Exercise 6.3 Page No: 211

1. Find the slope of tangent to the curve $y = 3x^4 - 4x$ at x = 4.

Solution:

Equation of the curve $y = 3x^4 - 4x$ ……….(1)

Slope of the tangent to the curve = Value of $\frac{dy}{dx}$ at the point (x, y).

$$\frac{dy}{dx} = 3\left(4x^3\right) - 4 = 12x^3 - 4$$

Slope of the tangent at point x = 4 to the curve (1)

= $12(4)^3 - 4$ = 764

2. Find the slope of tangent to the curve $y = \frac{x-1}{x-2}, x \neq 2$ at x = 10.

Solution:

Equation of the curve $y = \frac{x-1}{x-2}$ ……….(1)

Derivate y w.r.t. x,

$$\frac{dy}{dx} = \frac{(x-2)\frac{d}{dx}(x-1) - (x-1)\frac{d}{dx}(x-2)}{(x-2)^2}$$

= $\frac{(x-2)-(x-1)}{(x-2)^2}$

= $\frac{-1}{(x-2)^2}$ ……….(2)

Slope of the tangent at point x = 10 to the curve (1)

= $\frac{-1}{(10-2)^2}$

= $\frac{-1}{8^2} = \frac{-1}{64}$

3. Find the slope of tangent to the curve $y = x^3 - x + 1$ at the given point whose $x-$ coordinate is 2.
Solution:

Equation of the curve $y = x^3 - x + 1$ ……….(1)
Apply derivate w.r.t x,

$\frac{dy}{dx} = 3x^2 - 1$

Slope of the tangent at point $x = 2$ to the curve (1)

= $3(2)^2 - 1$ = 11

4. Find the slope of tangent to the curve $y = x^3 - 3x + 2$ at the given point whose x-coordinate is 3.
Solution:

Equation of the curve $y = x^3 - 3x + 2$……….(1)
Apply derivate w.r.t x,

$\frac{dy}{dx} = 3x^2 - 3$

Slope of the tangent at point $x = 3$ to the curve (1)

= $3(3)^2 - 3$ = 24

5. Find the slope of the normal to the curve $x = a\cos^3\theta, y = a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

Solution:

Equations of the curves are $x = a\cos^3\theta, y = a\sin^3\theta$

$x = a\cos^3\theta,$
Apply derivate w.r.t x,

$$\frac{dx}{d\theta} = a\frac{d}{d\theta}(\cos\theta)^3$$

= $a.3(\cos\theta)^2 \frac{d}{d\theta}(\cos\theta)$

$\Rightarrow \frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$ …..(1)

And,

$y = a\sin^3\theta$

Apply derivate w.r.t x,

$$\frac{dy}{d\theta} = a\frac{d}{d\theta}(\sin\theta)^3$$

$$a.3(\sin\theta)^2 \frac{d}{d\theta}(\sin\theta)$$

$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$ ………(2)

Using (1) and (2), we have

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta}$$

= $\frac{-\sin\theta}{\cos\theta} = -\tan\theta$

Now,

Slope of the tangent at $\theta = \frac{\pi}{4}$

= $-\tan\frac{\pi}{4} = -1$

And Slope of the normal at $\theta = \frac{\pi}{4}$

$= \frac{-1}{m} = \frac{-1}{-1} = 1$

6. Find the slope of the normal to the curve $x = 1 - a\sin\theta, y = b\cos^2\theta$ at $\theta = \frac{\pi}{2}$.

Solution:

Equations of the curves are $x = 1 - a\sin\theta$ and $y = b\cos^2\theta$.

$x = 1 - a\sin\theta$

Apply derivative w.r.t. x, we have

$\frac{dx}{d\theta} = 0 - a\cos\theta$

$\Rightarrow \frac{dx}{d\theta} = -a\cos\theta$

Again,

$y = b\cos^2\theta$

Apply derivative w.r.t. x, we have

$\frac{dy}{d\theta} = b\frac{d}{d\theta}(\cos\theta)^2$

$\frac{dy}{d\theta} = b.2\cos\theta\frac{d}{d\theta}\cos\theta = -2b\cos\theta\sin\theta$

Now,

$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{-2b\cos\theta\sin\theta}{-a\cos\theta}$

$= \frac{2b}{a}\sin\theta$

Again, Slope of the tangent at $\theta = \frac{\pi}{2}$

$= \frac{2b}{a}\sin\frac{\pi}{2} = \frac{2b}{a}$

And Slope of the normal at $\theta = \frac{\pi}{2}$

$= \frac{-1}{m} = \frac{-1}{2b/a}$

$= \frac{-a}{2b}$

7. Find the point at which the tangent to the curve $y = x^3 - 3x^2 - 9x + 7$ is parallel to the x-**axis.**

Solution:

Equation of the curve $y = x^3 - 3x^2 - 9x + 7$ ……….(1)

$\frac{dy}{dx} = 3x^2 - 6x - 9$

Since, the tangent is parallel to the x-axis, so, $\frac{dy}{dx} = 0$

$3x^2 - 6x - 9 = 0$

$x^2 - 2x - 3 = 0$

$(x-3)(x+1) = 0$

$x = 3, x = -1$

From equation (1), when x = 3.

$y = 27 - 27 - 27 + 7 = -20$

when $x = -1,\ y = -1 - 3 + 9 + 7 = 12$

Therefore, the required points are $(3, -20)$ and $(-1, 12)$.

8. Find the point on the curve $y = (x-2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

Solution: Let the given points are M (2, 0) and N (4, 4).

Slope of the chord, MN = $\frac{4-0}{4-2}=2$

$\left[\because m=\frac{y_2-y_1}{x_2-x_1}\right]$

Equation of the curve is $y=(x-2)^2$ (Given)

Slope of the tangent at (x, y)

= $\frac{dy}{dx}=2(x-2)$

If the tangent is parallel to the chord MN, then

Slope of tangent = Slope of chord

$2(x-2)=2$

$x=3$

Therefore, $y=(3-2)^2=1$

Therefore, the required point is (3, 1).

9. Find the point on the curve $y=x^3-11x+5$ at which the tangent is y = x -11.

Solution:

Equation of the curve $y=x^3-11x+5$ ……….(1)

Equation of the tangent $y=x-11$……….(2)

=> $x-y-11=0$

Slope of the tangent at (x, y)

= $\frac{dy}{dx}=3x^2-11$ [From equation (1)]

Slope of tangent = $\frac{-a}{b} = \frac{-1}{-1} = 1$

[From equation (2)]

Therefore,

$3x^2 - 11 = 1$

$x^2 = 4$

$x = \pm 2$

From equation (1), when $x = 2, \; y = 8 - 22 + 5 = -9$

And when $x = -2, \; y = -8 + 22 + 5 = 19$

We observed that, $(-2, 19)$ does not satisfy equation (2), therefore the required point is $(2, -9)$.

10. Find the equation of all lines having slope -1 that are tangents to the curve $y = \frac{1}{x-1}, x \neq 1$.

Solution:

Equation of the curve $y = \frac{1}{x-1} = (x-1)^{-1}$ ……….(1)

$\frac{dy}{dx} = (-1)(x-1)^{-2} \frac{d}{dx}(x-1)$

$= \frac{-1}{(x-1)^2}$ = Slope of the tangent at (x, y)

But according to given statement, slope = -1

$\frac{-1}{(x-1)^2} = -1$

$(x-1)^2 = 1$

$x - 1 = \pm 1$

$x=1+1=2$ or $x=1-1=0$

From equation (1), when x = 2

$$y=\frac{1}{2-1}=1$$

And when x = 0

$$y=\frac{1}{0-1}=-1$$

Points of contact are (2, 1) and (0, -1).

And Equation of two tangents are $y-1=-1(x-2)$

= $x+y-3=0$ and

$y-(-1)=-1(x-0)$ = $x+y+1=0$

11. Find the equations of all lines having slope 2 which are tangents to the curve $y=\frac{1}{x-3}, x\neq 3.$

Solution:

Equation of the curve $y=\frac{1}{x-3}=(x-3)^{-1}$

$$\frac{dy}{dx}=(-1)(x-3)^{-2}$$

= $\frac{-1}{(x-3)^2}$

= Slope of the tangent at (x,y)

But according to question, slope = 2

$\frac{-1}{(x-3)^2}$ = 2

$$(x-3)^2=\frac{-1}{2}$$

which is not possible.

Hence, there is no tangent to the given curve having slope 2.

12. Find the equations of all lines having slope 0 which are tangents to the curve $y=\frac{1}{x^2-2x+3}$.

Solution:

Equation of the curve $y=\frac{1}{x^2-2x+3}$ ……….(1)

$\frac{dy}{dx}=\frac{d}{dx}\left[\left(x^2-2x+3\right)^{-1}\right]$

= $-\left(x^2-2x+3\right)^{-2}.\left(2x-2\right)$

= $\frac{-2\left(x-1\right)}{\left(x^2-2x+3\right)^2}$

But according to question, slope = 0, so

$\frac{-2\left(x-1\right)}{\left(x^2-2x+3\right)^2}$ = 0

$-2\left(x-1\right)=0$

$x=1$

From equation (1), $y=\frac{1}{1-2+3}=\frac{1}{2}$

Therefore, the point on the curve which tangent has slope 0 is $\left(1,\frac{1}{2}\right)$.

Equation of the tangent is $y-\frac{1}{2}=0\left(x-1\right)$

$y-\frac{1}{2}=0$

Which implies, the value of y is ½.

13. Find the points on the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ at which the tangents are:

(i) parallel to x-axis (ii) parallel to y-axis

Solution:

Equation of the curve $\frac{x^2}{9}+\frac{y^2}{16}=1$ ……….(1)

Derivate y w.r.t. x, we have

$$\frac{2x}{9}+\frac{2y}{16}\frac{dy}{dx}=0$$

$$\frac{2y}{16}\frac{dy}{dx}=-\frac{2x}{9}$$

$$\frac{dy}{dx}=\frac{-32x}{18y}=\frac{-16x}{9y}$$ ……….(2)

(i) If tangent is parallel to x-axis, then Slope of tangent = 0

Which implies, $\frac{dy}{dx}$ = 0

$$\frac{-16x}{9y}=0$$

$$x=0$$

From equation (1), $\frac{y^2}{16}=1$

$$y^2=16$$

$$y=\pm 4$$

The points on curve

(1) where tangents are parallel to x-axis are $(0, \pm 4)$.

(ii) If the tangent parallel to y-axis.

Slope of the tangent = $\pm\infty$

$$\frac{dy}{dx} = \pm\infty$$

$$\frac{dx}{dy} = 0$$ (taking reciprocal)

From equation (2), $\frac{9y}{-16x} = 0$

$y = 0$

From equation (1), $\frac{x^2}{9} = 1$

$x^2 = 9$

$x = \pm 3$

Therefore, the points on curve (1) where tangents are parallel to y-axis are $(\pm 3, 0)$.

14. Find the equation of the tangents and normal to the given curves at the indicated points:

(i) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ **at (0, 5)**

(ii) $y = x^4 - 6x^3 + 13x^2 - 10x + 5$ **at (1, 3)**

(iii) $y = x^3$ **at (1, 1)**

(iv) $y = x^2$ **at (0, 0)**

(v) x = cos t, y = sin t at t = π/4

Solution:

(i) Equation of the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

On differentiating y w.r.t. x, we have

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

Now, value of $\frac{dy}{dx}$ at (0, 5)

At x=0,

$4(0)^3 - 18(0)^2 + 26(0) - 10 = -10 = m$ (say)

Slope of the normal at (0, 5) is $\frac{-1}{m} = \frac{-1}{-10} = \frac{1}{10}$

Equation of the tangent at (0, 5) is $y - 5 = 10(x - 0)$

$y - 5 = 10x$

$10x + y = 5$

And Equation of the normal at (0, 5) is $y - 5 = \frac{1}{10}(x - 0)$

$10y - 50 = x$

$x - 10y + 50 = 0$

(ii) Equation of the curve $y = x^4 - 6x^3 + 13x^2 - 10x + 5$

On differentiating y w.r.t. x, we have

$$\frac{dy}{dx} = 4x^3 - 18x^2 + 26x - 10$$

Now value of $\frac{dy}{dx}$ at (1, 3)

At $x = 1,$

$4(1)^3 - 18(1)^2 + 26(1) - 10 = 4 - 18 + 26 - 10 = 2 = m$ (say)

Slope of the normal at (1, 3) is $\frac{-1}{m} = \frac{-1}{2}$

Equation of the tangent at (1, 3) is $y-3=2(x-1)$

$y-3=2x-2$

$y=2x+1$

And Equation of the normal at (1, 3) is $y-3=\frac{-1}{2}(x-1)$

$2y-6=-x+1$

$x+2y-7=0$

(iii) Equation of the curve $y=x^3$ ……….(1)
On differentiating y w.r.t. x, we have

$\frac{dy}{dx}=3x^2$

Now, value of $\frac{dy}{dx}$ at (1, 1)

At $x=1,$

$3(1)^2=3$ = m (say)

Slope of the normal at (1, 1) is $\frac{-1}{m}=\frac{-1}{3}$

Equation of the tangent at (1, 1) is $y-1=3(x-1)$

$y-1=3x-3$ or $y=3x-2$

And Equation of the normal at (1, 1) is $y-1=\frac{-1}{3}(x-1)$

$3y-3=-x+1$

$x+3y-4=0$

(iv) Equation of the curve $y=x^2$ ……….(1)
On differentiating y w.r.t. x, we have

$\frac{dy}{dx}=2x$

Now value of $\frac{dy}{dx}$ at (0, 0)

At $x=0$, $2\times 0=0$ = m (say)

Equation of the tangent at (0, 0) is $y-0=0(x-0)$

$y=0$

And normal at (0, 0) is y-axis.

(v) Equation of the curves are $x=\cos t, y=\sin t$

$\therefore \frac{dx}{dt}=-\sin t$ and $\frac{dy}{dt}=\cos t$

$\therefore \frac{dy}{dx}=\frac{dy/dt}{dx/dt}=\frac{\cos t}{-\sin t}=-\cot t$

Slope of the tangent at $t=\frac{\pi}{4}$ = $-\cot\frac{\pi}{4}=-1=m$ (say)

Slope of the normal at $t=\frac{\pi}{4}$ is $\frac{-1}{m}=\frac{-1}{-1}=1$

Point (x,y) = $(\cos t, \sin t)$

= $\left(\cos\frac{\pi}{4}, \sin\frac{\pi}{4}\right)$

= $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

Equation of the tangent is $y-\frac{1}{\sqrt{2}}=-1\left(x-\frac{1}{\sqrt{2}}\right)$

$x+y=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$

$x+y=\sqrt{2}$

And Equation of the normal is $y-\frac{1}{\sqrt{2}}=1\left(x-\frac{1}{\sqrt{2}}\right)$

$y-\frac{1}{\sqrt{2}}=x-\frac{1}{\sqrt{2}}$

=> $y=x$

15. Find the equation of the tangent line to curve $y=x^2-2x+7$ which is:
(a) parallel to the line $2x-y+9=0$
(b) perpendicular to the line $5y-15x=13$

Solution:

Equation of the curve $y=x^2-2x+7$ ……….(1)

Slope of tangent = $\frac{dy}{dx}=2x-2$ …….(2)

(a) Slope of the line $2x-y+9=0$ is $\frac{-a}{b}=\frac{-2}{-1}=2$
Slope of tangent parallel to this line is also = 2

From equation (2), $2x-2=2$

=> $x=2$

From equation (1), $y=4-4+7=7$

Therefore, point of contact is (2, 7).

Equation of the tangent at (2, 7) is $y-7=2(x-2)$

$\Rightarrow y-7=2x-4$

$\Rightarrow y-2x-3=0$

(b) Slope of the line $-15x+5y=13$ is $\frac{-a}{b}=\frac{-(-15)}{5}=3$ = m

Slope of the required tangent perpendicular to this line = $\frac{-1}{m}=\frac{-1}{3}$

From equation (2), $2x-2=\frac{-1}{3}$

$6x-6=-1$

$x=\frac{5}{6}$

From equation (1), $y=\frac{25}{36}-\frac{5}{3}+7$

= $\frac{25-60+252}{36}$ = $\frac{217}{36}$

Therefore, point of contact is $\left(\frac{5}{6},\frac{217}{36}\right)$.

Equation of the required tangent is $y-\frac{217}{36}=\frac{-1}{3}\left(x-\frac{5}{6}\right)$

$3y-\frac{217}{12}=-x+\frac{5}{6}$

$x+3y=\frac{217}{12}+\frac{5}{6}$

$x+3y=\frac{217+10}{12}=\frac{227}{12}$

$12x+36y=227$. (Which is required equation)

16. Show that the tangents to the curve $y=7x^3+11$ at the points where $x=2$ and $x=-2$ are parallel.
Solution:

Equation of the curve $y=7x^3+11$

Slope of tangent at (x,y) = $\frac{dy}{dx}=21x^2$

At the point x = 2,

Slope of the tangent = $21(2)^2=21\times4=84$

At the point x = -2,

Slope of the tangent = $21(-2)^2=21\times4=84$

Since, the slopes of the two tangents are equal.

Therefore, tangents at x = 2 and x = -2 are parallel.

17. Find the points on the curve $y=x^3$ at which the slope of the tangent is equal to the $y-$coordinate of the point.
Solution:

Equation of the curve $y=x^3$ ………(1)

Slope of tangent at (x,y)

= $\frac{dy}{dx}=3x^2$ ……….(2)

As given, Slope of the tangent = y-coordinate of the point

$3x^2=x^3$

$3x^2-x^3=0$

$x^2(3-x)=0$

$x^2=0$ or $3-x=0$

$x=0$ or $x=3$

From equation (1), at $x=0$, $y=0$

The point is (0, 0).

And From equation (1), at $x=3$, $y=27$

The point is (3, 27).

Therefore, the desired points are (0, 0) and (3, 27).

18. For the curve $y=4x^3-2x^5$, find all point at which the tangent passes through the origin.

Solution:

Equation of the curve $y=4x^3-2x^5$ ……….(1)

Slope of the tangent at (x, y) passing through origin (0, 0)

$$\frac{dy}{dx}=12x^2-10x^4$$

And dy/dx = $\frac{y-0}{x-0}$

=> $\frac{y}{x}=12x^2-10x^4$

$$y=12x^3-10x^5$$

Substituting value of y in equation (1), we get,

$$12x^3-10x^5=4x^3-2x^5$$

$$8x^3-8x^5=0$$

$$8x^3\left(1-x^2\right)=0$$

$8x^3=0$ or $1-x^2=0$

$\Rightarrow$ $x=0$ or $x=\pm 1$

From equation (1),

at x = 0, the value of y = 0.

From equation (1), at $x = 1$,

The value of y is, $y = 4 - 2 = 2$

From equation (1), at $x = -1$,

The value of y is $y = -4 + 2 = -2$

Therefore, the required points are (0, 0), (1, 2) and (-1, -2).

19. Find the points on the curve $x^2 + y^2 - 2x - 3 = 0$ at which the tangents are parallel to x-axis.

Solution:

Equation of the curve $x^2 + y^2 - 2x - 3 = 0$ ……….(1)

On differentiating expression w.r.t. x, we have

$$2x + 2y\frac{dy}{dx} - 2 = 0$$

$$2y\frac{dy}{dx} = 2 - 2x$$

$$\frac{dy}{dx} = \frac{2(1-x)}{2y} = \frac{1-x}{y}$$

Since tangent is parallel to x-axis: $\frac{dy}{dx} = 0$

$$\frac{1-x}{y} = 0$$

=> $x = 1$

From equation (1), $1 + y^2 - 2 - 3 = 0$

=> $y^2 = 4$

$\Rightarrow y = \pm 2$

Therefore, the required points are (1, 2) and (1, -2).

20. Find the equation of the normal at the point (am^2, am^3) for the curve $ay^2 = x^3$.

Solution:

Equation of the curve $ay^2 = x^3$ ……….(1)

On differentiating expression w.r.t. x, we have

$$a\frac{d}{dx}y^2 = \frac{d}{dx}x^3$$

$$a.2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{2ay}$$

Slope of the tangent at the point (am^2, am^3)

$= \frac{3(am^2)^2}{2a.am^3} = \frac{3m}{2}$

Slope of the normal at the point $(am^2, am^3) = \frac{-2}{3m}$

Equation of the normal at (am^2, am^3) is

$$y - am^3 = \frac{-2}{3m}(x - am^2)$$

=> $3my - 3am^4 = -2x + 2am^2$

=> $2x + 3my - 2am^2 - 3am^4 = 0$

=> $2x + 3my - am^2(2 + 3m^2) = 0$

21. Find the equations of the normal to the curve $y = x^3 + 2x + 6$ which are parallel to the line x + 14y + 4 = 0.

Solution:

Equation of the curve $y = x^3 + 2x + 6$ ….(1)

Slope of the tangent at (x, y)

So, $\frac{dy}{dx} = 3x^2 + 2$

Slope of the normal to the curve at (x, y)

= $\frac{-1}{3x^2 + 2}$ ……….(2)

Since Slope of the normal = $\frac{-1}{14}$ (Given)

$\frac{-1}{3x^2 + 2} = \frac{-1}{14}$

=> $3x^2 + 2 = 14$

$3x^2 = 12$

$x^2 = 4$

$x = \pm 2$

From equation (1), at $x = 2$, $y = 8 + 4 + 6 = 18$

at $x = -2$, $y = -8 - 4 + 6 = -6$

Therefore, the points of contact are (2, 18) and $(-2, -6)$.

Equation of the normal at (2, 18) is $y - 18 = \frac{-1}{14}(x - 2)$

$\Rightarrow 14y - 252 = -x + 2$

$x+14y-254=0$

And Equation of the normal at $(-2,-6)$ is $y+6=\frac{-1}{14}(x+2)$

$14y+84=-x-2$

$x+14y+86=0$

Which is required equation.

22. Find the equation of the tangent and normal to the parabola $y^2=4ax$ at the point $(at^2, 2at)$.

Solution:

Equation of the parabola $y^2=4ax$ ……….(1)

Slope of the tangent at (x, y)

$= \frac{dy}{dx}y^2=4a\frac{d}{dx}(x)$

$\Rightarrow 2y\frac{dy}{dx}=4a$

$\Rightarrow \frac{dy}{dx}=\frac{4a}{2y}=\frac{2a}{y}$

Slope of the tangent at the point $(at^2, 2at)$ $= \frac{2a}{2at}=\frac{1}{t}$

Slope of the normal = $-t$

Equation of the tangent at the point $(at^2, 2at)$

$= y-2at=\frac{1}{t}(x-at^2)$

$ty-2at^2=x-at^2$

$ty = x + at^2$

And Equation of the normal at the point $(at^2, 2at)$

= $y - 2at = -t(x - at^2)$

Which implies, $tx + y = 2at + at^3$

23. Prove that the curves $x = y^2$ and $xy = k$ cut at right angles if $8k^2 = 1$.

Solution: Equations of the curves are $x = y^2$ …..(1) and

xy = k……….(2)
Substituting the value of x in equation (2), we get $y^2.y = k$

y = $k^{1/3}$

Put the value of y in equation (1), we get

$$x = \left(k^{1/3}\right)^2 = k^{2/3}$$

Therefore, the point of intersection (x, y) is

$\left(k^{2/3}, k^{1/3}\right)$……….(3)

Differentiating equation (1) w.r.t x

=> $\dfrac{dy}{dx} = \dfrac{1}{2y} = m_1$ ……….(4)

Differentiating equation (2) w.r.t

=> $\dfrac{dy}{dx} = \dfrac{-y}{x} = m_2$……….(5)

According to the question, $m_1 m_2 = -1$

Which implies,

$$\frac{1}{2y}\left(\frac{-y}{x}\right)=-1$$

$$\frac{1}{2x}=1$$

$$2x=1$$

$2k^{1/3}$ = 1 [using equation (3)]

Taking cube both the sides,

$$8k^2=1$$

Hence Proved.

24. Find the equation of the tangent and normal to the hyperbola $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at the point (x_0, y_0).

Solution:

Equation of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ……….(1)

On differentiating w.r.t. x, we get

$$\frac{2x}{a^2}-\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{-2y}{b^2}\cdot\frac{dy}{dx}=\frac{-2x}{a^2}$$

$$\frac{dy}{dx}=\frac{b^2x}{a^2y}$$ ……….(2)

Slope of tangent at (x_0, y_0) is $\frac{b^2x_0}{a^2y_0}$

Equation of the tangent at (x_0, y_0) is $y-y_0=\frac{b^2x_0}{a^2y_0}(x-x_0)$

$$yy_0 - y_0^2 = \frac{b^2}{a^2}\left(xx_0 - x_0^2\right)$$

$$\frac{yy_0}{b^2} - \frac{y_0^2}{b^2} = \frac{xx_0}{a^2} - \frac{x_0^2}{a^2}$$

$$\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = \frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} \quad \text{……….(3)}$$

Since (x_0, y_0) lies on the hyperbola (1), therefore, $\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} = 1$

From equation (3), $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$

Now, Slope of normal at (x_0, y_0) = $\frac{-a^2 y_0}{b^2 x_0}$

Therefore,

Equation of the normal at (x_0, y_0) is $y - y_0 = \frac{-a^2 x_0}{b^2 y_0}\left(x - x_0\right)$

$$b^2 x_0 y - b^2 x_0 y_0 = -a^2 y_0 x + a^2 x_0 y_0$$

$$b^2 x_0\left(y - y_0\right) = -a^2 y_0\left(x - x_0\right)$$

On dividing both sides by $a^2 b^2 x_0 y_0$, we get

$$\frac{y - y_0}{a^2 y_0} = -\frac{\left(x - x_0\right)}{b^2 x_0}$$

$$\frac{\left(x - x_0\right)}{b^2 x_0} + \frac{y - y_0}{a^2 y_0} = 0$$

Which is required equation.

25. Find the equation of the tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0.$

Solution:

Equation of the curve $y=\sqrt{3x-2}$ ……….(1)

Slope of the tangent at point (x, y) is $\frac{dy}{dx}=\frac{d}{dx}(3x-2)^{\frac{1}{2}}$

$$\frac{dy}{dx}=\frac{1}{2}(3x-2)^{-\frac{1}{2}}\frac{d}{dx}(3x-2)$$

$= \frac{1}{2\sqrt{3x-2}}.3$ ……….(2)

Again slope of the line $4x-2y+5=0$ is $\frac{-a}{b}=\frac{-4}{-2}=2$ ……….(3)

As given: Parallel lines have same slope.

By equation slopes of both the lines, we get

$\frac{1}{2\sqrt{3x-2}}.3$ = 2

$$4\sqrt{3x-2}=3$$

$$16(3x-2)=9$$

$$48x-32=9$$

$$48x=41$$

$$x=\frac{41}{48}$$

Substitute the value of x in equation (1),

$$y=\sqrt{3\left(\frac{41}{48}\right)-2}$$

$= \sqrt{\frac{41}{16} - 2}$

$= \sqrt{\frac{41-32}{16}} = \sqrt{\frac{9}{16}} = \frac{3}{4}$

Therefore, point of contact is $\left(\frac{41}{48}, \frac{3}{4}\right)$.

Now,

Equation of the required tangent is $y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right)$

$y - \frac{3}{4} = 2x - \frac{41}{24}$

$y = 2x + \frac{18-41}{24}$

$24y = 48x - 23$

$48x - 24y = 23$

Which is required equation.

Choose the correct answer in Exercises 26 and 27.

26. The slope of the normal to the curve $y = 2x^2 + 3\sin x$ at $x = 0$ is:

(A) 3 (B) 1/3 (C) -3 (D) -1/3

Solution:

Option (D) is correct.

Explanation:

Equation of the curve $y = 2x^2 + 3\sin x$ ……….(1)

Slope of the tangent at point (x, y) is $\frac{dy}{dx} = 4x + 3\cos x$

Slope of the tangent at $x=0$, $4(0)+3\cos 0=3=m$ (say)

Slope of the normal = $\frac{-1}{m}=\frac{-1}{3}$

27. The line $y=x+1$ is a tangent to the curve $y^2=4x$ at the point:

(A) (1, 2) (B) (2, 1) (C) (1, -2) (D) (-1, 2)

Solution:
Option (A) is correct.

Explanation:
Equation of the curve $y^2=4x$ ……….(1)

Slope of the tangent at point (x,y) is $2y\frac{dy}{dx}=4$

$\frac{dy}{dx}=\frac{2}{y}$ ……….(2)

Now,

Slope of the line $y=x+1$ is 1…..(3)

[as we know, $\frac{-a}{b}=\frac{-1}{-1}=1$]

From equation (2) and (3),

$\frac{2}{y}=1$

$y=2$

From equation (1), $4=4x$

$x=1$

Therefore, required point is (1, 2).