to 0} f(a+h)=f(a)$$

The equation (1) can also be written as

$$\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)=\lim_{h\to 0} f'(a)$$

As $h\to 0$, $h\neq 0$

Multiplying both the sides of above equation by h we get

$$\lim_{h\to 0}\left[h\left(\frac{f(a+h)-f(a)}{h}\right)\right]=\lim_{h\to 0}[hf'(a)]$$

$$\lim_{h\to 0}[f(a+h)-f(a)]=0[f'(a)]=0$$

$$\therefore \lim_{h\to 0} f(a+h)=f(a)$$

This proves that $f(x)$ is continuous at $x = a$.

Hence every differentiable function is continuous.

Note that a continuous function need not be differentiable.

This can be proved by an example.

Ex.: Let $f(x)=|x|$ be defined on R.

$$f(x) = -x \quad \text{for } x<0$$
$$= x \quad \text{for } x\geq 0$$

Consider, $\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}(-x)=0$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}(x)=0$$

For, $x = 0$, $f(0) = 0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^+}(x)=f(0)=0$$

Hence $f(x)$ is continuous at $x = 0$.

Now, we have to prove that $f(x)$ is not differentiable at $x = 0$ i.e. $f'(0)$ doesn't exist.

i.e. we have to prove that,

$$\lim_{h\to 0^-}\left(\frac{f(0+h)-f(0)}{h}\right)\neq\lim_{h\to 0^+}\left(\frac{f(0+h)-f(0)}{h}\right)$$

We have, L. H. D. at $x = 0$, is $f'(0^-)$

$$=\lim_{h\to 0^-}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$=\lim_{h\to 0^-}\left(\frac{-h}{h}\right)=\lim_{h\to 0^-}(-1)$$

$f'(0^-) = -1$ .... (I)

Now, R. H. D. at $x = 0$, is $f'(0^+)$

$$= \lim_{h\to 0^+}\left(\frac{f(0+h)-f(0)}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{h}{h}\right) = \lim_{h\to 0^+}(1)$$

$$f'(0^+) = 1 \ldots\ldots\ldots\ldots \text{(II)}$$

Therefore from (I) and (II), we get

$f'(0^-) \neq f'(0^+)$ that is

$$\lim_{h\to 0^-}\left(\frac{f(0+h)-f(0)}{h}\right) \neq \lim_{h\to 0^+}\left(\frac{f(0+h)-f(0)}{h}\right)$$

Though $f(x)$ is continuous at $x = 0$, it is not differentiable at $x = 0$.

SOLVED EXAMPLES

Ex. 1. Test whether the function $f(x) = (3x-2)^{\frac{2}{5}}$ is differentiable at $x = \frac{2}{3}$

Solution :

Given that, $f(x) = (3x-2)^{\frac{2}{5}}$

$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$

Note that $f\left(\frac{2}{3}\right) = 0$

For, $x = \frac{2}{3}$, $f'\left(\frac{2}{3}\right) = \lim_{h\to 0}\left(\frac{f\left(\frac{2}{3}+h\right)-f\left(\frac{2}{3}\right)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{\left[3\left(\frac{2}{3}+h\right)-2\right]^{\frac{2}{5}}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{[2+3h-2]^{\frac{2}{5}}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{(3h)^{\frac{2}{5}}}{h}\right) = 3^{\frac{2}{5}}\lim_{h\to 0}\left(\frac{h^{\frac{2}{5}}}{h}\right)$$

$$= 3^{\frac{2}{5}}\lim_{h\to 0}\left(\frac{1}{h^{\frac{3}{5}}}\right)$$

This limit does not exist.

$\therefore f(x)$ is not differentiable at $x = \frac{2}{3}$

Ex. 2. Examine the differentiability of

$$f(x) = (x-2)|x-2| \text{ at } x = 2$$

Solution : Given that $f(x) = (x-2)|x-2|$

That is $f(x) = -(x-2)^2$ for $x < 2$

$= (x-2)^2$ for $x \geq 2$

$$Lf'(2) = \lim_{h\to 0^-}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{-(2+h-2)^2-(2-2)^2}{h}\right)$$

$$= \lim_{h\to 0^-}\left(\frac{-h^2}{h}\right) = \lim_{h\to 0^-}(-h)$$

$$Lf'(2) = 0$$

$$Rf'(2) = \lim_{h\to 0^+}\left(\frac{f(2+h)-f(2)}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{(2+h-2)^2-(2-2)^2}{h}\right)$$

$$= \lim_{h\to 0^+}\left(\frac{h^2}{h}\right) = \lim_{h\to 0^+}(h)$$

$$Rf'(2) = 0$$

So, $Lf'(2) = Rf'(2) = 0$

Hence the function $f(x)$ is differentiable at $x = 2$.

Ex. 3. Show the function $f(x)$ is continuous at $x = 3$, but not differentiable at $x = 3$. if

$$f(x) = 2x + 1 \quad \text{for } x \le 3$$

$$= 16 - x^2 \quad \text{for } x > 3.$$

Solution : $f(x) = 2x + 1 \quad \text{for } x \le 3$

$$= 16 - x^2 \quad \text{for } x > 3.$$

For $x = 3$, $f(3) = 2(3) + 1 = 7$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3}(2x+1) = 2(3)+1 = 7$$

$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3}(16-x^2) = 16-(3)^2 = 7$$

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = f(3) = 7$$

$\therefore$ $f(x)$ is continuous at $x = 3$.

$$Lf'(3) = \lim_{h\to 0^-}\left(\frac{f(3+h)-f(3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2(3+h)+1-7}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{6+2h+1-7}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{2h}{h}\right) = \lim_{h\to 0}(2)$$

$$\text{Lf}'(3) = 2 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$Rf'(3) = \lim_{h\to 0^+}\left(\frac{f(3+h)-f(3)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{16-(3+h)^2-7}{h}\right) = \lim_{h\to 0}\left(\frac{16-9-6h-h^2-7}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{-h(6+h)}{h}\right) = \lim_{h\to 0}\left(-(6+h)\right)$$

$$\text{Rf}'(3) = -6 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

from (1) and (2), $Lf'(3) \ne Rf'(3)$

$\therefore$ $f(x)$ is not differentiable at $x = 3$.

Hence $f(x)$ is continuous at $x = 3$, but not differentiable at $x = 3$.

Ex. 4. Show that the function $f(x)$ is differentiable at $x = -3$ where, $f(x) = x^2 + 2$.

Solution :

For $x = -3$, $f'(-3) = \lim_{h\to 0}\left(\frac{f(-3+h)-f(-3)}{h}\right)$

$$= \lim_{h\to 0}\left(\frac{(-3+h)^2+2-11}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{9-6h+h^2+2-11}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{h^2-6h}{h}\right)$$

$$= \lim_{h\to 0}(h-6) \qquad (\because h\to 0,\ h\ne 0)$$

$$= -6$$

$f'(-3)$ exists so, $f(x)$ is differentiable at $x = -3$.

EXERCISE 9.1

(1) Find the derivatives of the following w. r. t. x by using method of first principle.

(a) x^2+3x-1 (b) $\sin(3x)$

(c) e^{2x+1} (d) 3^x (e) $\log(2x+5)$

(f) $\tan(2x+3)$ (g) $\sec(5x-2)$

(h) $x\sqrt{x}$

(2) Find the derivatives of the following w. r. t. x. at the points indicated against them by using method of first principle

(a) $\sqrt{2x+5}$ at $x = 2$ (b) $\tan x$ at $x = \pi/4$

(c) 2^{3x+1} at $x = 2$ (d) $\log(2x+1)$ at $x = 2$

(e) e^{3x-4} at $x = 2$ (f) $\cos x$ at $x = \frac{5\pi}{4}$

(3) Show that the function f is not differentiable at $x = -3$,

where $f(x) = x^2 + 2$ for $x < -3$

$= 2 - 3x$ for $x \geq -3$

(4) Show that $f(x) = x^2$ is continuous and differentiable at $x = 0$.

(5) Discuss the continuity and differentiability of

(i) $f(x) = x|x|$ at $x = 0$

(ii) $f(x) = (2x+3)|2x+3|$ at $x = -3/2$

(6) Discuss the continuity and differentiability of $f(x)$ at $x = 2$

$f(x) = [x]$ if $x \in [0,4)$. [where [*] is a greatest integer (floor) function]

(7) Test the continuity and differentiability of $f(x) = 3x + 2$ if $x > 2$

$= 12 - x^2$ if $x \leq 2$ at $x = 2$.

(8) If $f(x) = \sin x - \cos x$ if $x \leq \pi/2$
$= 2x - \pi + 1$ if $x > \pi/2$. Test the continuity and differentiability of f at $x = \pi/2$

(9) Examine the function

$$f(x) = x^2 \cos\left(\frac{1}{x}\right), \text{ for } x \neq 0$$

$$= 0, \quad \text{for } x = 0$$

for continuity and differentiability at $x = 0$.

9.2 RULES OF DIFFERENTIATION

9.2.1. Theorem 1. Derivative of Sum of functions.

If u and v are differentiable functions of x such that $y = u + v$, then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

Proof: Given that, $y = u + v$ where u and v are differentiable functions of x

Let there be a small increment in the value of x, say δx, then u changes to $(u + \delta u)$ and v changes to $(v + \delta v)$ respectively, correspondingly y changes to $(y + \delta y)$

$$\therefore (y+\delta y) = (u+\delta u) + (v+\delta v)$$

$$\therefore \delta y = (u+\delta u) + (v+\delta v) - y$$

$$\therefore \delta y = (u+\delta u) + (v+\delta v) - (u+v)$$

$$[\because y = u + v]$$

$$\therefore \delta y = \delta u + \delta v$$

As δx is small increment in x and $\delta x \neq 0$, dividing throughout by x we get,

$$\frac{\delta y}{\delta x} = \frac{\delta u + \delta v}{\delta x} = \frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}$$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + \frac{\delta v}{\delta x}\right)$$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) + \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) \quad \text{...... (I)}$$

Since u and v are differentiable function of x

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \text{ and } \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) = \frac{dv}{dx} \quad \text{..... (II)}$$

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{du}{dx} + \frac{dv}{dx} \quad \text{[From (I) and (II)]}$$

i.e. $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$

9.2.2 Theorem 2. Derivative of Difference of functions.

If u and v are differentiable functions of x such that $y = u - v$, then $\frac{dy}{dx} = \frac{du}{dx} - \frac{dv}{dx}$

[Left for students to prove]

Corollary : If $u, v, w.....$ are finite number of differentiable functions of x such that $y = k_1 u \pm k_2 v \pm k_3 w \pm$ .....

then $\frac{dy}{dx} = k_1 \frac{du}{dx} \pm k_2 \frac{dv}{dx} \pm k_3 \frac{dw}{dx}$ ......

9.2.3 Theorem 3. Derivative of Product of functions.

If u and v are differentiable functions of x such that $y = u.v$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Proof : Given that y = uv

Let there be a small increment in the value of x, say δx, then u changes to $(u + \delta u)$ and v changes to $(v + \delta v)$ respectively, correspondingly y changes to $(y + \delta y)$

$\therefore$ y+δy = $(u+\delta u)(v+\delta v)$

$\therefore \delta y = (u+\delta u)(v+\delta v) - y$

$\therefore \delta y = uv + u\delta v + v\delta u + \delta u \delta v - uv$

$\therefore \delta y = u\delta v + v\delta u + \delta u.\delta v$

As δx is small increment in x and $\delta x \neq 0$, dividing throughout by δx We get,

$$\frac{\delta y}{\delta x} = \frac{u\delta v + v\delta u + \delta u \delta v}{\delta x} = u\frac{\delta v}{\delta x} + v\frac{\delta v}{\delta x} + \frac{\delta u \delta v}{\delta x}$$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(u\frac{\delta v}{\delta x} + v\frac{\delta u}{\delta x} + \frac{\delta u \delta v}{\delta x}\right)$$

$$u\lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) + v\lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) + \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)\lim_{\delta x \to 0}(\delta v)$$

As $\delta x \to 0$, we get $\delta v \to 0$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = u\lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) + v\lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)$$

$$+ \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right)\lim_{\delta v \to 0}(\delta v) \quad .... (1)$$

Given that, u and v are differentiable functions of x

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \text{ and } \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) = \frac{dv}{dx} \quad ..... (2)$$

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = u\frac{dv}{dx} + v\frac{du}{dx} + \frac{du}{dx}(0)$$

[From (1) & (2)]

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = u\frac{dv}{dx} + v\frac{du}{dx}$$

i.e. $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

Corollary : If u, v and w are differentiable functions of x such that $y = uvw$ then

$$\frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

9.2.4 Theorem 4. Derivative of Quotient of functions.

If u and v are differentiable functions of x such that

$$y = \frac{u}{v} \text{ where } v \neq 0 \text{ then } \frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Proof : Given that, $y = \frac{u}{v}$, where u and v are differentiable functions of x

Let there be a small increment in the value of x say δx then u changes to $(u + \delta u)$ and v changes to $(v + \delta v)$ respectively, correspondingly y changes to $(y + \delta y)$

$$\therefore y + \delta y = \frac{u + \delta u}{v + \delta v}$$

$$\therefore \delta y = \frac{u + \delta u}{v + \delta v} - y$$

$$\therefore \delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v} \ldots\ldots \left[\because \; y = \frac{u}{v} \right]$$

$$\therefore \delta y = \frac{v(u+\delta u) - u(v+\delta v)}{(v+\delta v).v} = \frac{v.u + v.\delta u - u.v - u.\delta v}{(v+\delta v).v}$$

$$\delta y = \frac{v.\delta u - u.\delta v}{(v+\delta v).v}$$

As δx is small increment in x and $\delta x \neq 0$, dividing throughout by δx We get,

$$\frac{\delta y}{\delta x} = \frac{v.\delta u - u.\delta v}{\delta x.(v+\delta v).v} = \frac{v.\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2 + v.\delta v}$$

Taking the limit as $\delta x \to 0$, we get,

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \lim_{\delta x \to 0}\left(\frac{v.\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v^2 + v.\delta v}\right)$$

As $\delta x \to 0$, we get $\delta v \to 0$

$$= \frac{v.\lim\limits_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) - u.\lim\limits_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right)}{v^2 + v.\lim\limits_{\delta v \to 0}(\delta v)} \quad \ldots\ldots\ldots (1)$$

Since, u and v are differentiable functions of x

$$\lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x}\right) = \frac{du}{dx} \text{ and } \lim_{\delta x \to 0}\left(\frac{\delta v}{\delta x}\right) = \frac{dv}{dx} \quad \ldots\ldots (2)$$

$$\lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2 + v.(0)} \quad \ldots \text{[From (1) and (2)]}$$

$$\therefore \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

i.e. $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

SOLVED EXAMPLES

Find the derivatives of the following functions

Ex. 1. 1) $y = x^{\frac{3}{2}} + \log x - \cos x$

2) $f(x) = x^5 \operatorname{cosec} x + \sqrt{x}\tan x$ 3) $y = \frac{e^x - 5}{e^x + 5}$

4) $y = \frac{x \sin x}{x + \sin x}$

Solution :

1) Given, $y = x^{\frac{3}{2}} + \log x - \cos x$

Differentiate w.r.t.x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(x^{\frac{3}{2}} + \log x - \cos x\right)$$

$$= \frac{d}{dx}\left(x^{\frac{3}{2}}\right) + \frac{d}{dx}(\log x) - \frac{d}{dx}(\cos x)$$

$$= \frac{3}{2}x^{\frac{1}{2}} + \frac{1}{x} - (-\sin x)$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x} + \frac{1}{x} + \sin x$$

2) Given $f(x) = x^5 \operatorname{cosec} x + \sqrt{x}\tan x$

Differentiate w.r.t.x.

$$f'(x) = \frac{d}{dx}\left(x^5 \operatorname{cosec} x + \sqrt{x}\tan x\right)$$

$$= \frac{d}{dx}\left(x^5 \operatorname{cosec} x\right) + \frac{d}{dx}\left(\sqrt{x}\tan x\right)$$

$$= x^5 \times \frac{d}{dx}(\operatorname{cosec} x) + \operatorname{cosec} x \times \frac{d}{dx}(x^5) + \sqrt{x} \times \frac{d}{dx}(\tan x) + \tan x \times \frac{d}{dx}\sqrt{x}$$

$$= x^5 \times (-\operatorname{cosec} x \times \cot x) + \operatorname{cosec} x \times (5x^4) + \sqrt{x} \times (\sec^2 x) + \tan x \times \left(\frac{1}{2\sqrt{x}}\right)$$

$= -x^5 \ cosecx.cotx + 5x^4 \ cosecx + \sqrt{x} \times (sec^2x) + \dfrac{1}{2\sqrt{x}} \tan x$

3) Given that $y = \left(\dfrac{e^x - 5}{e^x + 5}\right)$

Differentiate w.r.t.x.

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{e^x-5}{e^x+5}\right)$$

$$= \frac{(e^x+5)\times\frac{d}{dx}(e^x-5)-(e^x-5)\times\frac{d}{dx}(e^x+5)}{(e^x+5)^2}$$

$$= \frac{(e^x+5).(e^x)-(e^x-5).(e^x)}{(e^x+5)^2}$$

$$= \frac{e^{2x}+5(e^x)-e^{2x}+5(e^x)}{(e^x+5)^2}$$

$$= \frac{10e^x}{(e^x+5)^2}$$

4) $y = \dfrac{x \sin x}{x + \sin x}$

Differentiate w.r.t.x.,

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x\sin x}{x+\sin x}\right)$$

$$= \frac{(x+\sin x)\times\frac{d}{dx}(x\sin x)-(x\sin x)\times\frac{d}{dx}(x+\sin x)}{(x+\sin x)^2}$$

$$= \frac{(x+\sin x)\times\left(x\frac{d}{dx}(\sin x)+\sin x\frac{d}{dx}(x)\right)-(x\sin x).(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{(x+\sin x).(x\cos x+\sin x)-(x\sin x).(1+\cos x)}{(x+\sin x)^2}$$

$$= \frac{x^2\cos x+x\sin x+x\sin x\cos x+\sin^2 x-x\sin x-x\sin x\cos x}{(x+\sin x)^2}$$

$$= \frac{x^2\cos x+\sin^2 x}{(x+\sin x)^2}$$

Ex. 2) If $f(x) = p \tan x + q \sin x + r$, $f(0) = -4\sqrt{3}$, $f\left(\dfrac{\pi}{3}\right) = -7\sqrt{3}$, $f'\left(\dfrac{\pi}{3}\right) = 3$ then find p, q and r.

Solution :

Given that $f(x) = p \tan x + q \sin x + r$ ...(1)

$f'(x) = p \sec^2 x + q \cos x$ ...(2)

$f(0) = -4\sqrt{3}$

put $x = 0$ in (1)

$f(0) = p \tan 0 + q \sin 0 + r = r \quad \therefore r = -4\sqrt{3}$

$f\left(\dfrac{\pi}{3}\right) = -7\sqrt{3}$,

$\therefore$ from (1) $p \tan\left(\dfrac{\pi}{3}\right) + q \sin\left(\dfrac{\pi}{3}\right) + r = -7\sqrt{3}$

$p\sqrt{3} + q\dfrac{\sqrt{3}}{2} - 4\sqrt{3} = -7\sqrt{3} \quad \therefore 2p + q = -6$ ...(3)

$f'\left(\dfrac{\pi}{3}\right) = 3$

$\therefore$ from (2), $p\sec^2\left(\dfrac{\pi}{3}\right) + q\cos\left(\dfrac{\pi}{3}\right) = 3$

$4p + \dfrac{q}{2} = 3 \quad \therefore 8p + q = 6$ ...(4)

(4) – (3) gives $6p = 12 \quad \therefore p = 2$, put $p = 2$ in (3), we get $q = -10 \quad \therefore p = 2$, $q = -10$ and $r = -4\sqrt{3}$

9.2.5 Derivatives of Algebraic Functions

Sr.No.	$f(x)$	$f'(x)$
01	**c**	**0**
02	x^n	nx^{n-1}
03	$\dfrac{1}{x^n}$	$-\dfrac{n}{x^{n+1}}$
04	$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$

9.2.6 Derivatives of Trigonometric functions

Sr.No.	$y = f(x)$	$dy/dx = f'(x)$
01	$\sin x$	$\cos x$
02	$\cos x$	$-\sin x$
03	$\tan x$	$\sec^2 x$
04	$\cot x$	$-\operatorname{cosec}^2 x$
05	$\sec x$	$\sec x \tan x$
06	$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$

9.2.7 Derivatives of Logarithmic and Exponential functions

Sr.No.	$y = f(x)$	$dy/dx = f'(x)$
01	$\log x$	$1/x$
02	e^x	e^x
03	a^x	$a^x \log a$

EXERCISE 9.2

(I) Differentiate the following *w.r.t.x*

1) $y = x^{\frac{4}{3}} + e^x - \sin x$

2) $y = \sqrt{x} + \tan x - x^3$

3) $y = \log x - \operatorname{cosec} x + 5^x - \dfrac{3}{x^{\frac{3}{2}}}$

4) $y = x^{\frac{7}{3}} + 5x^{\frac{4}{5}} - \dfrac{5}{x^{\frac{2}{5}}}$

5) $y = 7^x + x^7 - \dfrac{2}{3}x\sqrt{x} - \log x + 7^7$

6) $y = 3\cot x - 5e^x + 3\log x - \dfrac{4}{x^{\frac{3}{4}}}$

(II) Differentiate the following w.r.t.x.

1) $y = x^5 \tan x$

2) $y = x^3 \log x$

3) $y = (x^2 + 2)^2 \sin x$

4) $y = e^x \log x$

5) $y = x^{\frac{3}{2}} e^x \log x$

6) $y = \log e^{x^3} \log x^3$

(III) Differentiate the following w.r.t.x.

1) $y = x^2\sqrt{x} + x^4 \log x$

2) $y = e^x \sec x - x^{\frac{5}{3}} \log x$

3) $y = x^4 + x\sqrt{x}\cos x - x^2 e^x$

4) $y = (x^3 - 2)\tan x - x\cos x + 7^x . x^7$

5) $y = \sin x \log x + e^x \cos x - e^x\sqrt{x}$

6) $y = e^x \tan x + \cos x \log x - \sqrt{x}\, 5^x$

(IV) Differentiate the following w.r.t.x.

1) $y = \dfrac{x^2 + 3}{x^2 - 5}$

2) $y = \dfrac{\sqrt{x} + 5}{\sqrt{x} - 5}$

3) $y = \dfrac{xe^x}{x + e^x}$

4) $y = \dfrac{x \log x}{x + \log x}$

5) $y = \dfrac{x^2 \sin x}{x + \cos x}$

6) $y = \dfrac{5e^x - 4}{3e^x - 2}$

(V) (1) If $f(x)$ is a quadratic polynomial such that $f(0) = 3$, $f'(2) = 2$ and $f'(3) = 12$ then find $f(x)$

(2) If $f(x) = a\sin x - b\cos x$, $f'\left(\dfrac{\pi}{4}\right) = \sqrt{2}$ and $f'\left(\dfrac{\pi}{6}\right) = 2$, then find $f(x)$.

(VI) Fill in the blanks. (Activity Problems)

(1) $y = e^x . \tan x$

diff. *w.r.t.x.*

$$\frac{dy}{dx} = \frac{d}{dx}(e^x \tan x)$$

$$= \square \frac{d}{dx} \tan x + \tan x \frac{d}{dx} \square$$

$$= \square\ \square + \tan x$$

$$= e^x [\square + \square]$$

(2) $y = \dfrac{\sin x}{x^2 + 2}$

diff. *w.r.t.x.*

$$\frac{dy}{dx} = \frac{\square \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx} \square}{(x^2+2)^2}$$

$$= \frac{\square \square - \sin x \square}{(x^2+2)^2}$$

$$= \frac{\square - \square}{(x^2+2)^2}$$

3) $y = (3x^2 + 5)\cos x$

Diff. *w.r.t.x*

$$\frac{dy}{dx} = \frac{d}{dx}\left[(3x^2+5)\cos x\right]$$

$$= \left(3x^2+5\right)\frac{d}{dx}[\ \square\ ] + \cos x \frac{d}{dx}[\ \square\ ]$$

$$= \left(3x^2+5\right)[\ \square\ ] + \cos x\ [\ \square\ ]$$

$$\therefore \frac{dx}{dy} = \left(3x^2+5\right)[\ \square\ ] + [\ \square\ ]\cos x$$

4) Differentiate tan x and sec x w.r.t.x. using the formulae for differentiation of $\frac{u}{v}$ and $\frac{1}{v}$ respectively.

Brief idea of L' Hospital Rule

Consider the functions $f(x)$ and $g(x)$, If $\lim\limits_{x\to a} f(x) = 0$ and $\lim\limits_{x\to a} g(x) = 0$ and if $\lim\limits_{x\to a} f'(x) = p$ and $\lim\limits_{x\to a} g'(x) = q$ where $q \neq 0$ then $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \lim\limits_{x\to a}\dfrac{f'(x)}{g'(x)} = \dfrac{p}{q}$

If $\lim\limits_{x\to a} g'(x) = 0$, then provided $\lim\limits_{x\to a} f'(x) = 0$, we can study $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)}$ using the same rule.

Ex. 1 : $\lim\limits_{x\to 0}\left(\dfrac{\sin x}{x^2}\right)$

Here $f(x) = \sin x$, $\lim\limits_{x\to 0} f(x) = 0$ and

$g(x) = x^2$, $\lim\limits_{x\to 0} g(x) = 0$

$f'(x) = \cos x$, $\lim\limits_{x\to 0} f'(x) = \cos 0 = 1 \neq 0$

$g'(x) = 2x$, $\lim\limits_{x\to 0} g'(x) = 2(0) = 0$

Since $\lim\limits_{x\to 0} g'(x) = 0$. L' Hospital Rule cannot be applied.

Ex. 2 : $\lim\limits_{x\to 2}\left[\dfrac{x^2 - 7x + 10}{x^2 + 2x - 8}\right]$

Here $f(x) = x^2 - 7x + 10$, $\lim\limits_{x\to 2} f(x) = 0$ and $g(x) = x^2 + 2x - 8$ $\lim\limits_{x\to 2} g(x) = 0$

$f'(x) = 2x - 7$, $\lim\limits_{x\to 2} f'(x) = 2(2) - 7 = -3 \neq 0$

$g'(x) = 2x + 2$, $\lim\limits_{x\to 2} g'(x) = 2(2) + 2 = 6 \neq 0$,

So L 'Hospital' s rule is applicable.

$$\therefore \lim_{x\to 2}\left[\frac{x^2 - 7x + 10}{x^2 + 2x - 8}\right]$$

$$= \lim_{x\to 2}\left[\frac{2x - 7}{2x + 2}\right] = \frac{-3}{6} = -\frac{1}{2}$$

Let's Remember

- $f(x)$ is differentiable at $x = a$ if

  $Lf'(a) = Rf'(a)$

- Derivative by First Principle :

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

- Derivatives of standard functions :

$y = f(x)$	$\frac{dy}{dx} = f'(x)$
c(constant)	0
x^n	nx^{n-1}
$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\log_e x$	$\frac{1}{x}$
$\log_a x$	$\frac{1}{x \log a}$
e^x	e^x
a^x	$a^x \log a$
$u \pm v$	$u' \pm v'$
uv	$uv' + u'v$
$\frac{u}{v}$	$\frac{vu' - uv'}{v^2}$

MISCELLANEOUS EXERCISE-9

I) Select the appropriate option from the given alternative.

1) If $y = \frac{x-4}{\sqrt{x}+2}$, then $\frac{dy}{dx}$

(A) $\frac{1}{x+4}$ (B) $\frac{\sqrt{x}}{(\sqrt{x}+2)^2}$

(C) $\frac{1}{2\sqrt{x}}$ (D) $\frac{x}{(\sqrt{x}+2)^2}$

2) If $y = \frac{ax+b}{cx+d}$, then $\frac{dy}{dx} =$

(A) $\frac{ab-cd}{(cx+d)^2}$ (B) $\frac{ax-c}{(cx+d)^2}$

(C) $\frac{ac-bd}{(cx+d)^2}$ (D) $\frac{ad-bc}{(cx+d)^2}$

3) If $y = \frac{3x+5}{4x+5}$, then $\frac{dy}{dx} =$

(A) $-\frac{15}{(3x+5)^2}$ (B) $-\frac{15}{(4x+5)^2}$

(C) $-\frac{5}{(4x+5)^2}$ (D) $-\frac{13}{(4x+5)^2}$

4) If $y = \frac{5\sin x - 2}{4\sin x + 3}$, then $\frac{dy}{dx} =$

(A) $\frac{7\cos x}{(4\sin x+3)^2}$ (B) $\frac{23\cos x}{(4\sin x+3)^2}$

(C) $-\frac{7\cos x}{(4\sin x+3)^2}$ (D) $-\frac{15\cos x}{(4\sin x+3)^2}$

5) Suppose $f(x)$ is the derivative of $g(x)$ and $g(x)$ is the derivative of $h(x)$.
If $h(x) = a\sin x + b\cos x + c$ then $f(x) + h(x) =$

(A) 0 (B) c (C) $-$c (D) $-2(a\sin + b\cos x)$

6) If $f(x) = 2x + 6$ for $0 \le x \le 2$
$= ax^2 + bx$ for $2 < x \le 4$
is differentiable at $x = 2$ then the values of a and b are.

(A) $a = -\frac{3}{2}, b = 3$ (B) $a = \frac{3}{2}, b = 8$

(C) $a = \frac{1}{2}, b = 8$ (D) $a = -\frac{3}{2}, b = 8$

7) If $f(x) = x^2 + \sin x + 1$ for $x \le 0$

$= x^2 - 2x + 1$ for $x \le 0$ then

(A) f is continuous at $x = 0$, but not differentiable at $x = 0$

(B) f is neither continuous nor differentiable at $x = 0$

(C) f is not continuous at $x = 0$, but differentiable at $x = 0$

(D) f is both continuous and differentiable at $x = 0$

8) If, $f(x) = \frac{x^{50}}{50} + \frac{x^{49}}{49} + \frac{x^{48}}{48} + ... + \frac{x^2}{2} + x + 1$, then $f'(1) =$

(A) 48 (B) 49 (C) 50 (D) 51

II)

(1) Determine whether the following function is differentiable at $x = 3$ where,

$$f(x) = x^2 + 2 \quad , \quad \text{for } x \ge 3$$
$$= 6x - 7 \quad , \quad \text{for } x < 3.$$

(2) Find the values of p and q that make function $f(x)$ differentiable everywhere on R

$$f(x) = 3 - x \quad , \quad \text{for } x < 1$$
$$= px^2 + qx \quad , \quad \text{for } x \ge 1.$$

(3) Determine the values of p and q that make the function $f(x)$ differentiable on R where

$$f(x) = px^3 \quad , \quad \text{for } x < 2$$
$$= x^2 + q \quad , \quad \text{for } x \ge 2.$$

(4) Determine all real values of p and q that ensure the function

$$f(x) = px + q \text{ for, } x \le 1$$

$= \tan\left(\frac{\pi x}{4}\right)$, for $1 < x < 2$ is differentiable at $x = 1$.

(5) Discuss whether the function

$f(x) = |x+1| + |x-1|$ is differentiable $\forall x \in R$

(6) Test whether the function

$$f(x) = 2x - 3, \quad \text{for } x \ge 2$$
$$= x - 1, \text{ for } x < 2$$

is differentiable at $x = 2$.

(7) Test whether the function

$$f(x) = x^2 + 1, \quad \text{for } x \ge 2$$
$$= 2x + 1, \text{ for } x < 2$$

is differentiable at $x = 2$.

(8) Test whether the function

$$f(x) = 5x - 3x^2 \quad \text{for } x \ge 1$$
$$= 3 - x, \text{ for } x < 1$$

is differentiable at $x = 1$.

(9) If $f(2) = 4$, $f'(2) = 1$ then find

$$\lim_{x \to 2}\left[\frac{xf(2) - 2f(x)}{x - 2}\right]$$

10) If $y = \frac{e^x}{\sqrt{x}}$ find $\frac{dy}{dx}$ when $x = 1$.

❖ ❖ ❖

ANSWERS

1. COMPLEX NUMBER

EXERCISE 1.1

Q.1 i) 0 ii) $11i$

Q.2 i) $(3-i)$ ii) $(3+i)$

iii) $(-\sqrt{5}-\sqrt{7})$ iv) $\sqrt{5}\, i$

v) $5i$ vi) $(\sqrt{5}+i)$

vii) $(\sqrt{2}-\sqrt{3}\, i)$ viii) $\cos\theta - i\sin\theta$

Q.3 i) $a = 3,\ b = \frac{1}{2}$ ii) $a = 5,\ b = 0$

iii) $a = -3,\ b = 7$ iv) $a = \pm 2,\ b = \pm 6$

v) $a = \frac{3}{13},\ b = \frac{2}{13}$

vi) $a = \frac{3}{2},\ b = \frac{-1}{2}$

Q.4 i) $a = -4,\ b = -3$ ii) $a = 0,\ b = 1$

iii) $a = \frac{-7}{2},\ b = \frac{1}{2}$ iv) $a = \frac{3}{10},\ b = \frac{-1}{10}$

v) $a = -1,\ b = 0$ vi) $a = \frac{-8}{29},\ b = 0$

vi) $a = \frac{-1}{4},\ b = \frac{-1}{4}$

viii) $a = \frac{11}{19},\ b = \frac{2\sqrt{3}}{19}$

ix) $a = \frac{23}{13},\ b = \frac{15}{13}$

Q.6 $4 + 6i$

Q.7 i) $-i$ ii) 1

iii) i iv) 1

v) $-i$ vi) -1

vii) 1 viii) 0

Q.9 i) $2i$ ii) 0

Q.10 1

Q.11 1

Q.13 Yes, its value $= -2 \in R$

Q.14 0

Q.16 0

Q. 24 i) $x = 1,\ y = 2$ ii) $x = 0,\ y = 2$

iii) $x = 1,\ y = 2$ iv) $x + y = 3$

v) $x + y = 9$

EXERCISE 1.2

Q.1 i) $\pm(1-3i)$ ii) $\pm(4+3i)$

iii) $\pm(2-\sqrt{3}\, i)$ iv) $\pm(\sqrt{5}+\sqrt{2}\, i)$

v) $\pm(\sqrt{3}-i)$ ii) $\pm(\sqrt{5}+\sqrt{2}\, i)$

Q.2 i) $\frac{-1+\sqrt{7}i}{8}, \frac{-1-\sqrt{7}i}{8}$

ii) $\frac{\sqrt{3}+\sqrt{5}i}{4}, \frac{\sqrt{3}-\sqrt{5}i}{4}$

iii) $\frac{7+\sqrt{11}i}{6}, \frac{7-\sqrt{11}i}{6}$ iv) $2+3i, 2-3i$

Q.3 i) $-5i, 2i$ ii) $\frac{i}{2}, -2i$

iii) $-2i$ iv) $-2i$

Q.4 i) $-1+2i, 3-i$ ii) $3\sqrt{2}, -2i$

iii) $2+3i, 3-4i$ iv) $1-i, \frac{4}{5}\ \frac{-2i}{5}$

Q. 5 i) 7 ii) 2

iii) 7 iv) $200 - 264i$

v) 6

EXERCISE 1.3

Q.1 i) $\sqrt{74}, -\tan^{-1}\left(\frac{5}{7}\right)$

ii) i , $\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{3}}\right)$

iii) $17, -\tan^{-1}\left(\frac{15}{8}\right)$ iv) $3\sqrt{2}, \frac{3\pi}{2}$

v) $4\sqrt{2}, \frac{\pi}{4}$ vi) $\frac{11\pi}{6}$

vii) 3,0 viii) $\sqrt{2}, \frac{\pi}{4}$

ix) $2, \frac{\pi}{3}$ x) $5\sqrt{2}, \tan^{-1}7$

Q.2 $\theta = n\pi, n \in Z$

Q.4 i) $2\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right); 2e^{\frac{2\pi}{3}i}$

ii) $2\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right); 2e^{\frac{3\pi}{2}i}$

iii) $1(\cos\pi + i\sin\pi); e^{\pi i}$

iv) $\frac{1}{\sqrt{2}}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right); \frac{1}{\sqrt{2}}e^{\frac{7\pi}{4}i}$

v) $\frac{1}{\sqrt{2}}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right); \frac{1}{\sqrt{2}}e^{\frac{3\pi}{4}i}$

vi) $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right); \sqrt{2}e^{\frac{3\pi}{4}i}$

Q.5 i) $\frac{3}{2}+\frac{\sqrt{3}i}{2}$ ii) $(1-i)$

iii) $\frac{-7\sqrt{3}}{2}-\frac{7i}{2}$ iv) $\frac{1+\sqrt{3}i}{2}$

v) $\frac{-1+i\sqrt{3}}{2}$ vi) $\frac{-\sqrt{3}+i}{2}$

Q. 6 $\frac{1}{\sqrt{2}}, \frac{3\pi}{4}$

Q. 7 $\sqrt{2}(\cos\theta + i\sin\theta)$ where

$\tan\theta = \frac{\sqrt{3}+1}{\sqrt{3}-1}$

EXERCISE 1.4

Q.1 i) 1 ii) 1

iii) 1 iv) 1

Q.3 i) -1 ii) 0

iii) -1 iv) 0

v) 1

Q.6 i) $x^2+y^2=100$ ii) $(x-3)^2+y^2=4$

iii) $(x-5)^2+(y+6)^2=25$

iv) $x+2=0$ v) $y=-x$

vi) $2y-3=0$

Q.7 i) $\text{Cos}2\theta + i\,\text{Sin}2\theta$ ii) $\text{Cos}11\theta + i\,\text{Sin}11\theta$

iii) 1

Q.8 i) $-4+4i$ ii) $-8i$

iii) $-8+8\sqrt{3}\,i$ iv) $512\sqrt{3}-512i$

MISECLLANEOUS EXERCISE - 1

(I)

Q. No.	1	2	3	4	5
Ans	B	D	A	C	B

Q. No.	6	7	8	9	10
Ans	A	C	A	D	D

(II) 1)

i) $(3+8i)$ ii) $-4+0i$

iii) $(14-5i)$ iv) $+10i$

v) $-30+10i$ vi) $\frac{1}{2}+\frac{7}{2}i$

vii) $\frac{-35}{26}+\frac{45}{26}i$ viii) $\frac{1}{4}+\frac{\sqrt{15}}{4}i$

ix) $-i$ x) $\frac{40}{25}+\frac{56}{25}i$

2) i) $x=2,\ y=1$ ii) $x=17, y=19$

iii) $x=\frac{28}{61}, y=\frac{3}{61}$ iv) $x=4, y=-2$

3) i) $-i$ ii) 0

4) i) 1 ii) 0

5) i) $\pm(3+5i)$ ii) $\pm(4-i)$

iii) $\pm\ \sqrt{3}+i)$ iv) $\pm(3+3i)$

v) $\pm(2-i)$ vi) $\pm\sqrt{2}\ (2+i)$

16) i) $17,\ \tan^{-1}\left(\frac{15}{8}\right)$,

$17(\cos\theta+i\sin\theta$ where $\tan\theta=\frac{15}{8}$

ii) $\sqrt{37},\ \theta=\tan^{-1}\left(\frac{-1}{6}\right)$,

$\sqrt{37}(\cos\theta+i\sin\theta)$ where $\tan\theta=\frac{-1}{6}$

iii) $1,\ \frac{\pi}{3},\ 1\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

iv) $1,\ 5\frac{\pi}{4},\ 1\left(\cos 5\frac{\pi}{4}+i\sin\left(5\frac{\pi}{4}\right)\right)$

v) $2,\ \frac{\pi}{2},\ 2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

vi) $3,\ \frac{3\pi}{2},\ 3\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)$

vii) $1,\ \frac{\pi}{4},\ 1\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$

9) $x=1,\ y=2$

12) i) $2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right);\ 2e^{\frac{\pi}{3}i}$

ii) $\sqrt{38}(\cos\theta+i\sin\theta)$ where $\tan\theta=\frac{-\sqrt{2}}{6}$;

$\sqrt{38}\,e^{i\theta}$

iii) $3(\cos\theta+i\sin\theta)$ where $\tan\theta=-\sqrt{3}$;

$3e^{i\theta}$

16) i) $-i$ ii) 0 iii) -1

17) $\frac{1}{4}+\frac{9}{4}i$

20) 2

2. SEQUENCES AND SERIES

EXERCISE 2.1

1) i), ii) and iii) are G.Ps iv) and v) not G.Ps.

i) $2(3)^{n-1}$ ii) $(-5)^{n-1}$ iii) $\sqrt{5}\left(\frac{1}{5}\right)^{n-1}$

2) i) $t_7 = \frac{1}{81}$ ii) $t_6 = 7$ iii) $a = -7$ iv) $r = 3$

3) $t_{10} = 5^{10}$ 4) $x = \frac{4}{9}$ 5) $t_1 = \frac{4}{25}, \ r = \frac{5}{2}$

6) 3, 6, 12 or 12, 6, 3

7) $\frac{1}{27}, \frac{1}{3}, 3, 27$ or $27, 3, \frac{1}{3}, \frac{1}{27}$

8) 1, 2, 4, 8, 16 or 16, 8, 4, 2, 1

11) 800 12) $80\left(\frac{3}{4}\right)^6$ ft, $80\left(\frac{3}{4}\right)^n$ ft

13) i) 6, −3 ii) $3(2)^{19}; -3$ iii) $3(2)^{n-1}; 3(-1)^{n-1}$

14) i) $200(1.1)^3$ ii) $200(1.1)^{10}$ iii) $200(1.1)^n$

15) i) 10 ii) 4 iii) $4(5)^{n-1}$

EXERCISE 2.2

1) i) $3\left(2^n - 1\right)$ ii) $\frac{p^2}{p-q}\left[1-\left(\frac{q}{p}\right)^n\right]$ if $q < p$

and $\frac{p^2}{q-p}\left[\left(\frac{q}{p}\right)^n - 1\right]$ if $q > p$

iii) $\frac{7}{9}\left[1-\frac{1}{10^n}\right]$ iv) $\frac{-\sqrt{5}}{\left(\sqrt{5}+1\right)}\left[\left(-\sqrt{5}\right)^n - 1\right]$

2) i) $\frac{266}{243}$ ii) 3 3) i) 5 ii) $\frac{3}{5}$

4) i) 2 ii) 2046

5) i) $\frac{3}{81}\left[10\left(10^n - 1\right) - 9n\right]$

ii) $\frac{8}{81}\left[10\left(10^n - 1\right) - 9n\right]$

6) i) $\frac{4}{81}\left[9n - \left(1 - \frac{1}{10^n}\right)\right]$

ii) $\frac{7}{81}\left[9n - \left(1 - \frac{1}{10^n}\right)\right]$

7) i) $\frac{5}{9}\left[1-\left(\frac{1}{10}\right)^n\right]$ ii) $\frac{2}{9}\left[1-\left(\frac{1}{10}\right)^n\right]$

8) $t_n = 4(3)^{n-1}$

11) i) 6138 ` ii) $\frac{15}{2}\left[3^{10} - 1\right]$

12) 20.1 Lac 15) 10 years

EXERCISE 2.3

1) i) 1 ii) does not exist

iii) $-\frac{9}{4}$ iv) does not exist v) 90

2) i) $\frac{7}{9}$ ii) $\frac{22}{9}$ iii) $\frac{106}{45}$ iv) $\frac{2296}{45}$

3) 4 4) $-\frac{11}{6}$ 5) $\frac{15}{4}, \frac{15}{16}, \frac{15}{64} \ldots$

6) i) 4 ii) $-\frac{1}{4}$ iii) $\frac{8}{3}$ iv) $\frac{4}{9}$

7) i) $\frac{5}{3}$ ii) $\frac{4\sqrt{2}}{\sqrt{2}-1}$ 8) 25 m

EXERCISE 2.4

1) (i) and (iii) are H.P. ; (ii) is not H.P.

2) i) $\frac{1}{3n-1};\frac{1}{23}$ ii) $\frac{1}{2n+2};\frac{1}{18}$

iii) $\frac{1}{5n};\frac{1}{40}$

3) 5 4) $\frac{24}{5}$ 5) 60 6) $\frac{1}{9},\frac{1}{11}$

7) $-3, 9$ 8) 4, 16 9) 14, 56

EXERCISE 2.5

1) i) $\frac{2}{1-x}+\frac{2x\left(1-x^{n-1}\right)}{(1-x)^2}$

ii) $\frac{1}{1-x}+\frac{3x\left(1-x^{n-1}\right)}{(1-x)^2}$

iii) $-\frac{1}{2}+\frac{3-3^n}{4}$ iv) $3 - 6.2^{n-1}$

2) i) $\frac{4}{3}$ ii) $\frac{15}{4}$ iii) $\frac{3}{16}$

EXERCISE 2.6

1) $\frac{n\left(4n^2+9n-1\right)}{6}$ 2) $n(2n^2 + 2n + 1)$

3) $\frac{n(n+3)}{4}$ 4) $\frac{n(n+4)}{6}$

5) $\frac{16n(n+1)(2n+1)}{6}+\frac{16n(n+1)}{2}+3n$

6) $\frac{2n(n+1)(2n+1)}{3}$ 7) 2485

8) $n^2(n+1)^2+\frac{8.n(n+1)(2n+1)}{6}+\frac{3.n(n+1)}{2}$

9) 48

MISCELLANEOUS EXERCISE - 2

(I)

1	2	3	4	5	6	7	8	9	10
D	C	A	B	A	C	C	A	D	C

II)

1) 3072 2) $\frac{11}{81}$ 3) $\frac{3}{4}$

4) $a=\frac{49}{5}$; $r=\frac{5}{7}$ 5) 5,10,20 or 20,10,5

6) $\frac{1}{3}$, 1, 3, 9, 27 or 27, 9, 3, 1, $\frac{1}{3}$

8) $\frac{2}{81}\left[10\left(10^n-1\right)-9n\right]$

9) $\frac{2}{3}\left(1-\frac{1}{10^n}\right)$ 10) $\frac{n}{6}\left(10n^2+27n-1\right)$

11) $\frac{n(n+1)\left(3n^2-17n+8\right)}{12}$

12) $\frac{n(n+1)(2n+9)}{6}$ 13) $\frac{n(n+1)(2n+1)}{24}$

14) $6\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$

15) $2n(n+1)(n+3)(n+4)$

16) $\frac{2n(n+1)(2n+1)}{6}+\frac{3n(n+1)}{2}+n$

17) 14184 18) 9 19) 1275

20) 5 21) $\frac{7}{15}$ 22) $\frac{1}{3}$ 23) $2\left[1-\left(\frac{2}{3}\right)^n\right]$

24) 2 25) $2187\left[1-\left(\frac{2}{3}\right)^8\right]$ 26) 1

27) 10, 20 (28) A.P. 32) $\frac{8}{45}$ 33) 5

3. PERMUTIONS AND COMBINATIONS

EXERCISE 3.1

1) 50 ways
2) 12
3) i) 25 ii) 20
4) i) 100 ii) 48
5) 125
6) 124
7) 31
8) 90
9) 225
10) 23
11) 276
12) 207
13) 12
14) 216

EXERCISE 3.2

1) i) 40320 ii) 3628800
iii) 3628080 iv) 24

2) i) 665280 ii) 2
iii) 479001600 iv) 144
v) 84 vi) 29
vii) 57.93 viii) 20160

3) i) $\frac{10!}{4!}$ ii) $3^5 \times 5!$
iii) $\frac{9!}{5!}$ iv) $5^5 \times 5!$

4) i) 28 ii) 1
iii) 3003 iv) 6435

5) i) 1848 ii) 43/14
iii) 5 iv) 6
v) 8

6) i) 11 ii) 17
iii) 7 iv) 8
v) 5

10) i) $(2n + 1)(2n + 2)$ ii) $\frac{n+3}{n-2}$
iii) $\frac{-(n+1)}{n(n-2)!}$ iv) $(3n +2)n!$
v) $\frac{(n-1)}{n!}$ vi) $\frac{(n^2+1)}{(n+1)!}$
vii) $\frac{6}{(n+2)!}$ viii) $\frac{1}{(n+2)!}$

EXERCISE 3.3

1) $n = 9$
2) $m = 6,\ n = 2$
3) $r = 6$
5) a) 2401 b) 840
6) a) 30240 b) 151200
c) 43200 d) 5040
7) $\frac{12! \times 13!}{7}$
8) a) 1440 b) 720
c) 7! d) 240
e) 120 f) 120

9) 144

10) a) 1296 b) 360

11) 100

12) 720 a) 120, b) 600

13) 46800, 20800

14) 243

15) i) 2880 ii) 5040

16) i) 120 ii) 148 iii) 72

17) i) 720 ii) 144 iii) 288 iv) 144

EXERCISE 3.4

1) i) 120 ii) 60480
iii) 30240 iv) 5040
v) 302400

2) 144

3) a) 70 b) 37

4) $\dfrac{13!}{5!4!4!}$

5) $\dfrac{12!}{2!3!2!}$

6) $\dfrac{11!}{4!2!2!}$ a) 414960 b) $\dfrac{8!}{2!2!}$

7) 210 8) 60

9) $\dfrac{10!}{2!3!2!}$ 10) 1260, 1230

11) 180 12) 144

13) 360, 96 14) 180, 60

15) a) 1800 b) 72

EXERCISE 3.5

1) $7! = 5040$ 2) $20!$, $2 \cdot 18!$

3) a) $2 \cdot 23!$ b) $2! \cdot 22!$

4) $\dfrac{14!}{2}$

5) $2 \cdot 8!$

6) a) $5! \times 2! = 240$ b) 9605

7) $7! \times 6P_5$ 8) 144

9) $\dfrac{9!}{4!}$ 10) $13 \cdot 14!$

EXERCISE 3.6

1) a) 1365 b) 3160 c) ${}^{16}C_5$ d) ${}^{19}C_{15}$

2) a) $n = 2$ b) $n = 7$
c) $n = 9$

3) $r = 4$

4) a) $n = 10$, $r = 3$ b) $n = 10$, $r = 4$

5) $r = 8$ 6) 126

7) 39200 8) 120

9) 12 10) 190

11) ${}^{n}C_2 - n$; a) 35 b) 90 c) 54 d) 20

12) 210

13) a) 45 b) 40

14) a) 220 b) 216

15) 151200

16) i) $n = 20$ ii) $n = 4, 3$
iii) $n = 1, 5$ iv) $n = 4$
v) $n = 6$

17) $x = r!$ 18) $r = 7$

19) 14161

20) a) 2508 b) 1646 c) 4125

21) 16 22) 2275

23) 36873 ; 6885 24) 425

25) 51051

26) a) 84 b) 126

MISCELLANEOUS EXERCISE - 3

(I)

1	2	3	4	5	6	7	8	9	10
C	A	B	D	A	C	A	A	C	D

(II)

1) 45 2) 120

3) 720 ; AENREM 4) 990

5) 360 6) 5541965

7) 15 8) $\dfrac{30!}{7!\ 10!\ 13!}$

9) 127 10) ${}^9C_3 + {}^9C_4 + {}^9C_5$

11) 4095 12) 48

13) 1680 14) 63

15) $\dfrac{20!}{8!\ 7!\ 5!}$ 16) 896

17) 60

18) i) 66 ii) 11 iii) 220 iv) 55

4. METHOD OF INDUCTION AND BINOMIAL THEOREM

EXERCISE 4.1

Hints :

2) $p(n) = 3 + 7 + 11 + ... + (4n - 1) = n\ (2n + 1)$

5) $p(n) = 1^3 + 3^3 + 5^3 + ... + (2n - 1)^3 = n^2\ (2n^2 - 1)$

7) $p(n) = 1.3 + 3.5 + 5.7 + ... + (2n - 1)\ (2n + 1)$

$= \dfrac{n}{3}(4n^2 + 6n - 1)$

9) $p(n) = \dfrac{1}{3.5} + \dfrac{1}{5.7} + \dfrac{1}{7.9} + ... + \dfrac{1}{(2n+1)(2n+3)}$

$= \dfrac{n}{3(2n+3)}$

EXERCISE 4.2

1) i) $49 + 20\sqrt{6}$ ii) $145\sqrt{5} - 229\sqrt{2}$

2) i) $16x^8 + 96x^6 + 216x^4 + 216x^2 + 81$

ii) $64x^6 - 192x^4 + 240x^2 - 160 + \dfrac{60}{x^2} - \dfrac{12}{x^4} + \dfrac{1}{x^6}$

3) i) $32\sqrt{3}$ ii) 1364

5) i) 108243216 ii) 1.61051

6) i) 970.299 ii) 0.6561

7) i) 16 ii) 16

8) 1.1262

9) 1.051

10) 0.5314

EXERCISE 4.3

1) i) $4032x^{10}$ ii) $84480x^2$

iii) $\dfrac{10500}{x^3}$ iv) $\dfrac{5a^{16}}{3}$

v) $\dfrac{{}^{13}C_9 \cdot 3^4 \cdot 4^9}{a^5}$

2) i) $122472\sqrt{2}$ ii) 700000

iii) 48620 iv) $\dfrac{5}{16}$

v) $\dfrac{-105}{8192}$

3) i) $\dfrac{1792}{9}$ ii) -96096

iii) 405 iv) 84

v) 10500000

4) i) 924 ii) $35x^5$, $35x^2$

iii) $1120x^4$ iv) -252

v) $-462x^9$ and $462x^2$

5) $k = 5$

6) 51030

7) m = 8

EXERCISE 4.4

1) i) $1-4x+10x^2-20x^3+\ldots$

ii) $1-\frac{x}{3}-\frac{x^2}{9}-\frac{5x^3}{81}-\ldots$

iii) $1+3x^2+6x^4+10x^6+\ldots$

iv) $1-\frac{x}{5}+\frac{3x^2}{25}-\frac{11x^3}{125}+\ldots$

v) $1-x^2+x^4-x^6+\ldots$

2) i) $a^{-3}\left[1+\frac{3b}{a}+\frac{6b^2}{a^2}+\frac{10b^3}{a^3}+\ldots\right]$

ii) $a^{-4}\left[1-\frac{4b}{a}+\frac{10b^2}{a^2}-\frac{20b^3}{a^3}+\ldots\right]$

iii) $a^{\frac{1}{4}}\left[1+\frac{b}{4a}-\frac{3b^2}{32a^2}+\frac{7b^3}{128a^3}+\ldots\right]$

iv) $a^{-\frac{1}{4}}\left[1+\frac{b}{4a}-\frac{5b^2}{32a^2}+\frac{15b^3}{128a^3}+\ldots\right]$

v) $a^{-\frac{1}{3}}\left[1-\frac{b}{3a}+\frac{2b^2}{9a^2}-\frac{14b^3}{81a^3}+\ldots\right]$

3) i) $1-8x+40x^2+\ldots$

ii) $1-\frac{3x}{2}+\frac{27x^2}{8}+\ldots$

iii) $2^{\frac{1}{3}}\left(1-\frac{x}{2}-\frac{x^2}{4}\ldots\right)$

iv) $5^{-\frac{1}{2}}\left(1-\frac{2x}{5}+\frac{6x^2}{25}+\ldots\right)$

v) $5^{-\frac{1}{3}}\left(1+\frac{x}{5}+\frac{2x^2}{25}+\ldots\right)$

4) i) 9.9499 ii) 5.0133

iii) 2.0025 iv) 0.9057

v) 1.0625

MISCELLANEOUS EXERCISE - 4

(I)

1	2	3	4	5	6	7	8	9	10
B	C	C	B	A	D	B	D	B	D

(II)

4) $243x^{10} + 810x^8y + 1080x^6y^2 + 720x^4y^3 + 240x^2y^4 + 32y^5$

5) $\frac{16x^4}{81}-\frac{16x^2}{9}+6-\frac{9}{x^2}+\frac{81}{16x^4}$

6) $\frac{27}{2}x^4y^6$ 7) $\frac{1760}{x^3}$

8) i) -20 ii) $\frac{-63x^5}{8y^5}$

iii) $280x^8y^6$ and $560x^6y^8$

iv) $\frac{189}{16}x^6$ and $-\frac{21}{8}x^3$

9) i) 378 ii) 153

10) i) 2268 ii) 7920

12) ± 2 13) 2

14) $\frac{9}{7}$ 15) 2

18) $1+\frac{x}{3}+\frac{x^2}{6}+\frac{5x^3}{54}+\ldots$

19) $1+\frac{x}{4}+\frac{5x^2}{32}+\frac{15x^3}{128}+\ldots$

20) $5^{-\frac{1}{2}}\left[1-\frac{2x}{5}+\frac{6x^2}{25}-...\right]$

21) 9.9833

22) 0.2451

23) -80

24) $a = 3, b = -2, c = 57$

25) $n = 9$

26) $n = 6$; $k = -2$

5. SETS AND RELATIONS

EXERCISE 5.1

1) i) A = {M, O, V, E, N, T}

ii) B {−1, 0, 1, 2, 3, 4}

iii) C = {3, 5, 7, ...}

2) i) $\{x \mid x \in W, x \notin N\}$

ii) $\{x \mid -3 \le x \le 3, x \in Z\}$

iii) $\{x \mid x = \frac{n}{n^2+1}, n \in N \text{ and } n \le 7\}$

iv) $\{x \mid x = (-1)^{n-1} \times (n-1), n \in Z\}$

3) $A \cup B \cup C = \{\frac{-5}{3}, -1, \frac{-1}{2}, \frac{3}{2}, 3\}$

4) $A \cap B \cap C = \{\ \}$

6) i) 45 ii) 10 iii) 10 iv) 25

7) i) 32 ii) 63

8) i) 1750 ii) 250 iii) 1100

9) 42

10) i) 114 ii) 38 iii) 188

11) $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \{1,2,3\}\}$

12) i) $\{x \mid x \in R, -3<x<0\}$

ii) $\{x \mid x \in R, 6 \le x \le 12\}$

iii) $\{x \mid x \in R, x \ge 6\}$

iv) $\{x \mid x \in R, x \le 5\}$

iv) $\{x \mid x \in R, 1<x\le 5\}$

iv) $\{x \mid x \in R, -3 \le x<4\}$

13) 18

14) i) $(-8, 6]$ ii) $(-\infty, -4) \cup (5,\infty)$

iii) $(-\infty, 4) \cup [5,\infty)$ iv) $\left[\frac{1}{3}, \frac{1}{2}\right]$

15) i) $(-7, 6]$ ii) $[2, 9]$

iii) $(-7, 3] \cup [4, 9]$ iv) $[2, 3]$

v) $[4, 6]$ vi) $\{\ \}$

vii) $(3, 6]$ viii) $(-\infty, 2) \cup (9, \infty)$

ix) $[2, 4)$ iv) $(-7, 2)$

EXERCISE 5.2

1) $x = 2$, $y = -2$

2) $x = \frac{1}{6}, y = \frac{15}{2}$

3) $A\times B = \{(a,x), (b,x), (c,x), (a,y), (b,y), (c,y)\}$

$B\times A = \{(x,a), (x,b), (x,c), (y,a), (y,b), (y,c)$

$A\times A = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a)\ (c,b), (c,c)\}$

$B\times B = \{(x,x), (x,y), (y,x), (y,y)\}$

4) $P\times Q = \{(1,1), (1,4), (2,1), (2,4), (3,1), (3,4)\}$

$Q\times P = \{(1,1), (1,2), (1,3), (4,1), (4,2), (4,3)\}$

6) {(0,10), (6,8), (8,6), (10,10)}

8) i) $R_1 = \{(2,4), (3,9), (5,25), (7,49), (11,121), (13,169)\}$

Domain $R_1 = \{2,3,5,7,11,13\}$

Range $R_1 = \{4,9,25,49,121,169\}$

ii) $R_2 = \{(1,1), (2,\frac{1}{2}), (3,\frac{1}{3}), (4,\frac{1}{4}), (5,\frac{1}{5})\}$

Domain $R_2 = \{1,2,3,4,5\}$

Range $R_2 = \{1,\frac{1}{2},\frac{1}{3},\frac{1}{4},\frac{1}{5}\}$

iii) $R_3 = \{(1,3), (2,6), (3,9)\}$

Domain $R_3 = \{1,2,3\}$

Range $R_3 = \{3,6,9\}$

iv) $R_4 = \{(1,4), (1,6), (2,4), (2,6)\}$

Domain $R_4 = \{1,2\}$

Range $R_4 = \{4,6\}$

v) $R_5 = \{(0,3), (1,2), (2,1), (3,0)\}$

Domain $R_5 = \{0,1,2,3\}$

Range $R_5 = \{3,2,1,0\}$

vi) $R_6 = \{(1,4), (2,4), (3,4), (4,4), (5,4)\}$

Domain $R_6 = \{1,2,3,4,5\}$

Range $R_6 = \{4\}$

vii) $R_7 = \{(1,5), (2,4), (3,3), (4,2), (5,1)\}$

Domain $R_7 = \{1,2,3,4,5\}$

Range $R_7 = \{5,4,3,2,1\}$

viii) $R_8 = \{(1,3), (2,4), (3,5), (4,6)\}$

Domain $R_8 = \{1,2,3,4\}$

Range $R_8 = \{3,4,5,6\}$

MISCELLANEOUS EXERCISE - 5

I)

1	2	3	4	5	6	7	8	9	10
C	D	D	C	A	D	C	D	D	D

II)

1) i) $A = \{x \,/\, x = 10n, n \in N, n \leq 5\}$

ii) $B = \{x \,/\, x$ is the vowel of English alphabet$\}$

iii) $C = \{x \,/\, x$ is a day of a week$\}$

2) i) $\{1,2,4,6,7,9,11\}$ ii) $\{\ \}$

iii) $\{1,10\}$ iv) $\{2,4,6,7,11\}$

v) $\{1,2,3,4,5,6,7,8,9,10,11,12\}$

vi) $\{4,7\}$

3) 230 4) 12

5) i) $A \times A = \left\{\begin{matrix}(1,1),(1,2),(1,3),(2,1),(2,2),\\(2,3),(3,1),(3,2),(3,3)\end{matrix}\right\}$

$A\times B = \{(1,2),(1,4),(2,2),(2,4),(3,2),(3,4)\}$

$B\times A = \{(2,1),(2,2),(2,3),(4,1),(4,2),(4,4)\}$

$B\times B = \{(2,2),(2,4),(4,2),(4,4)\}$

$(A\times B)\cap(B\times A) = \{(2,2)\}$

ii)

$$A\times A\times A = \left\{\begin{matrix}(-1,-1,-1),(-1,-1,1),(-1,1,-1,\\(-1,1,1),(1,-1,-1),(1,-1,1),\\(1,1,-1),(1,1,1)\end{matrix}\right\}$$

6) i) Yes; $D = \{1\}$, $R = \{4,5,6\}$

ii) Yes; $D = \{1,2,3\}$, $R = \{4,5,6\}$

iii) Yes; $D = \{1,2,3\}$, $R = \{4,5,6\}$

iv) No.

7. i) $D = \{1,2,3,4\}$, $R = \{4\}$

ii) $D = \{-2,-1,0,1,2\}$, $R = \{0,1,2,3\}$

8. i) $\{\ \}$ ii) $A\times A$

9) reflexive, symmetric and transitive.

10) Yes

6. FUNCTION

EXERCISE 6.1

1) a) Yes b) No
c) No

2) a) No b) Yes
c) No d) Yes

3) a) Yes b) No
c) Yes d) Yes
e) No

4) a) 1 b) 19 c) $-\frac{1}{4}$ d) $x^2 - x - 1$
e) $x^2 + 3x + 1$ f) $h + 1$

5) a) $\frac{6}{5}$ b) ± 3 c) $\frac{1}{2}, -\frac{2}{3}$ d) 1,–2,3

6) a) 0, ± 3 b) $\frac{17 \pm \sqrt{33}}{2}$

7) 1) a = –2, b = 2

8) a) R; $\left[-\frac{11}{7}, \infty\right)$ b) $R - \{2\}$; $R - \{1\}$
c) $(-5, \infty)$; R^+ d) R; R e) [2,5] ; $[0, \frac{3}{2}]$
f) [3,7) ; [0,∞] g) [–4,4] ; [0,4]

9) a) A = s^2 b) A = $\frac{p^2}{16}$

10) a) A = πr^2 b) A = $\frac{\pi d^2}{4}$ c) A = $\frac{c^2}{4\pi}$

11) $x(30 - 2x)^2$; (0,15)

12) Not a function; f (0) has 2 values.

13) a) Injective but not surjective
b) neither injective nor surjective
c) Surjective but not injective
d) injective but not surjective
e) injective and surjective

16) $\frac{3}{16}$

17) a) $5 = \log_2 32$ b) $0 = \log_{54} 1$
c) $1 = \log_{23} 23$ d) $\frac{3}{2} = \log_9 27$
e) $-4 = \log_3\left(\frac{1}{81}\right)$ f) $-2 = \log_{10} 0.01$
g) $ln\ 7.3890 = 2$ h) $ln\ 1.6487 = \frac{1}{2}$
i) $ln\ 6 = -x$

18) a) $2^6 = 64$ b) $\frac{1}{25} = 5^{-2}$ c) $0.001 = 10^{-3}$
d) $8 = \left(\frac{1}{2}\right)^{-3}$ e) $e^0 = 1$ f) $e^1 = e$ g) $\frac{1}{2} = e^{-0.693}$

19) a) $(5,\infty)$ b) $(-\infty,2) \cup (3,\infty)$

20) a) $\log p + \log q - \log r - \log s$
b) $\frac{1}{2} \log x + \frac{1}{3} \log y$
c) $3\ ln\ a + 2\ ln\ (a - 2) - \frac{1}{2}\ ln\ (b^2 + 5)$
d) $2\left[\frac{1}{3} ln(x-2) + 4 ln(2x+1) - ln(x+4) - \frac{1}{2} ln(2x+4)\right]$

21) a) $\log\left(\frac{x^5 y^7}{z}\right)$ b) $\log\left(\sqrt[3]{x} - 2\sqrt{x}\right)$
c) $ln\left[\frac{x^2 - 4}{(x+5)^3}\right]$

22) $\frac{5a+b}{2}$

24) $a = \frac{15}{4}$, $b = 9$

25) a) 3 b) 11, –1 c) 8 d) 1

EXERCISE 6.2

1) a) $9x + 4$ b) 0 c) 238

d) $\frac{3x+5}{6x-1}$; $R - \left\{\frac{1}{6}\right\}$

2) $\{(2,4), (4,2), (5,4)\}$

3) a) $50x^2 - 40x + 11$ b) $10x^2 + 13$

c) $8x^4 + 24x^2 + 21$

d) $25x - 12$

5) a) f^{-1} does not exist

b) f^{-1} doesn not exist

c) $f^{-1}(x) = \frac{3x+7}{6}$

d) f^{-1} does not exist

e) $f^{-1} = \sqrt[3]{\frac{x-8}{9}}$

f) f^{-1} does not exist

6) a) 22 b) 7 c) 3

7) a) -18 b) -14 c) 5 d) 25

8) a) 10 b) -5

9) a) 25 b) -3 c) -15 d) 21

10) a) -5 b) 1.75 c) -4.4. d) 42

11) a) $(-\infty,-9], [1,\infty)$ b) $1{\cdot}5$, $4{\cdot}5$

c) $\{\ \}$ d) $[-3,3]$

f) $3 + r$; $0 \le r < 1$ g) $\{\ \}$

h) N, Z i) $n + 0.5$, $n \in Z$

j) $x = 0$

MISCELLANEOUS EXERCISE - 6

(I)

1	2	3	4	5	6	7	8	9	10
B	B	B	C	C	A	A	B	C	B

(II)

1) i) Function ; $\{2,4,6,8,10,12,14\}$; $\{1,2,3,4,5,6,7\}$

ii) Not a function

iii) Function ; $\{2,3,5\}$; $\{1,2\}$

2) i) not one one ii) one one

3) i) not onto ii) not onto

4) $f^{-1}(x) = \left(\frac{x+8}{5}\right)^{\frac{1}{3}}$

5) $f^{-1}(x) = \frac{5(x-2)}{3}$

6) 1, -3, does not exist

7) i) 2 ii) 0

8) $3x^4 - 12x^3 + 13x^2 - 2x + 5$

9) $a = 4$, $f(4) = 16$

10) $a = 3$, $b = -2$

11) i) $g \circ f = \{(1,6),(2,8),(3,10),(4,12)\}$

ii) $g \circ f = \{(1,1),(2,64),(3,64),(4,27)\}$

12) i) $f \circ g = x^2 - 16x + 69$, $gof = x^2 - 3$

ii) $f \circ g = 3x^2 - 2$, $gog = 9x^2 - 12x + 4$

iii) $f \circ g = 256x^2$, $gof = 16x^2$

15) $f \neq g$

19) 8 22) log4

23) $\log_{10}5$

26) $\frac{3}{2}$, $\frac{1}{2}$ 33) 2

34) 3 37) -8

39) a) $(-3,0)$, $(2,5)$ b) $\{-2,2,4\}$

c) $[-3,2]$, $[2,3]$ d) $[-7,7]$

e) $[\frac{13}{5},7)$ f) $[2,4)$

g) $x = 0$ h) $x = 6k$, $k \geq 0$

40) a) $(-\infty,-3) \cup (-3,2) \cup (2,\infty)$

b) $[3,4) \cup (4,5)$

c) $[-1,1]$ d) W

e) $\{1,2,3\}$ f) $[0,1]$

g) $(-\infty,3-\sqrt{3}) \cup (3+\sqrt{3},\infty)$

41) a) $[0,\infty)$ b) $[-\frac{1}{6},\frac{1}{6}]$

c) $(0,\infty)$ d) $(-1,0]$

e) $(1,\infty)$

42) a) $fog(x) = x = gof(x)$

b) $fog(x) = x = gof(x)$

43) a) $f(x) = 2x - 3$

b) $f(x) = x^2 + 2$

44) a) $\dfrac{x}{\sqrt{1+2x^2}}$ b) x

7. LIMITS

EXERCISE 7.1

I) 1) $-\dfrac{1}{\sqrt{3}}$ 2) 15 3) $-\dfrac{1}{25}$

II) 1) $\dfrac{2\sqrt{3}}{3}$ 2) $-\dfrac{3}{16}$ 3) $\dfrac{3}{125}$ 4) $\pm\dfrac{2}{\sqrt{3}}$

III) 1) $\dfrac{n(n+1)}{2}$ 2) $\dfrac{2}{3\sqrt[3]{7}}$ 3) 4 4) 4

5) $-\dfrac{1}{6}$ 6) 24 7) $\dfrac{3\sqrt{a+2}}{2}$

8) $294\sqrt{7}$ 9) n^2

EXERCISE 7.2

I) 1) $-\dfrac{1}{4}$ 2) $-\dfrac{1}{2}$ 3) $-\dfrac{1}{2}$ 4) $-\dfrac{1}{2}$ 5) 8

II) 1) $\dfrac{4}{3}$ 2) 0 3) 0 4) $2x-2$ 5) -3

6) Does not exist

III) 1) 3 2) -2 3) $\dfrac{1}{2}$ 4) 0 5) $-\dfrac{3}{a^2}$

EXERCISE 7.3

I) 1) $\dfrac{1}{2\sqrt{6}}$ 2) $-\dfrac{1}{18}$ 3) -1 4) $2\sqrt{2}$

II) 1) $\dfrac{2}{3\sqrt{3}}$ 2) -8 3) $\dfrac{1}{8\sqrt{3}}$ 4) $-\dfrac{1}{2a}$ 5) $-\dfrac{2}{3}$

III) 1) $\dfrac{7}{2}$ 2) 1 3) 24 4) $-\dfrac{1}{3}$ 5) $\dfrac{1}{3}$

EXERCISE 7.4

I) 1) $\dfrac{m}{n}$ 2) 0 3) 2 4) $\dfrac{1}{2}$

II) 1) $\dfrac{n}{m}$ 2) $-\dfrac{1}{4}$ 3) $\dfrac{1}{\sqrt{2}}$

III) 1) $\dfrac{a^2-b^2}{c^2}$ 2) $-\dfrac{1}{4\sqrt{2}}$ 3) $2\sqrt{2}$ 4) -3

EXERCISE 7.5

I) 1) $\dfrac{1}{2}$ 2) $5a^{\frac{4}{5}}.\cos a$ 3) $\dfrac{1}{8}$

4) $\dfrac{1}{3}$ 5) $\dfrac{2}{\pi}$

II) 1) $-\dfrac{1}{2\sqrt{3}}$ 2) $\dfrac{1}{16\sqrt{2}}$ 3) $\dfrac{1}{36}$

4) $\dfrac{\cos\sqrt{a}}{2\sqrt{a}}$ 5) $-\dfrac{1}{2}$

EXERCISE 7.6

I) 1) $\dfrac{\log\left(\dfrac{9}{5}\right)}{\log 4}$ 2) $\log\dfrac{15}{2}$ 3) $\log(abc)$

4) $\log\left(\dfrac{40}{9}\right)$ 5) $\log 2.$

II) 1) $(\log 3)^2$ 2) $e^{\frac{2}{3}}$ 3) $e^{14/3}$ 4) $-\dfrac{2}{3}$

5) e^8 6) $e^{\frac{2}{3}}$

III) 1) $\dfrac{1}{2}\log\dfrac{a}{b}$ 2) $\dfrac{(\log 2)^3}{\log 3}$ 3) $\log 3.\log 5$

4) $(\log 5)^2$ 5) $\dfrac{1}{2}\left[\log\left(\dfrac{7}{2}\right)\right]^2$

EXERCISE 7.7

I) 1) $\dfrac{a}{e}$ 2) 1 3) $\dfrac{7}{8}$

II) 1) 7 2) 2 3) 2

III) 1) 15 2) $\dfrac{256}{81}$ 3) $\dfrac{1}{2}$ 4) $\left(\dfrac{3}{2}\right)^{30}$ 5) 4

MISCELLANEOUS EXERCISE - 7

I)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
C	B	A	D	C	C	C	C	A	D	B	D	C	B	B

II)

1) $\dfrac{5}{3}$ 2) Does not exist 3) $2\pi r$

4) Does not exist 5) 3 6) 21 7) $\dfrac{1}{2}$ 8) 1

9) $\dfrac{-1}{10}\log 2$ 10) $2(\log a)^2$ 11) $\cos a$ (12) $\dfrac{1}{2}$

13) $\dfrac{ab}{2}\log\dfrac{b}{a}$ 14) $\dfrac{(\log 5)^2}{\log 2}$

15) $\dfrac{(2)^2(7)^3}{(5)^5}=\dfrac{1372}{3125}$ 16) $a\sin a+\cos a$

17) $2\sqrt{2}$ 18) $(\log 2)^2$ 19) $(\log 2)^2$ 20) $\dfrac{1}{2}$

21) Does not exist 22) $\dfrac{n(n+1)(4n-1)}{6}$

23) $\dfrac{1}{256}$ 24) 2

8. CONTINUITY

EXERCISE 7.7

1) (i) Continuous at $x=-2$ (ii) Continuous at $x=\dfrac{\pi}{4}$ (iii) Discontinuous at $x=3$

2) (i) Discontinuous. (ii) Continuous. (iii) Continuous

3) Discontinuous at $x=-2$, $x=-1$, $x=0$, $x=1$.

4) Continuous.

5) (i) Discontinuous. (ii) Continuous. (iii) Continuous (iv) Continuous (v) Discontinuous at $x=2$

6) (i) Removable (ii) Jump (iii) Jump (iv) Removable

7) (i) Extension = 0 i.e. $f(0)=0$

(ii) Extension = 7/2, i.e. $f(0)=\dfrac{7}{2}$

(iii) Extension = –2/3, i.e. $f(-1)=-\dfrac{2}{3}$

8) (i) Continuous (ii) Discontinuous (iii) Continuous

9) (i) Removable, $f(0) = 3/2$

(ii) Removable, $f(0) = 5/3$

(iii) Removable, $f(0) = e^{-5/3}$

(iv) Irremovable

(v) Irremovable

10) (i) $-\frac{1}{4\sqrt{3}}$ (ii) $-\frac{4}{3}$ (iii) $4(\log 2)^2$

11) (i) $\frac{3}{2}$ (ii) $(\log 5)^2$ (iii) $a = -18/5,\ b = 7.$

(iv) $a = 2,\ b = -4.$

(v) $a = 1/2$ and $b = 1/2$

12) Continuous

13) Continuous

[*Clue* : $(\sin x + \cos x)^3 = [(\sin x + \cos x)^2]^{3/2}$

$= (1 + \sin 2x)^{3/2}$, *Let* $(1 + \sin 2x) = t$]

14) $p = -3$ and $q = 4$

MISCELLANEOUS EXERCISE - 8

(I)

1	2	3	4	5	6	7	8	9	10
A	D	D	B	A	B	A	B	C	C

(II) (1) Continuous on its domain except at $x = 5$ and $x = 6$.

(2) Continuous on its domain except at $x = 5$

(3) Discontinuous at $x = 0$

(4) Continuous at $x = 1$

(5) Discontinuous

(6) Discontinuous at $x = -1, 0, 1.$

(7) Continuous on R, except at $x = 5.$

(III) (1) Removable.

(2) Jump

(3) Continuous.

(IV) (1) Removable

$$f(x) = \frac{(x+3)(x^2-6x+8)}{x^2-x-12}$$
$$= -5 \quad \text{for } x = -3$$
$$= 2 \quad \text{for } x = 4$$

(2) Irremovable

(V) (1) e^6 (2) 125

(VI) (1) $a = 2 \quad b = 4$

(2) $a = -\frac{4}{5},\ b = \frac{27}{5}$

(VII) (1) $f(1) = \frac{\pi}{2}$ (2) $f(\pi) = \frac{49}{10}$

(VIII) (1) $f(1) < 0$ and $f(2) > 0$

(2) $f(z) = 0$; $f(3) < 0$ and $f(4) > 0$

9. DIFFERENTIATION

EXERCISE 9.1

1) (a) $2x + 3$ b) $3\cos(3x)$ c) $2e^{2x+1}$ d) $3^x \log 3$

e) $\frac{2}{2x+5}$ f) $2\sec^2(2x + 3)$

g) $5\sec(5x - 2)\tan(5x - 2)$ h) $\frac{3\sqrt{x}}{2}$

2) a) $\frac{1}{3}$ b) 2 c) 384 log 2 d) $\frac{2}{5}$

e) $3e^2$ f) $\frac{1}{\sqrt{2}}$

5) i) Continuous and differentiable

ii) Continuous and differentiable

6) Neither continuous nor differentiable at $x = 2$

7) Continuous but not differentiable

8) Continuous but not differentiable

9) Continuous and differentiable

EXERCISE 9.2

(I) (1) $\frac{4}{3}x^{\frac{1}{3}}+e^{x}-\cos x$

(2) $\frac{1}{2\sqrt{x}}+\sec^{2}x-3x^{2}$

(3) $\frac{1}{x}+\operatorname{cosec}x.\cot x+5^{x}\log 5+\frac{9}{2x^{\frac{5}{2}}}$

(4) $\frac{7}{3}x^{\frac{4}{3}}+\frac{4}{x^{\frac{1}{5}}}+\frac{2}{x^{\frac{7}{5}}}$

(5) $7^{x}\log 7+7x^{6}-\sqrt{x}-\frac{1}{x}$

(6) $-3\operatorname{cosec}^{2}x-5e^{x}+\frac{3}{x}+\frac{3}{x^{\frac{7}{4}}}$

(II) (1) $x^{4}(x\sec^{2}x+5\tan x)$

(2) $x^{2}(1+3\log x)$

(3) $(x^{2}+2)[(x^{2}+2)\cos x+4x\sin x]$

(4) $e^{x}\left(\frac{1}{x}+\log x\right)$

(5) $\sqrt{x}e^{x}[1+x\log x+\frac{3}{2}\log x]$

(6) $3x^{2}(1+3\log x)$

(III)

(1) $\frac{5}{2}x^{\frac{3}{2}}+x^{3}(1+4\log x)$

(2) $e^{x}\sec x(\tan x+1)-x^{\frac{2}{3}}(1+\frac{5}{3}\log x)$

(3) $4x^{3}-x^{\frac{3}{2}}\sin x+\frac{3}{2}\sqrt{x}\cos x-xe^{x}(x+2)$

(4) $(x^{3}-2)\sec^{2}x+3x^{2}\tan x+x\sin x-\cos .x+x^{6}.7^{x}(7+x\log 7)$

(5) $\frac{\sin x}{x}+\cos x\log x+e^{x}(-\sin x+\cos x)-e^{x}\left(\frac{1+2x}{2\sqrt{x}}\right)$

(6) $e^{x}(\sec^{2}x+\tan x)+\frac{\cos x}{x}-\sin x\log x-5^{x}\left(\frac{2x\log 5+1}{2\sqrt{x}}\right)$

(IV) (1) $-\frac{16x}{(x^{2}-5)^{2}}$

(2) $-\frac{5}{\sqrt{x}\left(\sqrt{x}-5\right)^{2}}$

(3) $\frac{e^{x}(x^{2}+e^{x})}{(x+e^{x})^{2}}$

(4) $\frac{x+(\log x)^{2}}{(x+\log x)^{2}}$

(5) $\frac{x^{2}(1+\sin x+x\cos x)+x\sin 2x}{(x+\cos x)^{2}}$

(6) $\frac{2e^{x}}{(3e^{x}-2)^{2}}$

(V) (1) $f(x)=5x^{2}-18x+3$

(2) $f(x)=(\sqrt{3}+1)\sin x+(\sqrt{3}-1)\cos x$

MISCELLANEOUS EXERCISE - 9

I.

(1) C	(2) D	(3) C	(4) B
(5) B	(6) D	(7) A	(8) C

II. (1) Differentiable

(2) p = -3 , q = 5

(3) p = 1/3 , q = -4/3

(4) $p=\pi/2$, $q=(2-\pi)/2$

(5) Not Differentiable

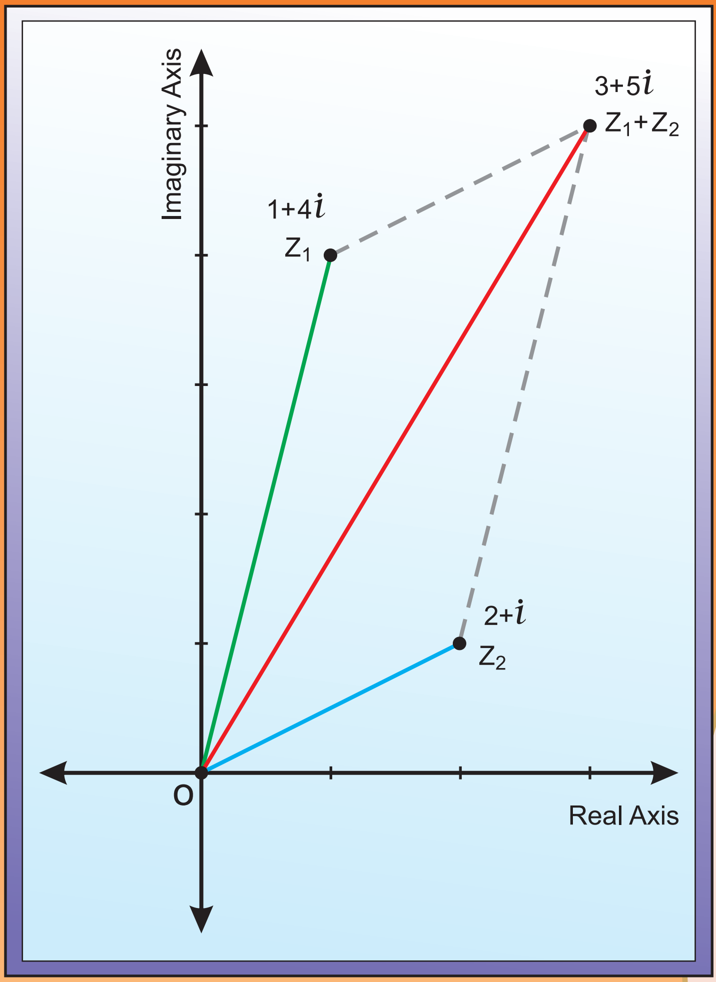
(6) Not Differentiable

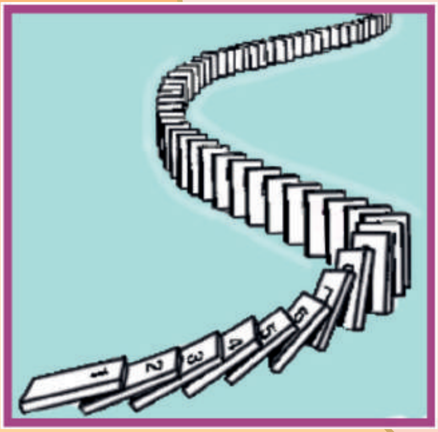
(7) Not Differentiable.

(8) Differentiable

(9) 2. Hint : Add and subtract 2f(2) in numerator.

(10) $\frac{e}{2}$





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