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Differentiate the functions with respect to x in Exercise 1 to 5.

1. $\cos x \cos 2x \cos 3x$

Solution: Let $y = \cos x \cos 2x \cos 3x$

Taking logs on both sides, we get

$$\log y = \log(\cos x \cos 2x \cos 3x)$$

$$= \log \cos x + \log \cos 2x + \log \cos 3x$$

Now,

$$\frac{d}{dx}\log y = \frac{d}{dx}\log \cos x + \frac{d}{dx}\log \cos 2x + \frac{d}{dx}\log \cos 3x$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}\frac{d}{dx}\cos x + \frac{1}{\cos 2x}\frac{d}{dx}\cos 2x + \frac{1}{\cos 3x}\frac{d}{dx}\cos 3x$$

$$\frac{1}{y}.\frac{dy}{dx} = \frac{1}{\cos x}(-\sin x) + \frac{1}{\cos 2x}(-\sin 2x)\frac{d}{dx}2x + \frac{1}{\cos 3x}(-\sin 3x)\frac{d}{dx}3x$$

$$\frac{1}{y}.\frac{dy}{dx} = -\tan x - (\tan 2x)2 - \tan 3x(3)$$

$$\frac{dy}{dx} = -y(\tan x + 2\tan 2x + 3\tan 3x)$$

$$\frac{dy}{dx} = -\cos x \cos 2x \cos 3x(\tan x + 2\tan 2x + 3\tan 3x)$$ [using value of y]

2. $\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

Solution: Let $y = \sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}$

$= \left(\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}\right)^{\frac{1}{2}}$

Taking logs on both sides, we get

$$\log y = \frac{1}{2}\left[\log(x-1)+\log(x-2)-\log(x-3)-\log(x-4)-\log(x-5)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left[\frac{1}{x-1}\frac{d}{dx}(x-1)+\frac{1}{x-2}\frac{d}{dx}(x-2)-\frac{1}{x-3}\frac{d}{dx}(x-3)-\frac{1}{x-4}\frac{d}{dx}(x-4)-\frac{1}{x-5}\frac{d}{dx}(x-5)\right]$$

$$\frac{dy}{dx} = \frac{1}{2}y\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$

$$\frac{dy}{dx} = \frac{1}{2}\sqrt{\frac{(x-1)(x-2)}{(x-3)(x-4)(x-5)}}\left[\frac{1}{x-1}+\frac{1}{x-2}-\frac{1}{x-3}-\frac{1}{x-4}-\frac{1}{x-5}\right]$$ [using the value of y]

3. $(\log x)^{\cos x}$

Solution: Let $y = (\log x)^{\cos x}$

Taking logs on both sides, we get

$$\log y = \log(\log x)^{\cos x} = \cos x \log(\log x)$$

$$\frac{d}{dx}\log y = \frac{d}{dx}\left[\cos x \log(\log x)\right]$$

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{d}{dx}\log(\log x) + \log(\log x)\frac{d}{dx}\cos x$$ [By Product rule]

$$\frac{1}{y}\frac{dy}{dx} = \cos x \frac{1}{\log x}\frac{d}{dx}(\log x) + \log(\log x)(-\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{\cos x}{\log x}.\frac{1}{\log x} - \sin x \log(\log x)$$

$$\frac{dy}{dx} = y\left[\frac{\cos x}{\log x} - \sin x \log(\log x)\right]$$

= $(\log x)^{\cos x}\left[\frac{\cos x}{\log x}-\sin x\log(\log x)\right]$

4. $x^x - 2^{\sin x}$

Solution: Let $y = x^x - 2^{\sin x}$

Put $u = x^x$ and $v = 2^{\sin x}$

$y = u - v$

$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}$ ……….(1)

Now, $u = x^x$

$\log u = \log x^x = x\log x$

$\frac{d}{dx}\log u = \frac{d}{dx}(x\log x)$

$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\log x + \log x\frac{d}{dx}x$

$\frac{1}{u}\frac{du}{dx} = 1+\log x$

$\frac{du}{dx} = u(1+\log x)$

$\frac{du}{dx}$ = $x^x(1+\log x)$ ……….(2)

Again, $v = 2^{\sin x}$

$\frac{dv}{dx}=\frac{d}{dx}2^{\sin x}$

$\frac{dv}{dx}=2^{\sin x}\log 2\frac{d}{dx}\sin x$ $\left[\because \frac{d}{dx}a^{f(x)} = a^{f(x)}\log a\frac{d}{dx}f(x)\right]$

$$\frac{dv}{dx} = 2^{\sin x}(\log 2).\cos x = \cos x.2^{\sin x}\log 2$$ ……….(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = x^x(1+\log x) - \cos x.2^{\sin x}\log 2$$

5. $(x+3)^2(x+4)^3(x+5)^4$

Solution: Let $y = (x+3)^2(x+4)^3(x+5)^4$

Taking logs on both sides, we get

$$\log y = 2\log(x+3) + 3\log(x+4) + 4\log(x+5)^4$$

Now,

$$\frac{d}{dx}\log y = 2\frac{d}{dx}\log(x+3) + 3\frac{d}{dx}\log(x+4) + 4\frac{d}{dx}\log(x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = 2\frac{1}{x+3}\frac{d}{dx}(x+3) + 3\frac{1}{x+4}\frac{d}{dx}(x+4) + 4\frac{1}{x+5}\frac{d}{dx}(x+5)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}$$

$$\frac{dy}{dx} = y\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

$$\frac{dy}{dx} = (x+3)^2(x+4)^3(x+5)^4\left(\frac{2}{x+3} + \frac{3}{x+4} + \frac{4}{x+5}\right)$$

(using value of y)

Differentiate the functions with respect to x in Exercise 6 to 11.

6. $\left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$

Solution: Let $y = \left(x+\frac{1}{x}\right)^x + x^{\left(x+\frac{1}{x}\right)}$

Put $\left(x+\frac{1}{x}\right)^{x}=u$ and $x^{\left(x+\frac{1}{x}\right)}=v$

$y=u+v$

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ……….(1)

Now $u=\left(x+\frac{1}{x}\right)^{x}$

$\log u=\log\left(x+\frac{1}{x}\right)^{x}=x\log\left(x+\frac{1}{x}\right)$

$\frac{1}{u}\frac{du}{dx}=x.\frac{1}{\left(x+\frac{1}{x}\right)}\frac{d}{dx}\left(x+\frac{1}{x}\right)+\log\left(x+\frac{1}{x}\right).1$

$\frac{1}{u}\frac{du}{dx}=x.\frac{1}{\left(x+\frac{1}{x}\right)}\left(x-\frac{1}{x^{2}}\right)+\log\left(x+\frac{1}{x}\right).1$

$\frac{du}{dx}=u\left[\frac{x^{2}-1}{x^{2}+1}+\log\left(x+\frac{1}{x}\right)\right]$

$=\left(x+\frac{1}{x}\right)^{x}\left[\frac{x^{2}-1}{x^{2}+1}+\log\left(x+\frac{1}{x}\right)\right]$ ……….(2)

Again $v=x^{\left(x+\frac{1}{x}\right)}$

$\log v=\log x^{\left(x+\frac{1}{x}\right)}=\left(x+\frac{1}{x}\right)\log x$

$\frac{1}{v}\frac{dv}{dx}=\left(x+\frac{1}{x}\right).\frac{1}{x}+\log x\left(\frac{-1}{x^{2}}\right)$

$$\frac{dv}{dx}=v\left[\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

$$\frac{dv}{dx}=x^{\left(x+\frac{1}{x}\right)}\left[\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$ ……….(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)^x\left[\frac{x^2-1}{x^2+1}+\log\left(x+\frac{1}{x}\right)\right]+x^{\left(x+\frac{1}{x}\right)}\left[\frac{1}{x}\left(x+\frac{1}{x}\right)-\frac{1}{x^2}\log x\right]$$

7. $(\log x)^x+x^{\log x}$

Solution: Let $y=(\log x)^x+x^{\log x}$ = $u+v$ where $u=(\log x)^x$ and $v=x^{\log x}$

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$ ………. (1)

Now $u=(\log x)^x$

$$\log u=\log(\log x)^x=x\log(\log x)$$

$$\frac{d}{dx}\log u=\frac{d}{dx}\left[x\log(\log x)\right]$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{d}{dx}\left[\log(\log x)\right]+\log(\log x)\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{\log x}\frac{d}{dx}\log x+\log(\log x).1$$

$$\frac{1}{u}\frac{du}{dx}=x\frac{1}{\log x}\frac{1}{x}+\log(\log x)$$

$$\frac{du}{dx}=u\left[\frac{1}{\log x}+\log(\log x)\right]$$

$$\frac{du}{dx}=(\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$$ ………. (2)

Again $v = x^{\log x}$

$$\log v = \log x^{\log x} = \log x.\log x = (\log x)^2$$

$$\frac{d}{dx}\log v = \frac{d}{dx}(\log x)^2$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x \frac{d}{dx}(\log x)$$

$$\frac{1}{v}\frac{dv}{dx} = 2\log x.\frac{1}{x}$$

$$\frac{dv}{dx} = v\left(\frac{2}{x}\log x\right) = x^{\log x}.\frac{2}{x}\log x$$

$$\frac{dv}{dx} = 2x^{\log x - 1}\log x$$ ……….(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1}{\log x} + \log(\log x)\right] + 2x^{\log x - 1}\log x$$

$$\frac{dy}{dx} = (\log x)^x\left[\frac{1 + \log x \log(\log x)}{\log x}\right] + 2x^{\log x - 1}\log x$$

$$\frac{dy}{dx} = (\log x)^{x-1}\left(1 + \log x \log(\log x)\right) + 2x^{\log x - 1}\log x$$

8. $(\sin x)^x + \sin^{-1}\sqrt{x}$

Solution: Let $y = (\sin x)^x + \sin^{-1}\sqrt{x}$ = $u + v$ where $u = (\sin x)^x$ and $v = \sin^{-1}\sqrt{x}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$ ………. (1)

Now $u = (\sin x)^x$

$$\log u = \log(\sin x)^x = x\log(\sin x)$$

$$\frac{d}{dx}\log u = \frac{d}{dx}\left[x\log(\sin x)\right]$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{d}{dx}\left[\log(\sin x)\right] + \log(\sin x)\frac{d}{dx}x$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\frac{d}{dx}\sin x + \log(\sin x).1$$

$$\frac{1}{u}\frac{du}{dx} = x\frac{1}{\sin x}\cos x + \log(\sin) = x\cot x + \log\sin x$$

$$\frac{du}{dx} = u\left[x\cot x + \log\sin x\right]$$

$$\frac{du}{dx} = (\sin x)^x\left[x\cot x + \log\sin x\right]$$ ………. (2)

Again $v = \sin^{-1}\sqrt{x}$

$$\log v = \log\sin^{-1}\sqrt{x}$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-\left(\sqrt{x}\right)^2}}\frac{d}{dx}\sqrt{x} \quad \left[\because \frac{d}{dx}\sin^{-1}f(x) = \frac{1}{\sqrt{1-\left(f(x)\right)^2}}\frac{d}{dx}f(x)\right]$$

$$\frac{dv}{dx} = \frac{1}{\sqrt{1-x}}.\frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}\sqrt{1-x}}$$

$$= \frac{1}{2\sqrt{x-x^2}}$$ ……….(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = (\sin x)^x\left[x\cot x + \log\sin x\right] + \frac{1}{2\sqrt{x-x^2}}$$

9. $x^{\sin x}+(\sin x)^{\cos x}$

Solution: Let $y=x^{\sin x}+(\sin x)^{\cos x}$

Put $u=x^{\sin x}$ and $v=(\sin x)^{\cos x}$, we get $y=u+v$

$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ………. (1)

Now $u=x^{\sin x}$

$\log u=\log x^{\sin x}$ = $\sin x\log x$

$\frac{d}{dx}\log u=\frac{d}{dx}(\sin x\log x)$

$\frac{1}{u}\frac{du}{dx}=\sin x\frac{d}{dx}\log x+\log x\frac{d}{dx}\sin x$

$\frac{1}{u}\frac{du}{dx}=\sin x\frac{1}{x}+\log x(\cos x)$

$\frac{du}{dx}=u\left(\frac{\sin x}{x}+\cos x\log x\right)$

$\frac{du}{dx}=x^{\sin x}\left(\frac{\sin x}{x}+\cos x\log x\right)$ ….. (2)

Again $v=(\sin x)^{\cos x}$

$\log v=\log(\sin x)^{\cos x}$ = $\cos x\log\sin x$

$\frac{d}{dx}\log v=\frac{d}{dx}\left[\cos x\log(\sin x)\right]$

$\frac{1}{v}\frac{dv}{dx}=\cos x\frac{d}{dx}\log\sin x+\log\sin x\frac{d}{dx}\cos x$

$$\frac{1}{v}\frac{dv}{dx} = \cos x \frac{1}{\sin x}\frac{d}{dx}\sin x + \log \sin x(-\sin x)$$

$$\frac{1}{v}\frac{dv}{dx} = \cot x.\cos x - \sin x \log \sin x$$

$$\frac{dv}{dx} = v(\cot x.\cos x - \sin x \log \sin x)$$

$$\frac{dv}{dx} = (\sin x)^{\cos x}(\cot x.\cos x - \sin x \log \sin x)$$ (using value of v) ……….(3)

Put values from (2) and (3) in (1),

$$\frac{dy}{dx} = x^{\sin x}\left(\frac{\sin x}{x} + \cos x \log x\right) + (\sin x)^{\cos x}(\cot x.\cos x - \sin x \log \sin x)$$

10. $x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Solution: Let $x^{x\cos x} + \frac{x^2+1}{x^2-1}$

Put $u = x^{x\cos x}$ and $v = \frac{x^2+1}{x^2-1}$, we get $y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$ ………. (1)

Now $u = x^{x\cos x}$

$\log u = \log x^{x\cos x}$ = $x\cos x \log x$

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x\cos x \log x)$$

$$\frac{1}{u}\frac{du}{dx} = \frac{d}{dx}(x).\cos x \log x + x\frac{d}{dx}(\cos x)\log x + x\cos x\frac{d}{dx}(\log x)$$

$$\frac{1}{u}\frac{du}{dx} = 1.\cos x \log x + x(-\sin x)\log x + x\cos x\frac{1}{x}$$

$$\frac{du}{dx} = u\left(\cos x \log x - x\sin x \log x + \cos x\right)$$

$$\frac{du}{dx} = x^{x\cos x}\left(\cos x \log x - x\sin x \log x + \cos x\right)$$ ………. (2)

Again $v = \frac{x^2+1}{x^2-1}$

$$\frac{dv}{dx} = \frac{\left(x^2-1\right)\frac{d}{dx}\left(x^2+1\right) - \left(x^2+1\right)\frac{d}{dx}\left(x^2-1\right)}{\left(x^2-1\right)^2}$$

$$\frac{dv}{dx} = \frac{\left(x^2-1\right)2x - \left(x^2+1\right)2x}{\left(x^2-1\right)^2}$$

$$\frac{dv}{dx} = \frac{2x^3 - 2x - 2x^3 - 2x}{\left(x^2-1\right)^2}$$

$$\frac{dv}{dx} = \frac{-4x}{\left(x^2-1\right)^2}$$ ……….(3)

Put the values from (2) and (3) in (1),

$$\frac{dy}{dx} = x^{x\cos x}\left(\cos x \log x - x\sin x \log x + \cos x\right) + \frac{-4x}{\left(x^2-1\right)^2}$$

11. $\left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Solution: Let $y = \left(x\cos x\right)^x + \left(x\sin x\right)^{\frac{1}{x}}$

Put $u = \left(x\cos x\right)^x$ and $v = \left(x\sin x\right)^{\frac{1}{x}}$, we get $y = u + v$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$ ………. (1)

Now $u = \left(x\cos x\right)^x$

$\log u = \log(x\cos x)^x = x\log(x\cos x)$

$\log u = x(\log x + \log \cos x)$

$\frac{d}{dx}\log u = \frac{d}{dx}\{x(\log x + \log \cos x)\}$

$\frac{1}{u}\frac{du}{dx} = x\left[\frac{1}{x} + \frac{1}{\cos x}.(-\sin x)\right] + (\log x + \log \cos x).1$

$\frac{du}{dx} = u[1 - x\tan x + \log(x\cos x)]$

$\frac{du}{dx} = (x\cos x)^x[1 - x\tan x + \log(x\cos x)]$ ………. (2)

Again $v = (x\sin x)^{\frac{1}{x}}$

$\log v = \log(x\sin x)^{\frac{1}{x}} = \frac{1}{x}\log(x\sin x)$

$\log v = \frac{1}{x}(\log x + \log \sin x)$

$\frac{d}{dx}\log v = \frac{d}{dx}\left\{\frac{1}{x}(\log x + \log \sin x)\right\}$

$\frac{1}{v}\frac{dv}{dx} = \frac{1}{x}\left[\frac{1}{x} + \frac{1}{\sin x}.\cos x\right] + (\log x + \log \sin x)\left(\frac{-1}{x^2}\right)$

$\frac{dv}{dx} = v\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x\sin x)}{x^2}\right]$

$\frac{dv}{dx} = (x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^2} + \frac{\cot x}{x} - \frac{\log(x\sin x)}{x^2}\right]$ ……….(3)

Put the values from (2) and (3) in (1)

$$\frac{dy}{dx}=(x\cos x)^{x}\left[1-x\tan x+\log(x\cos x)\right]+(x\sin x)^{\frac{1}{x}}\left[\frac{1}{x^{2}}+\frac{\cot x}{x}-\frac{\log(x\sin x)}{x^{2}}\right]$$

Find $\frac{dy}{dx}$ in the following Exercise 12 to 15

12. $x^{y}+y^{x}=1$

Solution: Given: $x^{y}+y^{x}=1$

$u+v=1,$ where $u=x^{y}$ and $v=y^{x}$

$$\frac{d}{dx}u+\frac{d}{dx}v=\frac{d}{dx}1$$

$$\frac{du}{dx}+\frac{dv}{dx}=0$$ .... (1)

Now $u=x^{y}$

$$\log u=\log x^{y}=y\log x$$

$$\frac{d}{dx}\log u=\frac{d}{dx}(y\log x)$$

$$\frac{1}{u}\frac{du}{dx}=y\frac{d}{dx}\log x+\log x\frac{dy}{dx}$$

$$\frac{1}{u}\frac{du}{dx}=y.\frac{1}{x}+\log x.\frac{dy}{dx}$$

$$\frac{du}{dx}=u\left(\frac{y}{x}+\log x.\frac{dy}{dx}\right)$$

$$\frac{du}{dx}=x^{y}\left(\frac{y}{x}+\log x.\frac{dy}{dx}\right)=x^{y}\frac{y}{x}+x^{y}\log x.\frac{dy}{dx}$$

$$\frac{du}{dx}=x^{y-1}y+x^{y}\log x.\frac{dy}{dx}$$ .... (2)

Again $v=y^{x}$

$$\log v = \log y^x = x\log y$$

$$\frac{d}{dx}\log v = \frac{d}{dx}(x\log y)$$

$$\frac{1}{v}\frac{dv}{dx} = x\frac{d}{dx}\log y + \log y\frac{d}{dx}x$$

$$\frac{1}{v}\frac{dv}{dx} = x.\frac{1}{y}\frac{dy}{dx} + \log y.1$$

$$\frac{dv}{dx} = v\left(\frac{x}{y}\frac{dy}{dx} + \log y\right)$$

$$\frac{dv}{dx} = y^x\left(\frac{x}{y}\frac{dy}{dx} + \log y\right) = y^x\frac{x}{y}\frac{dy}{dx} + y^x\log y$$

$$\frac{dv}{dx} = y^{x-1}x\frac{dy}{dx} + y^x\log y \qquad \text{……….(3)}$$

Put values from (2) and (3) in (1),

$$x^{y-1}y + x^y\log x.\frac{dy}{dx} + y^{x-1}x\frac{dy}{dx} + y^x\log y = 0$$

$$\frac{dy}{dx}\left(x^y\log x + y^{x-1}x\right) = -x^{y-1}y - y^x\log y$$

$$\frac{dy}{dx} = \frac{-\left(x^{y-1}y - y^x\log y\right)}{x^y\log x + y^{x-1}x}$$

13. $y^x = x^y$

Solution:

Apply log on both sides, we get

$\log y^x = \log x^y$

$x \log y = y \log x$

Now, apply differentiation with respect to 'x' on both sides, we get

$$x\left[\frac{1}{y} \times \frac{dy}{dx}\right] + logy(1) = y(\frac{1}{x}) + logx(\frac{dy}{dx})$$

$$\frac{x}{y}\frac{dy}{dx} - logx\frac{dy}{dx} = \frac{y}{x} - logy$$

By taking dy/dx common,

$$\frac{dy}{dx}\left(\frac{x}{y} - logx\right) = \frac{y - xlogy}{x}$$

$$\frac{dy}{dx}\left(\frac{x - ylogx}{y}\right) = \frac{y - xlogy}{x}$$

By cross multiplying, we get

$$\frac{dy}{dx} = \frac{y(y - xlogy)}{x(x - ylogx)}$$

14. $(\cos x)^y = (\cos y)^x$

Solution: Given: $(\cos x)^y = (\cos y)^x$

$$\log(\cos x)^y = \log(\cos y)^x$$

$$y \log \cos x = x \log \cos y$$

$$\frac{d}{dx}(y \log \cos x) = \frac{d}{dx}(x \log \cos y)$$

$$y\frac{d}{dx}\log \cos x + \log \cos x\frac{dy}{dx} = x\frac{d}{dx}\log \cos y + \log \cos y\frac{d}{dx}x$$

$$y\frac{1}{\cos x}\frac{d}{dx}\cos x + \log \cos x\frac{dy}{dx} = x\frac{1}{\cos y}\frac{d}{dx}\cos y + \log \cos y$$

$$y\frac{1}{\cos x}(-\sin x) + \log \cos x\frac{dy}{dx} = x\frac{1}{\cos y}\left(-\sin y\frac{dy}{dx}\right) + \log \cos y$$

$$-y\tan x + \log \cos x.\frac{dy}{dx} = -x\tan y.\frac{dy}{dx} + \log \cos y$$

$$x\tan y\frac{dy}{dx} + \log \cos x.\frac{dy}{dx} = y\tan x + \log \cos y$$

$$\frac{dy}{dx}(x\tan y+\log\cos x)=y\tan x+\log\cos y$$

$$\frac{dy}{dx}=\frac{y\tan x+\log\cos y}{x\tan y+\log\cos x}$$

15. $xy=e^{x-y}$

Solution: Given: $xy=e^{x-y}$

$$\log xy=\log e^{x-y}$$

$$\log x+\log y=(x-y)\log e$$

$$\log x+\log y=(x-y)\quad [\because \log e=1]$$

$$\frac{d}{dx}\log x+\frac{d}{dx}\log y=\frac{d}{dx}(x-y)$$

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}=1-\frac{dy}{dx}$$

$$\frac{1}{y}\cdot\frac{dy}{dx}+\frac{dy}{dx}=1-\frac{1}{x}$$

$$\frac{dy}{dx}\left(\frac{1}{y}+1\right)=\frac{x-1}{x}$$

$$\frac{dy}{dx}\left(\frac{1+y}{y}\right)=\frac{x-1}{x}$$

$$\frac{dy}{dx}=\frac{y(x-1)}{x(1+y)}$$

16. Find the derivative of the function given by $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ and hence $f'(1)$.

Solution: Given: $f(x)=(1+x)(1+x^2)(1+x^4)(1+x^8)$ ………. (1)

$$\log f(x) = \log(1+x) + \log(1+x^2) + \log(1+x^4) + \log(1+x^8)$$

$$\frac{1}{f(x)}\frac{d}{dx}f(x) = \frac{1}{1+x}\frac{d}{dx}(1+x) + \frac{1}{1+x^2}\frac{d}{dx}(1+x^2) + \frac{1}{1+x^4}\frac{d}{dx}(1+x^4) + \frac{1}{1+x^8}\frac{d}{dx}(1+x^8)$$

$$\frac{1}{f(x)}f'(x) = \frac{1}{1+x}.1 + \frac{1}{1+x^2}.2x + \frac{1}{1+x^4}.4x^3 + \frac{1}{1+x^8}8x^7$$

$$f'(x) = f(x)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Put the value of f(x) from (1),

$$f'(x) = (1+x)(1+x^2)(1+x^4)(1+x^8)\left[\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8}\right]$$

Now, Find for f'(1):

$$f'(1) = (1+1)(1+1^2)(1+1^4)(1+1^8)\left[\frac{1}{1+1} + \frac{2\times1}{1+1^2} + \frac{4\times1^3}{1+1^4} + \frac{8\times1^7}{1+1^8}\right]$$

$$f'(1) = (2)(2)(2)(2)\left[\frac{1}{2} + \frac{2}{2} + \frac{4}{2} + \frac{8}{2}\right]$$

$$f'(1) = 16\left[\frac{15}{2}\right]$$

= 8 x 15

= 120

17. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ in three ways mentioned below:
(i) by using product rule.
(ii) by expanding the product to obtain a single polynomial
(iii) by logarithmic differentiation.
Do they all give the same answer?

Solution: Let $y = (x^2 - 5x + 8)(x^3 + 7x + 9)$
(i) using product rule:

$$\frac{dy}{dx}=\left(x^2-5x+8\right)\frac{d}{dx}\left(x^3+7x+9\right)+\left(x^3+7x+9\right)\frac{d}{dx}\left(x^2-5x+8\right)$$

$$\frac{dy}{dx}=\left(x^2-5x+8\right)\left(3x^2+7\right)+\left(x^3+7x+9\right)\left(2x-5\right)$$

$$\frac{dy}{dx}=3x^4+7x^2-15x^3-35x+24x^2+56+2x^4-5x^3+14x^2-35x+18x-45$$

$$\frac{dy}{dx}=5x^4-20x^3+45x^2+11$$

(ii) Expand the product to obtain a single polynomial

$$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$$

$$y=x^5+7x^3+9x^2-5x^4-35x^2-45x+8x^3+56x+72$$

$$y=x^5-5x^4+15x^3-26x^2+11x+72$$

$$\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

(iii) Logarithmic differentiation

$$y=\left(x^2-5x+8\right)\left(x^3+7x+9\right)$$

$$\log y=\log\left(x^2-5x+8\right)+\log\left(x^3+7x+9\right)$$

$$\frac{d}{dx}\log y=\frac{d}{dx}\log\left(x^2-5x+8\right)+\frac{d}{dx}\log\left(x^3+7x+9\right)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\frac{d}{dx}\left(x^2-5x+8\right)+\frac{1}{x^3+7x+9}\frac{d}{dx}\left(x^3+7x+9\right)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x^2-5x+8}\left(2x-5\right)+\frac{1}{x^3+7x+9}\left(3x^2+7\right)$$

$$\frac{dy}{dx}=y\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\frac{dy}{dx}=y\left[\frac{2x-5}{x^2-5x+8}+\frac{3x^2+7}{x^3+7x+9}\right]$$

$$\frac{dy}{dx}=y\left[\frac{(2x-5)(x^3+7x+9)+(3x^2+7)(x^2-5x+8)}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$\frac{dy}{dx}=y\left[\frac{2x^4+14x^2+18x-5x^3-35x-45+3x^4-15x^3+24x^2+7x^2-35x+56}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$\frac{dy}{dx}=y\left[\frac{5x^4-20x^3+45x^2-52x+11}{(x^2-5x+8)(x^3+7x+9)}\right]$$

$$\frac{dy}{dx}=(x^2-5x+8)(x^3+7x+9)\left[\frac{5x^4-20x^3+45x^2-52x+11}{(x^2-5x+8)(x^3+7x+9)}\right]$$ [using value of y]

$$\frac{dy}{dx}=5x^4-20x^3+45x^2-52x+11$$

Therefore, the value of dy/dx is same obtained by three different methods.

18. If u, v and w are functions of x, then show that

$$\frac{d}{dx}(u.v.w)=\frac{du}{dx}v.w+u.\frac{dv}{dx}.w+u.v\frac{dw}{dx}$$

in two ways–first by repeated application of product rule, second by logarithmic differentiation.

Solution: Given u, v and w are functions of x.

To Prove: $\frac{d}{dx}(u.v.w)=\frac{du}{dx}.v.w+u.\frac{dv}{dx}.w+u.v.\frac{dw}{dx}$

Way 1: By repeated application of product rule

L.H.S.

$$\frac{d}{dx}(u.v.w)=\frac{d}{dx}[(uv).w]$$

$$=uv\frac{d}{dx}w+w\frac{d}{dx}(uv)$$

$$=uv\frac{dw}{dx}+w\left[u\frac{d}{dx}v+v\frac{d}{dx}u\right]$$

$= uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$

$= \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

= R.H.S

Hence proved.

Way 2: By Logarithmic differentiation

Let $y = uvw$

$\log y = \log(u.v.w)$

$\log y = \log u + \log v + \log w$

$\frac{d}{dx}\log y = \frac{d}{dx}\log u + \frac{d}{dx}\log v + \frac{d}{dx}\log w$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}$

$\frac{dy}{dx} = y\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$

Put y=uvw, we get

$\frac{d}{dx}(u.v.w) = uvw\left[\frac{1}{u}\frac{du}{dx} + \frac{1}{v}\frac{dv}{dx} + \frac{1}{w}\frac{dw}{dx}\right]$

$\frac{d}{dx}(u.v.w) = \frac{du}{dx}.v.w + u.\frac{dv}{dx}.w + u.v.\frac{dw}{dx}$

Hence proved.