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1. Show that the function given by f(x) = 3x + 17 is strictly increasing on R.

Solution: Given function: $f(x) = 3x + 17$
Derivate w.r.t x:
$f'(x) = 3(1) + 0 = 3 > 0$ that is, positive for all x ∈ R

Therefore, f(x) is strictly increasing on R.

2. Show that the function given by f(x) = e^{2x} is strictly increasing on R.

Solution: Given function: f(x) = e^{2x}

$f'(x) = e^{2x}\frac{d}{dx}2x = e^{2x}(2) = 2e^{2x} > 0$ that is, positive for all x ∈ R

Therefore, f(x) is strictly increasing on R.

3**. Show that the function given by f(x) = sin x is**

(a) strictly increasing $\left(0, \frac{\pi}{2}\right)$.

(b) strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

(c) neither increasing nor decreasing in $(0, \pi)$.

Solution: Given function: $f(x) = \sin x$
$f'(x) = \cos x$

(a) Since, $f'(x) = \cos x > 0$, that is, positive in 1st quadrant, that is, in $\left(0, \frac{\pi}{2}\right)$.

Therefore, $f(x)$ is strictly increasing in $\left(0, \frac{\pi}{2}\right)$.

(b) Since, $f'(x) = \cos x < 0$, that is, negative in 2nd quadrant, that is, in $\left(\frac{\pi}{2}, \pi\right)$.

$f(x)$ is strictly decreasing in $\left(\frac{\pi}{2}, \pi\right)$.

(c) Since $f'(x) = \cos x$ > 0, that is, positive in 1st quadrant, that is, in $\left(0, \frac{\pi}{2}\right)$ and

$f'(x) = \cos x$ < 0, that is, negative in 2nd quadrant, that is, in $\left(\frac{\pi}{2}, \pi\right)$ and $f'\left(\frac{\pi}{2}\right) = \cos\frac{\pi}{2} = 0$

Therefore, $f'(x)$ does not have same sign in the interval $(0, \pi)$.

So, $f(x)$ is neither increasing nor decreasing in $(0, \pi)$.

4. Find the intervals in which the function f given by f(x) = $2x^2 - 3x$ is
(a) strictly increasing,
(b) strictly decreasing.
Solution: Given function: f(x) = $2x^2 - 3x$

$f'(x) = 4x - 3$ ……….(1)

Now $4x - 3 = 0$

$\Rightarrow x = \frac{3}{4}$

Therefore, we have two intervals $\left(-\infty, \frac{3}{4}\right)$ and $\left(\frac{3}{4}, \infty\right)$.

(a) For interval $\left(\frac{3}{4}, \infty\right)$, picking x =1, then from equation (1), $f'(x)$ > 0.

Therefore, f is strictly increasing in $\left(\frac{3}{4}, \infty\right)$.

(b) For interval $\left(-\infty, \frac{3}{4}\right)$, picking $x = 0.5$, then from equation (1), $f'(x)$ < 0.

Therefore, f is strictly decreasing in $\left(-\infty, \frac{3}{4}\right)$.

5. Find the intervals in which the function f given by $f(x)=2x^3-3x^2-36x+7$ is
(a) strictly increasing,
(b) strictly decreasing.

Solution: (a) Given function: $f(x)=2x^3-3x^2-36x+7$

$f'(x)=6x^2-6x-36$ = $6(x^2-x-6)$

$f'(x)=6(x+2)(x-3)$ ……….(1)

Now $6(x+2)(x-3)=0$

$x+2=0$ or $x-3=0$

$x=-2$ or $x=3$

The value of x is either -2 or 3.

Therefore, we have sub-intervals are $(-\infty,-2),(-2,3)$ and $(3,\infty)$.

For interval $(-\infty,-2)$, picking x = -3, from equation (1),

$f'(x)=(+)(-)(-)=(+)$ > 0

Therefore, f is strictly increasing in $(-\infty,-2)$.

For interval $(-2,3)$, picking x = 2 , from equation (1),

$f'(x)=(+)(+)(-)=(-)$ < 0

Therefore, f is strictly decreasing in $(-2,3)$.

For interval $(3,\infty)$, picking x = 4, from equation (1),

$f'(x)=(+)(+)(+)=(+)$ > 0

Therefore, f is strictly increasing in $(3,\infty)$.

So, (a) f is strictly increasing in $(-\infty,-2)$ and $(3,\infty)$.

(b) f is strictly decreasing in $(-2,3)$.

6. Find the intervals in which the following functions are strictly increasing or decreasing:

(a) x^2+2x-5

(b) $10-6x-2x^2$

(c) $-2x^3-9x^2-12x+1$

(d) $6-9x-x^2$

(e) $(x+1)^3(x-3)^3$

Solution:

(a) Given function: $f(x)=x^2+2x-5$

$\Rightarrow f'(x)=2x+2=2(x+1)$ ……….(1)

Now $2(x+1)=0$

$\Rightarrow x=-1$

Therefore, we have two sub-intervals $(-\infty,-1)$ and $(-1,\infty)$.

For interval $(-\infty,-1)$ picking x = -2, from equation (1), $f'(x)=(-)$ < 0

Therefore, f is strictly decreasing.

For interval $(-1,\infty)$ picking x = 0 , from equation (1), $f'(x)=(+)$ > 0

Therefore, f is strictly increasing.

(b) Given function: $f(x)=10-6x-2x^2$

$\Rightarrow f'(x)=-6-4x$ = $-2(3+2x)$ ……….(1)

Now $-2(3+2x)=0$

$x=\frac{-3}{2}$

Therefore, we have two sub-intervals $\left(-\infty,\frac{-3}{2}\right)$ and $\left(\frac{-3}{2},\infty\right)$.

For interval $\left(-\infty,\frac{-3}{2}\right)$ picking x = -2 , from equation (1),

$f'(x)=(-)(-)=(+)$ > 0

Therefore, f is strictly increasing.

For interval $\left(\frac{-3}{2},\infty\right)$ picking x = -1 , from equation (1),

$f'(x)=(-)(+)=(-)$ < 0

Therefore, f is strictly decreasing.

(c) Given function: $f(x)=-2x^3-9x^2-12x+1$
Derivate w.r.t. x,
$f'(x)=-6x^2-18x-12$

$f'(x)=-6(x^2+3x+2)$

= $-6(x+1)(x+2)$ ……….(1)

Now,

$-6(x+1)(x+2)$ = 0

$\Rightarrow x=-1$ or $x=-2$

Therefore, we have three disjoint intervals $(-\infty,-2)$, $(-2,-1)$ and $(-1,\infty)$.

For interval $(-\infty, -2)$, from equation (1),

$f'(x) = (-)(-)(-) = (-) < 0$

Therefore, f is strictly decreasing.

For interval $(-2, -1)$, from equation (1),

$f'(x) = (-)(-)(+) = (+) > 0$

Therefore, f is strictly increasing.

For interval $(-1, \infty)$, from equation (1),

$f'(x) = (-)(+)(+) = (-) < 0$

Therefore, f is strictly decreasing.

(d) Given function: $f(x) = 6 - 9x - x^2$

$f'(x) = -9 - 2x$

Now $-9 - 2x = 0$

$x = \frac{-9}{2}$

Therefore, we have three disjoint intervals $\left(-\infty, \frac{-9}{2}\right)$ and $\left(\frac{-9}{2}, \infty\right)$.

For interval $\left(-\infty, \frac{-9}{2}\right)$, $x < \frac{-9}{2}$

Therefore, f is strictly increasing.

For interval $\left(\frac{-9}{2}, \infty\right)$, $x > \frac{-9}{2}$

Therefore, f is strictly decreasing.

(e) Given function: $f(x)=(x+1)^3(x-3)^3$

$f'(x)=(x+1)^3.3(x-3)^2+(x-3)^3.3(x+1)^2$

$f'(x)=3(x+1)^2(x-3)^2(x+1+x-3)$

$f'(x)=3(x+1)^2(x-3)^2(2x-2)$

$f'(x)=6(x+1)^2(x-3)^2(x-1)$

Here, factors $(x+1)^2$ and $(x-3)^2$ are non-negative for all x.

Therefore, f(x) is strictly increasing if $f'(x)>0$

$x-1>0$

$x>1$

And f(x) is strictly decreasing if $f'(x)<0$

$x-1<0$

$x<1$

So, f is strictly increasing in $(1,\infty)$ and f is strictly decreasing in $(-\infty,1)$.

7. Show that $y=\log(1+x)-\frac{2x}{2+x}, x>-1$ is an increasing function of x throughout its domain.

Solution: Given function: $y=\log(1+x)-\frac{2x}{2+x}$

Derivate y w.r.t. x, we have

$$\frac{dy}{dx}=\frac{1}{1+x}\frac{d}{dx}(1+x)-\left[\frac{(2+x)\frac{d}{dx}(2x)-2x\frac{d}{dx}(2+x)}{(2+x)^2}\right]$$

$=\frac{1}{1+x}-\left[\frac{(2+x)2-2x}{(2+x)^2}\right]$

$=\frac{1}{1+x}-\frac{(4+2x-2)}{(2+x)^2}$

$=\frac{1}{1+x}-\frac{4}{(2+x)^2}$

This implies,

$$\frac{dy}{dx}=\frac{(2+x)^2-4(1+x)}{(1+x)(2+x)^2}$$

$=\frac{x^2}{(1+x)(2+x)^2}$ ……….(1)

Domain of the given function is given to be $x>-1$

$\Rightarrow\ x+1>0$

Also $(2+x)^2>0$ and $x^2\geq 0$

From equation (1), $\frac{dy}{dx}\geq 0$ for all x in domain $x>-1$ and f is an increasing function.

8. Find the value of x **for which** $y=\{x(x-2)\}^2$ **is an increasing function.**

Solution: Given function: $f(x)=y=(x(x-2))^2$

Derivate y w.r.t. x, we get

$\Rightarrow \frac{dy}{dx} = 2x(x-2)\frac{d}{dx}[x(x-2)]$

$\Rightarrow \frac{dy}{dx} = 2x(x-2)\left[x\frac{d}{dx}(x-2)+(x-2)\frac{d}{dx}x\right]$

[Applying Product Rule]

$\Rightarrow \frac{dy}{dx} = 2x(x-2)[x+x-2]$

= $2x(x-2)(2x-2)$

= $4x(x-2)(x-1)$ ……….(1)

$\Rightarrow x=0, x=2, x=1$

Therefore, we have $(-\infty,0),(0,1),(1,2),(2,\infty)$

For $(-\infty,0)$ picking $x=-1$,

$\frac{dy}{dx} = (-)(-)(-) = (-)$ ≤ 0

$\therefore$ $f(x)$ is decreasing.

For $(0,1)$ picking $x=\frac{1}{2}$,

$\frac{dy}{dx} = (+)(-)(-) = (+)$ ≥ 0

$\therefore$ $f(x)$ is increasing.

For $(1,2)$ picking $x=1.5$,

$\frac{dy}{dx} = (+)(-)(+) = (-)$ ≤ 0

f(x) is decreasing.

For $(2, \infty)$ picking $x = 3$,

$\frac{dy}{dx} = (+)(+)(+) = (+)$ ≥ 0

f(x) is increasing.

9. Prove that $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$ **is an increasing function of** θ **in** $\left[0, \frac{\pi}{2}\right]$.

Solution: Given function: $y = \frac{4\sin\theta}{(2+\cos\theta)} - \theta$

Derivate y w.r.t. θ,

$$\frac{dy}{d\theta} = \frac{(2+\cos\theta).4\cos\theta - 4\sin\theta(-\sin\theta)}{(2+\cos\theta)^2} - 1$$

$$= \frac{8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}{(2+\cos\theta)^2} - 1$$

$$\frac{dy}{d\theta} = \frac{8\cos\theta + 4(\cos^2\theta + \sin^2\theta) - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$= \frac{8\cos\theta + 4 - (2+\cos\theta)^2}{(2+\cos\theta)^2}$$

$$\Rightarrow \frac{dy}{d\theta} = \frac{(8\cos\theta + 4) - (4 + 4\cos\theta + \cos^2\theta)}{(2+\cos\theta)^2}$$

$$= \frac{\cos\theta(4-\cos\theta)}{(2+\cos\theta)^2}$$

Since $0 \leq \theta \leq \frac{\pi}{2}$ and we have $0 \leq \cos\theta \leq 1$, therefore $4 - \cos\theta > 0$.

$\frac{dy}{d\theta} \geq 0$ for $0 \leq \theta \leq \frac{\pi}{2}$

So, y is an increasing function of θ in $\left[0, \frac{\pi}{2}\right]$.

10. Prove that the logarithmic function is strictly increasing on $(0, \infty)$.

Solution: Given function: $f(x) = \log x$

$f'(x) = \frac{1}{x}$ for all x in $(0, \infty)$.

Therefore, $f(x)$ is strictly increasing on $(0, \infty)$.

11. Prove that the function f **given by** $f(x) = x^2 - x + 1$ **is neither strictly increasing nor strictly decreasing on** $(-1, 1)$.

Solution: Given function: $f(x) = x^2 - x + 1$

$f'(x) = 2x - 1$

$f(x)$ is strictly increasing if $f'(x) > 0$

$2x - 1 > 0$

$x > \frac{1}{2}$

that is, increasing on the interval $\left(\frac{1}{2}, 1\right)$

$f(x)$ is strictly decreasing if $f'(x) < 0$

$2x - 1 < 0$

$x < \frac{1}{2}$

that is, decreasing

on the interval $\left(-1,\frac{1}{2}\right)$

So, f(x) is neither strictly increasing nor decreasing on the interval $(-1,1)$.

12. Which of the following functions are strictly decreasing on $\left(0,\frac{\pi}{2}\right)$?

(Note for users: since intervals are not defined in NCERT class 12 book. So we have used open brackets here)

(A) cos x (B) cos 2x (C) cos 3x (D) tan x

Solution:

(A) Let $f(x)=\cos x$

$f'(x)=-\sin x$

Since $0<x<\frac{\pi}{2}$ in $\left(0,\frac{\pi}{2}\right)$, therefore $\sin x>0$

$\Rightarrow -\sin x<0$

Therefore, $f(x)$ is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$.

(B) $f(x)=\cos 2x$

$f'(x)=-2\sin 2x$

Since $0<x<\frac{\pi}{2}$

$0<2x<\pi$ therefore $\sin 2x>0$

$\Rightarrow -2\sin 2x<0$

Therefore, $f(x)$ is strictly decreasing on $\left(0,\frac{\pi}{2}\right)$.

(C) $f(x)=\cos 3x$

$f'(x)=-3\sin 3x$

Since $0 < x < \frac{\pi}{2}$

$0 < 3x < \frac{3\pi}{2}$

For $0 < 3x < \pi \;\; \sin 3x > 0$

$\Rightarrow \; -3\sin 3x < 0$

Therefore, $f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{3}\right)$.

For $\pi < 3x < \frac{3\pi}{2} \;\; \sin 3x < 0$

$-3\sin 3x > 0$

Therefore, $f(x)$ is strictly increasing on $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$.

So, $f(x)$ is neither strictly increasing not strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

(D) Let $f(x) = \tan x$

$f'(x) = \sec^2 x$ > 0

Therefore, $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

13. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ is strictly decreasing:

(A) (0, 1) **(B)** $\left(\frac{\pi}{2}, \pi\right)$ **(C)** $\left(0, \frac{\pi}{2}\right)$ **(D) None of these**

(Note for users: since intervals are not defined in NCERT class 12 book. So we have used open brackets here)

Solution: Given function: $f(x) = x^{100} + \sin x - 1$

$f'(x) = 100x^{99} + \cos x$

(A) On (0, 1), $x > 0$ therefore $100x^{99} > 0$

And for $\cos x$

(0, 1 radian) = $(0, 57^\circ \text{ nearly})$ > 0

Therefore, $f(x)$ is strictly increasing on (0, 1).

(B) For $100x^{99}$ $x \in \left(\frac{\pi}{2}, \pi\right)$

For interval: $\left(\frac{11}{7}, \frac{22}{7}\right)$ = (1.5, 3.1) > 1 and So, $100x^{99}$ > 100

For $\cos x$ $\left(\frac{\pi}{2}, \pi\right)$ is in 2nd quadrant and So, $\cos x$ is negative and between -1 and 0.

Therefore, $f(x)$ is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

(C) On $\left(0, \frac{\pi}{2}\right)$ = (0, 1.5) both terms of given function are positive.

Therefore, $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

(D) Option (D) is the correct answer.

14. Find the least value of "a" such that the function f given by $f(x) = x^2 + ax + 1$ strictly increasing on (1, 2).

Solution: $f(x) = x^2 + ax + 1$

Apply derivative:

$f'(x) = 2x + a$

Since $f(x)$ is strictly increasing on (1, 2), therefore $f'(x) = 2x + a$ > 0 for all x in (1, 2)

On (1, 2) $1 < x < 2$

$2 < 2x < 4$

$2 + a < 2x + a < 4 + a$

Therefore, Minimum value of $f'(x)$ is $2 + a$ and maximum value is $4 + a$.

Since $f'(x)$ > 0 for all x in (1, 2)

$2 + a > 0$ and $4 + a > 0$

$a > -2$ and $a > -4$

Therefore, least value of a is -2.

15. Let I be any interval disjoint from [-1,1] Prove that the function f given by

$$f(x) = x + \frac{1}{x}$$

is strictly increasing on I.

Solution: Given function:

$$f(x) = x + \frac{1}{x} = x + x^{-1}$$

Apply derivative:

$$f'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2} = \frac{x^2 - 1}{x^2}$$

$$f'(x) = \frac{(x-1)(x+1)}{x^2}$$ ……….(1)

For every x either $x < -1$ or $x > 1$

for $x < -1$, $x = -2$,

$$f'(x) = \frac{(-)(-)}{(+)} = (+)$$ > 0

Again for, $x > 1$, $x = 2$,

$$f'(x) = \frac{(+)(+)}{(+)} = (+)$$ > 0

$f'(x) > 0$ for all $x \in I\ (-\infty, \infty)$, So, f(x) is strictly increasing on I.

16. Prove that the function f given by f(x) = log sin x is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution: Given function:
f(x) = log sin x

Apply derivative:

$$f'(x) = \frac{1}{\sin x}\frac{d}{dx}\sin x = \frac{1}{\sin x}\cos x = \cot x$$

On the interval $\left(0, \frac{\pi}{2}\right)$ that is, in 1st quadrant,

$f'(x) = \cot x > 0$

Therefore, $f(x)$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

On the interval $\left(\frac{\pi}{2}, \pi\right)$ that is, in 2nd quadrant,

$f'(x) = \cot x < 0$

Therefore, $f(x)$ is strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

17. Prove that the function f given by f(x) = log cos x is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$.

Solution: Given function: f(x) = log cos x

$\Rightarrow \frac{1}{\cos x}\frac{d}{dx}\cos x = \frac{1}{\cos x}(-\sin x) = -\tan x$

On the interval $\left(0, \frac{\pi}{2}\right)$,

In 1st quadrant, $\tan x$ is positive,

thus $f'(x) = -\tan x$ < 0

Therefore, $f(x)$ is strictly decreasing on $\left(0, \frac{\pi}{2}\right)$.

On the interval $\left(\frac{\pi}{2}, \pi\right)$,

In 2nd quadrant, $\tan x$ is negative

thus $f'(x) = -\tan x$ > 0

Therefore, f(x) is strictly increasing on $\left(\frac{\pi}{2}, \pi\right)$.

18. Prove that the function given by $f(x) = x^3 - 3x^2 + 3x - 100$ is increasing in R.

Solution: Given function:

$f(x) = x^3 - 3x^2 + 3x - 100$

Apply derivative:

$f'(x) = 3x^2 - 6x + 3 = 3(x^2 - 2x + 1)$

$f'(x) = 3(x-1)^2 \geq 0$ for all x in R.

Therefore, f(x) is increasing on R.

19. The interval in which $y = x^2 e^{-x}$ is increasing in:

(A) $(-\infty, \infty)$ **(B)** $(-2, 0)$ **(C)** $(2, \infty)$ **(D) (0, 2)**

Solution:
Option (D) is correct.

Explanation:
Given function:

$y = x^2 e^{-x}$

Apply derivative:

$$\frac{dy}{dx} = x^2 \frac{d}{dx} e^{-x} + e^{-x} \frac{d}{dx} x^2$$

= $x^2 e^{-x}(-1) + e^{-x}(2x)$

=> $\frac{dy}{dx} = -x^2 e^{-x} + 2xe^{-x}$

= $xe^{-x}(-x+2)$

So, $\frac{dy}{dx} = \frac{x(2-x)}{e^x}$

In option (D), $\frac{dy}{dx} > 0$ for all x in the interval (0, 2).