Exercise 1.2 Page No: 10

1. Show that the function f : R_* → R_* defined by f(x) = 1/x is one-one and onto, where R_* is the set of all non-zero real numbers. Is the result true, if the domain R_* is replaced by N with co-domain being same as R_*?

Solution:

Given: f : R_* → R_* defined by f(x) = 1/x

Check for One-One

$f(x_1) = \frac{1}{x_1}$ and $f(x_2) = \frac{1}{x_2}$

If $f(x_1) = f(x_2)$ then $\frac{1}{x_1} = \frac{1}{x_2}$

This implies $x_1 = x_2$

Therefore, f is one-one function.

Check for onto

f(x) = 1/x
or y = 1/x
or x = 1/y
f(1/y) = y
Therefore, f is onto function.

Again, If $f(x_1) = f(x_2)$

Say, $n_1, n_2 \in R$

$$\frac{1}{n_1} = \frac{1}{n_2}$$

So $n_1 = n_2$
Therefore, f is one-one

Every real number belonging to co-domain may not have a pre-image in N. for example, 1/3 and 3/2 are not belongs N. So N is not onto.

2. Check the injectivity and surjectivity of the following functions:

(i) f : N → N given by f(x) = x^2

(ii) f : Z → Z given by f(x) = x^2

(iii) f : R → R given by f(x) = x^2

(iv) f : N → N given by f(x) = x^3

(v) f : Z → Z given by f(x) = x^3

Solution:
(i) f : N → N given by f(x) = x^2

For x, y ∈ N => f(x) = f(y) which implies $x^2 = y^2$
⇨ x = y
Therefore f is injective.

There are such numbers of co-domain which have no image in domain N.

Say, 3 ∈ N, but there is no pre-image in domain of f. such that f(x) = x^2 = 3.

f is not surjective.

Therefore, f is injective but not surjective.

(ii) Given, f : Z → Z given by f(x) = x^2

Here, Z = {0, ±1, ±2, ±3, ±4, ...}

f(-1) = f(1) = 1

But -1 not equal to 1.

f is not injective.

There are many numbers of co-domain which have no image in domain Z.

For example, -3 ∈ co-domain Z, but -3 ∉ domain Z
f is not surjective.

Therefore, f is neither injective nor surjective.

(iii) f : R → R given by $f(x) = x^2$

$f(-1) = f(1) = 1$

But -1 not equal to 1.

f is not injective.

There are many numbers of co-domain which have no image in domain R.

For example, $-3 \in$ co-domain R, but there does not exist any x in domain R where $x^2 = -3$
f is not surjective.

Therefore, f is neither injective nor surjective.

(iv) f : N → N given by $f(x) = x^3$

For $x, y \in N$ => $f(x) = f(y)$ which implies $x^3 = y^3$
⇨ $x = y$
Therefore f is injective.

There are many numbers of co-domain which have no image in domain N.

For example, $4 \in$ co-domain N, but there does not exist any x in domain N where $x^3 = 4$.
f is not surjective.

Therefore, f is injective but not surjective.

(v) f : Z → Z given by $f(x) = x^3$

For $x, y \in Z$ => $f(x) = f(y)$ which implies $x^3 = y^3$
⇨ $x = y$
Therefore f is injective.

There are many numbers of co-domain which have no image in domain Z.

For example, $4 \in$ co-domain N, but there does not exist any x in domain Z where $x^3 = 4$.
f is not surjective.

Therefore, f is injective but not surjective.

3. Prove that the Greatest Integer Function f : R → R, given by f(x) = [x], is neither one-one nor onto, where [x] denotes the greatest integer less than or equal to x.

Solution:

Function f : R → R, given by f(x) = [x]
f(x) = 1, because 1 ≤ x ≤ 2

f(1.2) = [1.2] = 1
f(1.9) = [1.9] = 1
But 1.2 ≠ 1.9

f is not one-one.

There is no fraction proper or improper belonging to co-domain of f has any pre-image in its domain.

For example, f(x) = [x] is always an integer

for 0.7 belongs to R there does not exist any x in domain R where f(x) = 0.7
f is not onto.

Hence proved, the Greatest Integer Function is neither one-one nor onto.

4. Show that the Modulus Function f : R → R, given by f(x) = | x |, is neither one-one nor onto, where | x | is x, if x is positive or 0 and |x | is – x, if x is negative.

Solution:

f : R → R, given by f(x) = | x |, defined as

$$f(x) = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

f contains values like (-1, 1),(1, 1),(-2, 2)(2,2)

f(-1) = f(1), but -1 1

f is not one-one.

R contains some negative numbers which are not images of any real number since f(x) = |x| is always non-negative. So f is not onto.

Hence, Modulus Function is neither one-one nor onto.

5. Show that the Signum Function f : R → R, given by

$$f(x)=\begin{cases}1, \text{ if } x>0\\ 0, \text{ if } x=0\\ \quad 1, \text{ if } x<0\end{cases}$$

is neither one-one nor onto.

Solution: Signum Function f : R → R, given by

$$f(x)=\begin{cases}1, \text{ if } x>0\\ 0, \text{ if } x=0\\ \quad 1, \text{ if } x<0\end{cases}$$

f(1) = f(2) = 1

This implies, for n > 0, $f(x_1) = f(x_2) = 1$

$x_1 \neq x_2$

f is not one-one.

f(x) has only 3 values, (-1, 0 1). Other than these 3 values of co-domain R has no any pre-image its domain.

f is not onto.

Hence, Signum Function is neither one-one nor onto.

6. Let A = {1, 2, 3}, B = {4, 5, 6, 7} and let f = {(1, 4), (2, 5), (3, 6)} be a function from A to B. Show that f is one-one.

Solution:
A = {1, 2, 3}
B = {4, 5, 6, 7} and
f = {(1, 4), (2, 5), (3, 6)}

f(1) = 4, f(2) = 5 and f(3) = 6

Here, also distinct elements of A have distinct images in B.

Therefore, f is one-one.

7. In each of the following cases, state whether the function is one-one, onto or bijective. Justify your answer.
(i) f : R → R defined by f(x) = 3 – 4x
(ii) f : R → R defined by f(x) = 1 + x^2

Solution:
(i) f : R → R defined by f(x) = 3 – 4x
If x_1, x_2 ∈ R then

$f(x_1) = 3 – 4x_1$ and
$f(x_2) = 3 – 4x_2$

If $f(x_1) = f(x_2)$ then $x_1 = x_2$

Therefore, f is one-one.

Again,
f(x) = 3 – 4x
or y = 3 – 4x
or x = (3-y)/4 in R

f((3-y)/4) = 3 – 4((3-y)/4) = y

f is onto.

Hence f is onto or bijective.

(ii) f : R → R defined by f(x) = 1 + x^2

If x_1, x_2 ∈ R then

$f(x_1) = 1 + x_1^2$ and
$f(x_2) = 1 + x_2^2$

If $f(x_1) = f(x_2)$ then $x_1^2 = x_2^2$

This implies $x_1 \neq x_2$

Therefore, f is not one-one

Again, if every element of co-domain is image of some element of Domain under f, such that f(x) = y

f(x) = 1 + x^2

y = f(x) = 1 + x^2

or x = $\pm\sqrt{1-y}$

Therefore, f($\sqrt{1-y}$) = 2 - y $\neq$ y

Therefore, f is not onto or bijective.

8. Let A and B be sets. Show that f : A × B → B × A such that f(a, b) = (b, a) is bijective function.

Solution:

Step 1: Check for Injectivity:

Let (a_1, b_1) and (a_2, b_2) ∈ A x B such that

f (a_1, b_1) = (a_2, b_2)

This implies, (b_1 , a_1) and (b_2, a_2)

$b_1 = b_2$ and $a_1 = a_2$

(a_1, b_1) = (a_2, b_2) for all (a_1, b_1) and (a_2, b_2) ∈ A x B

Therefore, f is injective.

Step 2: Check for Surjectivity:

Let (b, a) be any element of B x A. Then a ∈ A and b ∈ B

This implies (a, b) ∈ A x B

For all (b, a) ∈ B x A, their exists (a, b) ∈ A x B

Therefore, f: A x B -> B x A is bijective function.

9. Let f : N → N be defined by

$$f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases} \quad \text{for all } n\in \mathbf{N}$$

State whether the function f is bijective. Justify your answer

Solution:

$$f(n)=\begin{cases}\dfrac{n+1}{2}, & \text{if } n \text{ is odd}\\ \dfrac{n}{2}, & \text{if } n \text{ is even}\end{cases} \quad \text{for all } n\in \mathbf{N}$$

For n = 1, 2

f(1) = (n+1)/2 = (1+1)/2 = 1 and

f(2) = (n)/2 = (2)/2 = 1

f(1) = f(2), but 1 ≠ 2

f is not one-one.

For a natural number, “a” in co-domain N

If n is odd

n = 2k + 1 for k ∈ N , then 4k + 1 ∈ N such that

f(4k+1) = (4k+1+1)/2 = 2k + 1

If n is even

n= 2k for some k ∈ N such that
f(4k) = 4k/2 = 2k
f is onto

Therefore, f is onto but not bijective function.

10. Let A = R – {3} and B = R – {1}. Consider the function f : A → B defined by
f(x) = (x-2)/(x-3)
Is f one-one and onto? Justify your answer.

Solution: A = R – {3} and B = R – {1}

f : A → B defined by f(x) = (x-2)/(x-3)

Let (x, y) ∈ A then

$f(x) = \frac{x\ -2}{x-3}\ and\ f(y) = \frac{y-2}{y-3}$

For f(x) = f(y)

$$\frac{x-2}{x-3} = \frac{y-2}{y-3}$$
$$(x-2)(y-3) = (y-2)(x-3)$$
$$xy - 3x - 2y + 6 = xy - 3y - 2x + 6$$
$$-3x - 2y = -3y - 2x$$
$$-3x + 2x = -3y + 2y$$
$$-x = -y$$
$$x = y$$

Again, f(x) = (x-2)/(x-3)
or y = f(x) = (x-2)/(x-3)
y = (x-2)/(x-3)
y(x – 3) = x – 2
xy – 3y = x – 2
x(y – 1) = 3y – 2

or x = (3y-2)/(y-1)

Now, f((3y-2)/(y-1)) = $\frac{\frac{3y-2}{y-1}-2}{\frac{3y-2}{y-1}-3}$ = y

f(x) = y

Therefore, f is onto function.

11. Let f : R → R be defined as $f(x) = x^4$. Choose the correct answer.
(A) f is one-one onto (B) f is many-one onto
(C) f is one-one but not onto (D) f is neither one-one nor onto.

Solution:
f : R → R be defined as $f(x) = x^4$

let x and y belongs to R such that, f(x) = f(y)

$x^4 = y^4$ or $x = \pm y$

f is not one-one function.

Now, $y = f(x) = x^4$ Or $x = \pm y^{1/4}$

$f(y^{1/4}) = y$ and $f(-y^{1/4}) = -y$

Therefore, f is not onto function.

Option D is correct.

12. Let f : R → R be defined as f(x) = 3x. Choose the correct answer.
(A) f is one-one onto (B) f is many-one onto
(C) f is one-one but not onto (D) f is neither one-one nor onto.

Solution: f : R → R be defined as f(x) = 3x

let x and y belongs to R such that f(x) = f(y)

3x = 3y or x = y

f is one-one function.

Now, y = f(x) = 3x

Or x = y/3

f(x) = f(y/3) = y

Therefore, f is onto function.

Option (A) is correct.