1. Using differentials, find the approximate value of each of the following:

(a) \( \left( \frac{17}{81} \right)^{\frac{1}{4}} \)

(b) \( (33)^{-\frac{1}{2}} \)

Solution: (a)

Consider \( y = x^{\frac{1}{4}} \) \( \ldots \ldots \) (1)

\[ \frac{dy}{dx} = \frac{1}{4x^{\frac{3}{4}}} \]

\[ dy = \frac{dx}{4 \left( \frac{1}{x^{\frac{3}{4}}} \right)} \]

\[ = \frac{\Delta x}{4 \left( \frac{1}{x^{\frac{3}{4}}} \right)} \] \( \ldots \ldots \) (2)

Changing \( x \) to \( x + \Delta x \) and \( y \) to \( y + \Delta y \) in equation (1), we have

\[ y + \Delta y = (x + \Delta x)^{\frac{1}{4}} = \left( \frac{17}{81} \right)^{\frac{1}{4}} = \left( \frac{16}{81} + \frac{1}{81} \right)^{\frac{1}{4}} \] \( \ldots \ldots \) (3)

Here \( x = \frac{16}{81} \) and \( \Delta x = \frac{1}{81} \)

\[ x^{\frac{1}{4}} = \left( \frac{16}{81} \right)^{\frac{1}{4}} = \frac{2}{3} \]

So,

From equation (3),
Consider \( y = x^{-\frac{1}{3}} \) \( \quad (1) \)

\[
\frac{\Delta y}{\Delta x} \approx \frac{1}{x^{\frac{4}{3}}} + \frac{\Delta x}{4 x^{\frac{1}{3}}} \quad (2)
\]

\[
(17^{\frac{1}{3}}) = \frac{2}{3} + \frac{1}{81} \times \frac{27}{32}
\]

\[
\frac{2}{3} + \frac{1}{96} = \frac{65}{96} = 0.677
\]

(b) \( (33)^{-\frac{1}{3}} \)

Consider \( y = x^{-\frac{1}{3}} \) \( \quad (1) \)

\[
\frac{dy}{dx} = \frac{-1}{5x^{\frac{2}{3}}} \quad (1)
\]

\[
\frac{\Delta y}{\Delta x} = \frac{-dx}{5 \left( \frac{1}{x^{\frac{2}{3}}} \right)} \quad (2)
\]

Changing \( x \) to \( x + \Delta x \) and \( y \) to \( y + \Delta y \) in equation (1), we have

\[
y + \Delta y = (x + \Delta x)^{-\frac{1}{3}} = (33)^{-\frac{1}{3}} = (32 + 1)^{-\frac{1}{3}} \quad (3)
\]

Here \( x = 32 \) and \( \Delta x = 1 \)
\[ x^{\frac{1}{3}} = (32)^{\frac{1}{3}} = \frac{1}{2} \]

From equation (3),

\[ (33)^{\frac{1}{3}} = y + \Delta y \sim y + \frac{\Delta y}{5 \left( x^3 \right)^{\frac{1}{3}}} \]

\[ \Rightarrow (33)^{\frac{1}{3}} \sim \frac{1}{2} - \frac{1}{5(2)^{\frac{1}{3}}} \]

\[ \sim \frac{1}{2} - \frac{1}{5} \times \frac{1}{64} \sim \frac{1}{2} - \frac{1}{320} = \frac{159}{320} = 0.497 \]

2. Show that the function given by \[ f(x) = \frac{\log x}{x}, x > 0 \] has maximum value at \( x = e \).

Solution: Here

\[ f(x) = \frac{\log x}{x}, x > 0 \] ...........(1)

\[ f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2} \] ...........(2)

\[ f''(x) = \frac{x^2 \left( -\frac{1}{x} \right) - (1 - \log x) \cdot 2x}{x^4} = \frac{-x - 2x + 2x \log x}{x^4} \]

And

\[ \frac{2x \log x - 3x}{x^4} = \frac{2 \log x - 3}{x^3} \]

Which implies,

\[ f''(x) = \frac{x (2 \log x - 3)}{x^4} = \frac{2 \log x - 3}{x^3} \] ...........(3)

Now \( f'(x) = 0 \)

\[ \frac{1 - \log x}{x^2} = 0 \]

\[ 1 - \log x = 0 \]
From equation (3),

\[ f''(x) = \frac{2\log e - 3}{e^3} = \frac{2 - 3}{e^3} = \frac{-1}{e^3} < 0 \]

Point of local maxima and maximum value of \( f(x) \) at \( x = e \).

3. The two equal sides of an isosceles triangle with fixed base \( b \) are decreasing at the rate of 3 cm per second. How fast is the area decreasing when the two equal sides are equal to the base?

Solution: Consider \( BC = b \) be the fixed base and \( AB = AC = x \) be the two sides of isosceles triangle.

\[ \text{Area of triangle} = \frac{1}{2} \times BC \times AM \]

\[ = \frac{b}{2} \sqrt{x^2 - \frac{b^2}{4}} \]

\[ = \frac{b}{4} \sqrt{4x^2 - b^2} \]

\[ \frac{d\Delta}{dt} = \frac{d}{dt} \left( \frac{b}{4} \sqrt{4x^2 - b^2} \right) \]
= \frac{b}{4} \times \frac{d}{dx} \left( \sqrt{4x^2 - b^2} \right) \times \frac{dx}{dt} \quad \text{[By chain rule]}

\frac{dA}{dt} = \frac{b}{4} \times \frac{1}{2\sqrt{4x^2 - b^2}} \times 8x \times (-3)

\frac{-3bx}{\sqrt{4x^2 - b^2}} \quad \text{cm}^2/\text{s}

Now, when \ x = b.

\frac{dA}{dt} = \frac{-3b \cdot b}{\sqrt{4b^2 - b^2}} = \frac{-3b^2}{\sqrt{3}b} = \frac{-\sqrt{3}b}{\text{cm}^2/\text{s}}

Therefore, the area is decreasing at the rate of \ \sqrt{3}b \ \text{cm}^2/\text{s}.

4. Find the equation of the normal to the curve \ x^2 = 4y \ at \ the \ point \ (1, 2).

Solution: Equation of the curve is \ x^2 = 4y \ \text{........(1)}

Differentiate w.r.t. \ x,

\[ 2x = 4 \frac{dy}{dx} \]

\[ \Rightarrow \frac{dy}{dx} = \frac{x}{2} = m \ (\text{say}) \]

Slope of the normal to the curve at (1, 2) is \ -1/m = -2/x

Equation of the normal to the curve (1) at (1, 2) is

\[ x + y = 3 \]

5. Show that the normal at any point \ \theta \ to \ the \ curve

\[ x = a \cos \theta + a \theta \sin \theta, \ y = a \sin \theta - a \theta \cos \theta \]

is at a constant distance from the origin.

Solution: The parametric equations of the curve are

\[ x = a \cos \theta + a \theta \sin \theta, \ y = a \sin \theta + a \theta \cos \theta \]

\[ x = a (\cos \theta + \theta \sin \theta), \ y = a (\sin \theta - \theta \cos \theta) \]
The slope of the tangent at a point \( (x, y) \) is given by:

\[
\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta
\]

The slope of the normal at any point \( \theta \) is:

\[
-1 = \tan^{-1}(-\cot \theta) = -\frac{\cos \theta}{\sin \theta}
\]

And the equation of the normal at any point \( \theta \), that is, at \( (x, y) \), is:

\[
\left[a(\cos \theta + \theta \sin \theta), a(\sin \theta - \theta \cos \theta)\right]
\]

\[
y - a(\sin \theta - \theta \cos \theta) = -\frac{\cos \theta}{\sin \theta} \left[x - a(\cos \theta + \theta \sin \theta)\right]
\]

\[
y \sin \theta - a \sin^2 \theta + a \theta \cos \theta \sin \theta = -x \cos \theta + a \cos^2 \theta + a \theta \sin \theta \cos \theta
\]

\[
x \cos \theta + y \sin \theta = a \left(\sin^2 \theta + \cos^2 \theta\right)
\]

\[
x \cos \theta + y \sin \theta = a
\]

\[
x \cos \theta + y \sin \theta - a = 0
\]

The distance of the normal from the origin (0, 0) is:

\[
\frac{|0 - 0 - a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = a
\]

which is a constant.
6. Find the intervals in which the function \( f \) given by
\[
f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}
\]
is
(i) increasing (ii) decreasing.

**Solution:** Given function is:
\[
f(x) = \frac{4\sin x - 2x - x\cos x}{2 + \cos x}
\]

On differentiating:
\[
f'(x) = \frac{(2 + \cos x) \frac{d}{dx}(4\sin x) - 4\sin x \frac{d}{dx}(2 + \cos x)}{(2 + \cos x)^2} - 1
\]

\[
f'(x) = \frac{(2 + \cos x)(4\cos x) - 4\sin x(-\sin x)}{(2 + \cos x)^2} - 1
\]

\[
= \frac{8\cos x + 4\cos^2 x + 4\sin^2 x}{(2 + \cos x)^2} - 1
\]

Which implies,
\[
f'(x) = \frac{8\cos x + 4}{(2 + \cos x)^2} - 1
\]

\[
= \frac{8\cos x + 4 - (2 + \cos x)^2}{(2 + \cos x)^2}
\]
\[
\frac{8 \cos x + 4 - 4 \cos^2 x - 4 \cos x}{(2 + \cos x)^2}
\]

\[f'(x) = \frac{4 \cos x - \cos^2 x}{(2 + \cos x)^2}\]

\[\Rightarrow \cos x \frac{4 - \cos x}{(2 + \cos x)^2} \quad \ldots \ldots \quad (1)\]

Now \(4 - \cos x > 0\) for all real \(x\) as \(-1 \leq \cos x \leq 1\). Also \((2 + \cos x)^2 > 0\)

(i) \(f(x)\) is increasing if \(f'(x) \geq 0\), that is, from equation (1), \(\cos x \geq 0\)

\(x\) lies in I and IV quadrants, that is, \(f(x)\) is increasing for \(0 \leq x \leq \frac{\pi}{2}\)

and \(\frac{3\pi}{2} \leq x \leq 2\pi\)

and (2) \(f(x)\) is decreasing if \(f'(x) \leq 0\), that is, from equation (1), \(\cos x \leq 0\)

\(\Rightarrow x\) lies in II and III quadrants, that is, \(f(x)\) is decreasing for \(\frac{\pi}{2} \leq x \leq \frac{3\pi}{2}\)

7. Find the intervals in which the function \(f\) given by \(f(x) = x^3 + \frac{1}{x}, x \neq 0\)

is

(i) increasing (ii) decreasing.

Solution:
(i)
\[
f'(x) = x^3 + \frac{1}{x^2}, x \neq 0
\]

\[
f'(x) = 3x^2 - 3x^{-4} = 3\left(x^2 - \frac{1}{x^4}\right)
\]
\[
\frac{3}{x^3} - 1 = \frac{3}{x^4} \left( x^2 + x^3 \right) - 1^3
\]

\[\Rightarrow f'(x) = \frac{3}{x^4} \left( x^2 - 1 \right) \left( x^2 + x + 1 \right)\]

\[\Rightarrow \frac{3}{x^4} \left( x^2 + x + 1 \right) (x+1)(x-1) = 0\]

Here, \( \frac{3}{x^4} \left( x^2 + x + 1 \right) \) is positive for all real \( x \neq 0 \)

\( x+1=0 \) or \( x-1=0 \) [Turning points]

Therefore, \( x = -1 \) or \( x = 1 \) divide the real line into three sub intervals \( (-\infty, -1), (-1, 1) \) and \( (1, \infty) \)

For \( (-\infty, -1) \), from equation (1) at \( x = -2 \) (say),

\[f'(x) = (+)(-) = (+)\]

Therefore, \( f(x) \) is increasing at \( x = -1 \)

For \( (-1, 1) \), from equation (1) at \( x = \frac{1}{2} \) (say)

\[f'(x) = (+)(+) = (-)\]

Therefore, \( f(x) \) is decreasing at \( x = -1, 1 \)

For \( (1, \infty) \), from equation (1) at \( x = 2 \) (say),

\[f'(x) = (+)(+) = (+)\]
Therefore, \( f(x) \) is increasing at \( x = 1 \)

Therefore, \( f(x) \) is (1) an increasing function for \( x \leq -1 \) and for \( x \geq 1 \) and (2) decreasing function for \(-1 \leq x \leq 1\).

8. Find the maximum area of an isosceles triangle inscribed in the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) with its vertex at one end of the major axis.

Solution: Equation of the ellipse is

\[
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)
\]

Comparing equation (1) with \( \cos^2 \theta + \sin^2 \theta = 1 \), we have

\[
\frac{x}{a} = \cos \theta \quad \text{and} \quad \frac{y}{b} = \sin \theta
\]

or \( x = a \cos \theta \) and \( y = b \sin \theta \)

Any point on ellipse is \( P(a \cos \theta, b \sin \theta) \).

Draw PM perpendicular to x-axis and produce it to meet the ellipse in the point Q.

\[
OM = a \cos \theta \quad \text{and} \quad PM = b \sin \theta
\]

We know that the ellipse (1) is symmetrical about x-axis, therefore,

PM = QM and So triangle APQ is isosceles.

Area of \( \triangle APQ \) = \( \frac{1}{2} \times \text{Base} \times \text{Height} \)
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[ \frac{1}{2} PQ \cdot AM = \frac{1}{2} PM \cdot AM = PM (OA - OM) \]

\[ \Rightarrow z = b \sin \theta (a - a \cos \theta) = ab (\sin \theta - \sin \theta \cos \theta) \]

\[ \Rightarrow z = \frac{ab}{2} (2 \sin \theta - 2 \sin \theta \cos \theta) = \frac{ab}{2} (2 \sin \theta - \sin 2\theta) \]

\[ \frac{dz}{d\theta} = \frac{ab}{2} (2 \cos \theta - 2 \cos 2\theta) = ab (\cos \theta - \cos 2\theta) \]

\[ \frac{d^2z}{d\theta^2} = ab (-\sin \theta + 2 \sin 2\theta) \]

Now, \[ \frac{dz}{d\theta} = 0 \]

\[ \Rightarrow ab (\cos \theta - \cos 2\theta) = 0 \]

\[ \Rightarrow \cos \theta - \cos 2\theta = 0 \]

\[ \Rightarrow \cos \theta = \cos 2\theta \]

\[ \Rightarrow \cos \theta = \cos (360^\circ - 2\theta) \]

\[ \Rightarrow \theta = 2\theta \text{ or } \theta = 360^\circ - 2\theta \]

that is, \( \theta = 0 \text{ or } 3\theta = 360^\circ \)

\[ \Rightarrow \theta = 120^\circ \]

\( \theta = 0 \text{ is impossible} \)
\[ \therefore \theta = 120^\circ \]

At \( \theta = 120^\circ \), \( \frac{d^2z}{d\theta^2} = ab\left(-\sin 120^\circ + 2\sin 240^\circ\right) \)

\[ = ab\left(\frac{-\sqrt{3}}{2} - \frac{2\sqrt{3}}{2}\right) = ab\left(\frac{-3\sqrt{3}}{2}\right) \text{ [Negative]} \]

\[ \therefore z \text{ is maximum at } \theta = 120^\circ \]

\[ \therefore \text{From equation (1), Maximum area} \]

\[ = \frac{ab}{2} \left(2\sin 120^\circ - \sin 240^\circ\right) \]

\[ = \frac{ab}{2} \left(2\frac{\sqrt{3}}{2} - \frac{-\sqrt{3}}{2}\right) \]

\[ = \frac{ab}{2} \left(\frac{3\sqrt{3}}{2}\right) = \frac{3\sqrt{3}}{4}ab \]

9. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m³. If building of tank costs Rs. 70 per sq. meter for the base and Rs. 45 per square meter for sides. What is the cost of least expensive tank?

**Solution:**

Depth of tank = 2 m

Consider \( x \) m be the length and \( y \) m be the breadth of the base of the tank.

Volume of tank = 8 cubic meters

\[ x.y.2 = 8 \]

\[ xy = 4 \]

\[ y = \frac{4}{x} \]
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

Cost of building the base of the tank at the rate of Rs. 70 per sq. meter is 70 \(xy\).

And cost of building the four walls of the tank at the rate of Rs. 45 per sq. meter is

\[45(x^2 + x^2 + y^2 + y^2)\]

\[= (180x + 180y)\]

Consider \(z\) be the total cost of building the tank.

\[z = 70xy + 180x + 180y\]

\[
\frac{dz}{dx} = 0 + 180 - \frac{720}{x^2}
\]

\[
\frac{d^2z}{dx^2} = \frac{1440}{x^3}
\]

Now

\[
\frac{dz}{dx} = 0
\]

\[\Rightarrow 180 - \frac{720}{x^2} = 0\]

\[\Rightarrow \frac{720}{x^2} = 180\]

\[\Rightarrow 180x^2 = 720\]

\[\Rightarrow x^2 = 4\]

\[\Rightarrow x = 2 \text{ [Length cannot be negative]}\]

At \(x = 2\),

\[
\frac{d^2z}{dx^2} = \frac{1440}{8} = 180
\]

[Positive]

\[\therefore z\text{ is minimum at } x = 2.\]
Minimum cost
\[= 280 + 180 \times 2 + \frac{720}{2} \]
\[= 280 + 360 + 360 = \text{Rs. 1000} \]

10. The sum of the perimeter of a circle and square is \(k\), where \(k\) is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

**Solution:** Consider \(x\) be the radius of the circle and \(y\) be the side of square. According to question, Perimeter of circle + Perimeter of square = \(k\)

\[2\pi x + 4y = k \]
\[\Rightarrow 4y = k - 2\pi x \]
\[\Rightarrow y = \frac{k - 2\pi x}{4} \quad \ldots(1) \]

Consider \(z\) be the sum of areas of circle and square.

\[z = \pi x^2 + y^2 \]

\[= \pi x^2 + \left(\frac{k - 2\pi x}{4}\right)^2 \quad \text{[From equation (1)]} \]
\[\Rightarrow z = \frac{16\pi x^2 + k^2 + 4\pi^2 x^2 - 4k\pi x}{16} \]
\[= \frac{1}{16} \left[ (16\pi + 4\pi^2) x^2 - 4k\pi x + k^2 \right] \]

\[\frac{dz}{dx} = \frac{1}{16} \left[ (16\pi + 4\pi^2) 2x - 4k\pi \right] \quad \text{and} \]
\[\frac{d^2z}{dx^2} = \frac{1}{16} (16\pi + 4\pi^2) 2 \]

Now \(\frac{dz}{dx} = 0\)
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[ \frac{1}{16} \left[ (16\pi + 4\pi^2) 2x - 4k\pi \right] = 0 \]

\[ (16\pi + 4\pi^2) 2x - 4k\pi = 0 \]

\[ 4\pi(4 + \pi) 2x = 4k\pi \]

\[ x = \frac{4k\pi}{4\pi(4 + \pi)2} = \frac{k}{2(4 + \pi)} \]

At \( x = \frac{k}{2(4 + \pi)} \) [Positive]

\[ \therefore \text{is minimum when} \]

\[ \therefore \text{From equation (1),} \]

\[ y = \frac{1}{4} \left[ \frac{k - 2\pi k}{2(4 + \pi)} \right] \]

\[ = \frac{1}{4} \left[ \frac{k - \pi k}{4 + \pi} \right] \]

\[ = \frac{1}{4} \left[ \frac{k(4 + \pi) - \pi k}{4 + \pi} \right] \]

\[ y = \frac{4k + \pi k - \pi k}{4(4 + \pi)} \]

\[ = \frac{4k}{4(4 + \pi)} \]

\[ = 2 \frac{k}{2(4 + \pi)} = 2x \]

Therefore, sum of areas is minimum when side of the square is double the radius of the circle.
11. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.

Solution: Consider \( x \) m be the radius of the semi-circular opening. Then one side of rectangle part of window is \( 2x \) and \( y \) m be the other side.

Perimeter of window = Semi-circular arc AB + Length (AD + DC + BC)

\[
\Rightarrow \frac{1}{2}(2\pi x) + y + 2x + y = 10
\]

\[
\Rightarrow \pi x + 2x + 2y = 10
\]

\[
\Rightarrow 2y = 10 - \pi x - 2x
\]

\[
\Rightarrow y = \frac{10 - (\pi + 2)x}{2} \quad \text{.........(1)}
\]

Area of window \((z) = \) Area of semi-circle + Area of rectangle

\[
= \frac{1}{2}(\pi x^2) + 2xy
\]

\[
= \frac{1}{2}(\pi x^2) + 2x\left[\frac{10 - (\pi + 2)x}{2}\right]
\]

\[
= \frac{1}{2}\left[\pi x^2 + 20x - 2(\pi + 2)x^2\right]
\]
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[ z = \frac{1}{2} \left[ \pi x^2 + 20x - 2\pi x - 4x^2 \right] \]

\[ = \frac{1}{2} \left[ -\pi x^2 - 4x^2 + 20x \right] \]

\[ \frac{dz}{dx} = \frac{1}{2} \left[ -2\pi x - 8x + 20 \right] \text{ and} \]

\[ \frac{d^2 z}{dx^2} = \frac{1}{2} \left( -2\pi - 8 \right) = -\left( \pi + 4 \right) \]

Now \( \frac{dz}{dx} = 0 \)

\[ \Rightarrow \frac{1}{2} \left[ -2\pi x - 8x + 20 \right] = 0 \]

\[ \Rightarrow -2\pi x - 8x + 20 = 0 \]

\[ \Rightarrow -2x(\pi + 4) = -20 \]

\[ x = \frac{20}{2(\pi + 4)} \]

\[ \Rightarrow x = \frac{10}{\pi + 4} \]

\[ \Rightarrow x = \frac{10}{\pi + 4} \]

At \( x = \frac{10}{\pi + 4} \) \( \frac{d^2 z}{dx^2} = -\left( \pi + 4 \right) \) [Negative]

\[ \therefore z \text{ is maximum at } x = \frac{10}{\pi + 4} \]
From equation (1),

\[
y = \frac{10 - (\pi + 2) \cdot \frac{10}{\pi + 4}}{2}
\]

\[
= \frac{10(\pi + 4) - 10(\pi + 2)}{2(\pi + 4)}
\]

\[
\Rightarrow y = \frac{10\pi + 10 - 10\pi - 20}{2(\pi + 4)}
\]

\[
= \frac{20}{2(\pi + 4)} = \frac{10}{\pi + 4} \text{ m}
\]

Therefore, Length of rectangle = \(\frac{20}{\pi + 4}\) m and Width of rectangle = \(\frac{10}{\pi + 4}\) m

And Radius of semi-circle = \(\frac{10}{\pi + 4}\) m

12. A point on the hypotenuse of a triangle is at distances a and b from the sides of the triangle. Show that the maximum length of the hypotenuse is \(\sqrt{a^2 + b^2}\).

Solution: Consider a right triangle ABC.
P be a point on the hypotenuse AC such that PL \(\perp\) AB = a and PM \(\perp\) BC = b and

Consider \(\angle BAC = \angle MPC = \theta\), then in right angled \(\triangle ALP\), \(\frac{AP}{PL} = \cos \theta\)
From triangle, \( AP = PL \cos \theta = a \cos \theta \)

And in right angled \( \triangle PMC \), \( PM = PM \sec \theta = b \sec \theta \)

Consider \( AC = z \), then

\[
z = AP + PC = \frac{PC}{PM} = \frac{a \cos \theta + b \sec \theta}{\cos \theta} \quad \text{, } 0 < \theta < \frac{\pi}{2} \quad \text{.........(1)}
\]

Now

\[
\frac{dz}{d\theta} = -a \sec \theta \cot \theta + b \sec \theta \tan \theta
\]

Now

\[
\frac{dz}{d\theta} = 0
\]

\[
\Rightarrow -a \sec \theta \cot \theta + b \sec \theta \tan \theta = 0
\]

\[
\Rightarrow \frac{b \sin \theta}{\cos^2 \theta} = \frac{a \cos \theta}{\sin^2 \theta}
\]

\[
\Rightarrow b \sin^2 \theta = a \cos^2 \theta \Rightarrow \frac{a}{b} = \frac{\sin^2 \theta}{\cos^2 \theta}
\]

\[
\Rightarrow \frac{a}{b} = \tan^2 \theta
\]

\[
\Rightarrow \tan \theta = \left( \frac{a}{b} \right)^{\frac{1}{2}} \quad \text{.........(2)}
\]

\[
\frac{d^2z}{d\theta^2} = a \left[ \cos \theta \left( -\cos \sec^2 \theta \right) + \cot \theta \left( -\cos \sec \cot \theta \right) \right] + b \left[ \sec \theta \sec^2 \theta + \tan \theta \sec \theta \tan \theta \right]
\]

And

\[
\frac{d^2z}{d\theta^2} = a \left( \cos^3 \theta + \cos \sec \cot \theta \right) + b \left( \sec^3 \theta + \sec \theta \tan^2 \theta \right)
\]

\[
\Rightarrow \frac{d^2z}{d\theta^2} > 0 \quad \text{[ } a > 0, b > 0 \text{ and } \theta \text{ is +ve as } 0 < \theta < \frac{\pi}{2} \text{ ]}
\]
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[ z \text{ is minimum when } \tan \theta = \left(\frac{a}{b}\right)^{\frac{1}{3}} \]

\[ \sec^2 \theta = 1 + \tan^2 \theta = 1 + \left(\frac{a}{b}\right)^{\frac{2}{3}} = \frac{b^{\frac{2}{3}} + a^{\frac{2}{3}}}{b^{\frac{2}{3}}} \]

\[ \sec \theta = \left(\frac{b^{\frac{2}{3}} + a^{\frac{2}{3}}}{b^{\frac{2}{3}}} \right)^{\frac{1}{2}} \]

\[ \Rightarrow \cos ec^2 \theta = 1 + \cot^2 \theta = 1 + \left(\frac{b}{a}\right)^{\frac{2}{3}} \]

\[ \frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}} = \frac{1}{a^{\frac{1}{3}}} \]

\[ \cos ec \theta = \left(\frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right)^{\frac{1}{2}} \]

\[ \Rightarrow \]

Putting these values in equation (1),

\[ a \cdot \left(\frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{1}{3}}} \right)^{\frac{1}{2}} + b \cdot \left(\frac{b^{\frac{2}{3}} + a^{\frac{2}{3}}}{b^{\frac{1}{3}}} \right)^{\frac{1}{2}} \]

Minimum length of hypotenuse =

\[ \left(\frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{a^{\frac{2}{3}}} \right)^{\frac{1}{2}} \left(\frac{a^{\frac{2}{3}} + b^{\frac{2}{3}}}{b^{\frac{2}{3}}} \right)^{\frac{1}{2}} \]
13. Find the points at which the function \( f \) given by \( f(x) = (x-2)^4 (x+1)^3 \) has:

(i) local maxima
(ii) local minima
(iii) point of inflexion.

Solution:

\[ f(x) = (x-2)^4 (x+1)^3 \]  
\[ f'(x) = (x-2)^4 \frac{d}{dx} (x+1)^3 + d \frac{d}{dx} (x-2)^4 (x+1)^3 \]
\[ = (x-2)^4 (x+1)^3 + 4(x-2)^3 (x+1)^3 \]
\[ = (x-2)^3 (x+1)^2 \left[ 3(x-2) + 4(x+1) \right] \]
\[ = (x-2)^3 (x+1)^2 (7x-2) \]

Now \( f'(x) = 0 \)
\[ \Rightarrow (x-2)^3 (x+1)^2 (7x-2) = 0 \]
\[ \Rightarrow x-2 = 0 \text{ or } x+1 = 0 \text{ or } 7x-2 = 0 \]
\[ \Rightarrow x = 2 \text{ or } x = -1 \text{ or } x = \frac{2}{7} \]

Now, for values of \( x \) close to \( \frac{2}{7} \) and to the left of \( \frac{2}{7} \), \( f'(x) > 0 \).

Also for values of \( x \) close to \( \frac{2}{7} \) and to the right of \( \frac{2}{7} \), \( f'(x) < 0 \).

Therefore, \( x = \frac{2}{7} \) is the point of local maxima.
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

Now, for values of \( x \) close to 2 and to the left of \( 2, f'(x) < 0 \). Also for values of \( x \) close to 2 and to the right of \( 2, f'(x) > 0 \).

Therefore, \( x = 2 \) is the point of local minima.

Now as the values of \( x \) varies through -1, \( f'(x) \) does not change its sign. Therefore, \( x = -1 \) is the point of inflexion.

14. Find the absolute maximum and minimum values of the function \( f \) given by

\[ f(x) = \cos^2 x + \sin x, x \in [0, \pi]. \]

**Solution:**

\[ f(x) = \cos^2 x + \sin x, x \in [0, \pi] \]  \[ (1) \]

\[ f'(x) = 2 \cos x \frac{d}{dx} \cos x + \cos x \]

\[ = -2 \cos x \sin x + \cos x = \cos x(-2 \sin x + 1) \]

Now \( f'(x) = 0 \)

\[ \Rightarrow \cos x(-2 \sin x + 1) = 0 \]

\[ \Rightarrow \cos x = 0 \text{ or } -2 \sin x + 1 = 0 \]

\[ \Rightarrow x = \frac{\pi}{2} \text{ or } 2 \sin x = 1 \]

\[ \Rightarrow \sin x = \frac{1}{2} \]

Here \( x = \frac{\pi}{6} \) is a turning point

\[ f\left(\frac{\pi}{2}\right) = \cos^2 \frac{\pi}{2} + \sin \frac{\pi}{2} \]

Now \( = 0 + 1 = 1 \)
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[ f\left(\frac{\pi}{6}\right) = \cos^2 \frac{\pi}{6} + \sin \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4} \]

\[ f(0) = \cos^2 0 + \sin 0 = 1 + 0 = 1 \]

\[ f(\pi) = \cos^2 \pi + \sin \pi = (-1)^2 + 0 = 1 \]

Therefore, absolute maximum is \(\frac{5}{4}\) and absolute minimum is 1.

15. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius \(r\) is \(\frac{4r}{3}\).

Solution: Consider \(x\) be the radius of base of cone and \(y\) be the height of the cone inscribed in a sphere of radius \(r\).

In right angled triangle OBD,

\[ OD^2 + BD^2 = OB^2 \]

\[ \Rightarrow (y-r)^2 + x^2 = r^2 \]

\[ \Rightarrow y^2 + r^2 - 2ry + x^2 = r^2 \]

\[ \Rightarrow x^2 = 2ry - y^2 \quad \text{...........(1)} \]

Volume of cone \(V = \frac{1}{3} \pi x^2 y = \frac{1}{3} \pi (2ry - y^2) y \) [From equation (1)]

\[ \Rightarrow V = \frac{\pi}{3} (2ry^2 - y^3) \]
Now \( \frac{dV}{dy} = 0 \)

\[ \Rightarrow \frac{\pi}{3} \left( 4ry - 3y^2 \right) = 0 \]

\[ \Rightarrow \frac{\pi}{3} \left( 4r - 3y \right) = 0 \]

\[ \Rightarrow 4r - 3y = 0 \]

\[ \Rightarrow y = \frac{4r}{3} \]

At \( y = \frac{4r}{3} \)

\[ \frac{d^2V}{dy^2} = \frac{\pi}{3} (4r - 8r) \]

\[ = -\frac{4\pi r}{3} \quad \text{[Negative]} \]

Volume is maximum at \( y = \frac{4r}{3} \)

16. Consider \( f \) be a function defined on \([a, b]\) such that \( f''(x) > 0 \), for all \( x \in (a, b) \). Then prove that \( f \) is an increasing function on \((a, b)\).

**Solution:** Consider \( I \) be the interval \((a, b)\)

Given: \( f''(x) > 0 \) for all \( x \) in an interval \( I \). Consider \( x_1, x_2 \in I \) with \( x_1 < x_2 \)

By Lagrange’s Mean Value Theorem, we have,

\[ \frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c), \quad \text{where} \quad x_1 < c < x_2 \]

\[ \Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{where} \quad x_1 < c < x_2 \]
Now \( x_1 < x_2 \)
\[ \Rightarrow x_2 - x_1 > 0 \] \quad \cdots (1)

Also, \( f'(x) > 0 \) for all \( x \) in an interval \( I \)
\[ \Rightarrow f'(c) > 0 \]

\[ \therefore \text{From equation (1), } f(x_2) - f(x_1) > 0 \]
\[ \Rightarrow f(x_1) < f(x_2) \]

Thus, for every pair of points \( x_1, x_2 \in I \) with \( x_1 < x_2 \)
\[ \Rightarrow f(x_1) < f(x_2) \]

Therefore, \( f(x) \) is strictly increasing in \( I \).

17. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius \( R \) is \( \frac{2R}{\sqrt{3}} \). Also find the maximum volume.

**Solution:** Consider \( x \) be the radius and \( y \) be the height of the cylinder inscribed in a sphere having centre "O" and radius "R", for \((x > 0, y > 0)\)

\[ \text{In right triangle OAM, } AM^2 + OM^2 = OA^2 \]
\[ x^2 + \left( \frac{y}{2} \right)^2 = R^2 \]

\[ x^2 = R^2 - \frac{y^2}{4} \quad \ldots \ldots (1) \]

Volume of cylinder (V) = \( \pi x^2 y \) \quad \ldots \ldots (2)

\[ V = \pi \left( R^2 - \frac{y^2}{4} \right) y \]

\[ = \pi \left( R^2 y - \frac{y^3}{4} \right) \quad \ldots \ldots (3) \]

\[ \frac{dV}{dy} = \pi \left( R^2 - \frac{3y^2}{4} \right) \quad \text{and} \quad \frac{d^2V}{dy^2} = \pi \left( -\frac{3y}{2} \right) = -\frac{3\pi y}{2} \]

Now \( \frac{dV}{dy} = 0 \)

\[ \pi \left( R^2 - \frac{3y^2}{4} \right) = 0 \]

\[ R^2 - \frac{3y^2}{4} = 0 \]

\[ \Rightarrow R^2 = \frac{3y^2}{4} \]

\[ \Rightarrow y^2 = \frac{4R^2}{3} \]

\[ \Rightarrow y = \frac{2R}{\sqrt{3}} \]

\[ y = \frac{2R}{\sqrt{3}} \quad \frac{d^2V}{dy^2} = -\frac{3\pi}{2} \left( \frac{2R}{\sqrt{3}} \right) \]

At \( y = \frac{2R}{\sqrt{3}} \), \( \frac{d^2V}{dy^2} = -\frac{3\pi}{2} \left( \frac{2R}{\sqrt{3}} \right) \)
\[ V = \frac{2R}{\sqrt{3}} \text{ [Negative]} \]

V is maximum at \[ y = \frac{2R}{\sqrt{3}} \]

From equation (3),

\[ \text{Maximum value of cylinder} = \pi \left[ R^2 \frac{2R}{\sqrt{3}} \left(1 - \frac{1}{3}\right) \right] \]

\[ = \frac{4\pi R^3}{3\sqrt{3}} \]

18. Show that the height of the cylinder of greatest volume which can be inscribed in a right circular cone of height \( h \) and having semi-vertical angle \( \alpha \) is one-third that of the cone and the greatest volume of the cylinder is \( \frac{4}{27} \pi h^3 \tan^2 \alpha \).

Solution: Consider \( r \) be the radius of the right circular cone of height \( h \). Say \( x \) be the radius of the inscribed cylinder with height \( y \).

\[ \frac{PQ}{RC} = \frac{AP}{AR} \]

\[ \frac{x}{y} = \frac{h-y}{h} \]

\[ \Rightarrow hx = rh - ry \]
Volume of cylinder \( V = \frac{\pi h}{r} (r - x) \) .......(3)

\[ \frac{dV}{dx} = \frac{\pi h}{r} (2r - 3x^2) \]
\[ \frac{d^2V}{dx^2} = \frac{\pi h}{r} (2r - 6x) \]

Now \( \frac{dV}{dx} = 0 \)

\[ \frac{\pi h}{r} (2r - 3x^2) = 0 \]
\[ 2r - 3x^2 = 0 \]
\[ 2r - 3x = 0 \]
\[ x = \frac{2r}{3} \]

And

\[ x = \frac{2r}{3}, \quad \frac{d^2V}{dx^2} = \frac{\pi h}{r} \left( 2r - \frac{12r}{3} \right) \]

At

\[ \frac{\pi h}{r} (-2r) = -2\pi h \] [Negative]

\[ V \text{ is maximum at } x = \frac{2r}{3} \]

From equation (3),
Maximum value of cylinder =

\[ \frac{\pi h}{r} \left[ \frac{4r^2}{9} - \frac{8r^2}{27} \right] \]

= \frac{\pi r^3}{r} \left[ \frac{4}{9} - \frac{8}{27} \right]

= \frac{\pi r^2}{27} \cdot \frac{4}{27}

= \frac{4}{27} \pi r^2 (h \tan \alpha)^2

= \frac{4}{27} \pi h \tan^2 \alpha \left[ \because \frac{r}{h} = \tan \alpha \right]

Choose the correct answer in the Exercises 19 to 24:

19. A cylindrical tank of radius 10 m is being filled with wheat at the rate of 314 cubic meter per hour. Then the depth of wheat is increasing at the rate of:

(A) 1 m/h  (B) 0.1 m/h  (C) 1.1 m/h  (D) 0.5 m/h

Solution:
Option (A) is correct.

Explanation:
Consider y be the depth of the wheat in the cylindrical tank whose radius is 10 m at time t. V = Volume of wheat in cylindrical tank at time t,

\[ V = \pi (10)^2 y = 100\pi y \] cubic meter

So, \( \frac{dV}{dt} = 100\pi \frac{dy}{dt} \) cubic meter /hr

We are given that \( \frac{dV}{dt} = 314 \) cubic meter /hr

\[ \frac{d}{dt} 100\pi y = 314 \]

So, \( 100\pi y = 314 \)
Therefore, \( y = 1 \) m/h

20. The slope of the tangent to the curve \( x = t^3 + 3t - 8, \quad y = 2t^2 - 2t - 5 \) at the point \((2, -1)\) is:

\[
\begin{align*}
(A) \quad \frac{22}{7} & \quad (B) \quad \frac{6}{7} & \quad (C) \quad \frac{7}{6} & \quad (D) \quad \frac{-6}{7}
\end{align*}
\]

**Solution:**

Option (B) is correct.

**Explanation:**

Equation of the curves are

\[
\begin{align*}
x &= t^3 + 3t - 8, \\
y &= 2t^2 - 2t - 5
\end{align*}
\]

Slope of the tangent to the given curve at point \((x, y)\) is

\[
\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3}
\]

At the given point \((2, -1)\), \( x = 2 \) and \( y = -1 \)

At \( x = 2 \), from equation (1), \( 2 = t^3 + 3t - 8 \)

\[
t^3 + 3t - 10 = 0
\]

\[
(t + 5)(t - 2) = 0
\]

\( t = -5, t = 2 \)

At \( y = -1 \), from equation (2), \( -1 = 2t^2 - 2t - 5 \)

\[
2t^2 - 2t - 4 = 0
\]

\[
t^2 - t - 2 = 0
\]
Here, common value of $t$ in the two sets of values is 2.

Again, from equation (3),

Slope of the tangent to the given curve at point $t = 2$ is \[ \frac{4(2) - 2}{2(2) + 3} = \frac{6}{7} \]

21. The line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$ if the value of $m$ is:

(A) 1 \quad (B) 2 \quad (C) 3 \quad (D) ½

Solution:

Option (A) is correct.

Explanation:

Equation of the curve is $y^2 = 4x$ .......(1)

Now

Now $y = mx + 1$
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[
\frac{2}{m} = mx + 1
\]

\[
mx = \frac{2}{m} - 1
\]

\[
x = \frac{2 - m}{m} \quad \text{(3)}
\]

Putting the values of \(x\) and \(y\) in equation (1), \[ \frac{4}{m^2} = \frac{4(2 - m)}{m^2} \]

\[ 2 - m = 1 \implies m = 1 \]

22. The normal at the point \((1, 1)\) on the curve \(2y + x^2 = 3\) is:

(A) \(x + y = 0\)  \hspace{1cm} (B) \(x - y = 0\)

(C) \(x + y + 1 = 0\)  \hspace{1cm} (D) \(x - y = 1\)

Solution:
Option (B) is correct.

Explanation:
Equation of the given curve is \(2y + x^2 = 3\) \(\text{........(1)}\)

\[ 2 \frac{dy}{dx} + 2x = 0 \]

\[ \frac{dy}{dx} = -x \]

Slope of the tangent to the given curve at point \((1, 1)\) is

\[ \frac{dy}{dx} = -x = -1 = m \quad \text{(say)} \]

Slope of the normal = \(\frac{-1}{m} = \frac{-1}{-1} = 1\)

Equation of the normal at \((1, 1)\) is \(y - 1 = 1(x - 1)\)
23. The normal to the curve $x^2 = 4y$ passing through (1, 2) is:

(A) $x + y = 3$  
(B) $x - y = 3$  
(C) $x + y = 1$  
(D) $x - y = 1$  

Solution:
Option (A) is correct.

Explanation:
Equation of the curve is $x^2 = 4y$ ...........(1)

$2x = 4 \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{x}{2}$

Slope of the normal at $(x, y)$ is

$- \frac{dx}{dy} = -\frac{2}{x}$ ...........(2)

Again slope of normal at given point $(1, 2) = \frac{y - 2}{x - 1}$ ...........(3)

From equation (2) and (3), we have

$-2 + 2 = xy - 2x$

$xy = 2$

$y = \frac{2}{x}$

From equation (1), $x^2 = \frac{8}{x}$
NCERT Solutions for Class 12 Maths Chapter 6 Application of Derivatives

\[ x^3 = 8 \]
\[ x = 2 \]
\[ y = \frac{2}{x} = \frac{2}{2} = 1 \]

Now, at point (2, 1), slope of the normal from equation (2) = \( \frac{-2}{x} = \frac{-2}{2} = -1 \)

Equation of the normal is \( y - 1 = -1(x - 2) \)
\[ y - 1 = -x + 2 \]
or \( x + y = 3 \)

24. The points on the curve \( 9y^2 = x^3 \), where the normal to the curve make equal intercepts with axes are:

(A) \( \left(4, \pm \frac{8}{3}\right) \)  
(B) \( \left(4, -\frac{8}{3}\right) \)  
(C) \( \left(4, \pm \frac{3}{8}\right) \)  
(D) \( \left(4, \frac{3}{8}\right) \)

Solution:

Option (A) is correct.

Explanation:

Equation of the curve is \( 9y^2 = x^3 \) \((1)\)

\[ 18y \frac{dy}{dx} = 3x^2 \]
\[ \frac{dy}{dx} = \frac{3x^2}{18y} = \frac{x^2}{6y} \]

Slope of the tangent to curve (1) at any point \((x, y)\) is
\[
\frac{dy}{dx} = \frac{x^2}{6y}
\]

Slope of the normal = negative reciprocal = \(\frac{-6y}{x^2} = \pm 1\)

As we know that, slopes of lines with equal intercepts on the axes are \(\pm 1\)

So, \(-6y = \pm x^2\)

If, \(-6y = x^2\)

\[y = \frac{-x^2}{6} \quad \text{........(2)}\]

From equation (1) and (2), we have \(x = 4\) and \(y = \frac{-8}{3}\)

If, \(-6y = -x^2\)

\[y = \frac{x^2}{6} \quad \text{........(2)}\]

From equation (1) and (2), we have \(x = 4\) and \(y = \frac{8}{3}\)

Required points are \(\left(4, \pm \frac{8}{3}\right)\)