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Prove the following

1.

$3\sin^{-1}x = \sin^{-1}(3x-4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Solution:

$3\sin^{-1}x = \sin^{-1}(3x-4x^3), x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

(Use identity: $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$)

Let $x = \sin\theta$. then

$\theta = \sin^{-1}x$

Now, RHS

$= \sin^{-1}(3x - 4x^3)$

$= \sin^{-1}(3\sin\theta - 4\sin^3\theta)$

$= \sin^{-1}(\sin 3\theta)$

$= 3\theta$

$= 3\sin^{-1}x$

= LHS

Hence Proved

2.

$3\cos^{-1}x = \cos^{-1}(4x^3-3x), x \in \left[\frac{1}{2}, 1\right]$

Solution:

$$3\cos^{-1} x = \cos^{-1}\left(4x^3 - 3x\right), x \in \left[\frac{1}{2}, 1\right]$$

Using identity: $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

Put x = cos θ

θ = $\cos^{-1}$ (x)

Therefore, cos 3 θ = $4x^3 - 3x$

RHS:

$\cos^{-1}\left(4x^3 - 3x\right)$

= $\cos^{-1}$ (cos 3 θ)

= 3 θ

= 3 $\cos^{-1}$ (x)

= LHS

Hence Proved.

3.

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$$

Solution:

$$\tan^{-1}\frac{2}{11} + \tan^{-1}\frac{7}{24} = \tan^{-1}\frac{1}{2}$$

Using identity: $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$

LHS = $\tan^{-1}\frac{2}{11}+\tan^{-1}\frac{7}{24}$

$= \tan^{-1}\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}$

$= \tan^{-1}\frac{48+77}{264-14}$

= tan $^{-1}$ (125/250)

= tan $^{-1}$ (1/2)

= RHS

Hence Proved

4.

$2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}=\tan^{-1}\frac{31}{17}$

Solution:

Use identity: $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$

LHS

= $2\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{7}$

= $\tan^{-1}\frac{2\times\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}+\tan^{-1}\frac{1}{7}$

= $\tan^{-1}(4/3) + \tan^{-1}(1/7)$

Again using identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$$

We have,

$$\tan^{-1} \frac{\frac{4}{3}+\frac{1}{7}}{1-\frac{4}{3}\times\frac{1}{7}}$$

= $tan^{-1}(\frac{28+3}{21-4})$

= $\tan^{-1}(31/17)$

RHS

Write the following functions in the simplest form:

5. $\tan^{-1} \frac{\sqrt{1+x^2}-1}{x}, x \neq 0$

Solution:

Let's say x = tan θ then θ = $\tan^{-1} x$

We get,

$$\tan^{-1} \frac{\sqrt{1+x^2}-1}{x} = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$$

$$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$$= \tan^{-1}\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2}\tan^{-1}x$$

This is simplest form of the function.

6. $\tan^{-1}\frac{1}{\sqrt{x^2-1}}, |x| > 1$

Solution:

Let us consider, x = sec θ, then θ = $\sec^{-1}$ x

$$\tan^{-1}\frac{1}{\sqrt{x^2-1}} = \tan^{-1}\frac{1}{\sqrt{\sec^2\theta - 1}}$$

$$= \tan^{-1}\frac{1}{\sqrt{\tan^2\theta}}$$

$$= \tan^{-1}\left(\frac{1}{\tan\theta}\right)$$

= $\tan^{-1}$(cot θ)

= $\tan^{-1}$ tan(π/2 - θ)

= (π/2 - θ)

= π/2 – $\sec^{-1}$ x

This is simplest form of the given function.

7. $\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right)$, $0 < x < \pi$

Solution:

$$\tan^{-1}\left(\sqrt{\frac{1-\cos x}{1+\cos x}}\right) = \tan^{-1}\left(\sqrt{\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)$$

$$= \tan^{-1}\left(\tan\frac{x}{2}\right) = \frac{x}{2}$$

8. $\tan^{-1}\left(\frac{\cos x - \sin x}{\cos x + \sin x}\right)$, $\frac{-\pi}{4} < x < \frac{3\pi}{4}$

Solution:

Divide numerator and denominator by cos x, we have

$$\boldsymbol{tan^{-1}\left(\frac{\frac{cox(x)}{cos(x)} - \frac{sin(x)}{cos(x)}}{\frac{cos(x)}{cos(x)} + \frac{sin(x)}{cos(x)}}\right)}$$

$$= \boldsymbol{tan^{-1}\left(\frac{1 - \frac{sin(x)}{cos(x)}}{1 + \frac{sin(x)}{cos(x)}}\right)}$$

$$= \tan^{-1}\left(\frac{1-\tan x}{1+\tan x}\right)$$

$$\tan^{-1}\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)$$

= tan $^{-1}$ tan(π/4 – x)

= π/4 – x

9. $\tan^{-1}\frac{x}{\sqrt{a^2-x^2}}, |x| < a$

Solution:

Put x = a sin θ, which implies sin θ = x/a and θ = sin^{-1}(x/a)

Substitute the values into given function, we get

$$\tan^{-1}\frac{x}{\sqrt{a^2-x^2}} = \tan^{-1}\left(\frac{a\sin\theta}{\sqrt{a^2-a^2\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\sqrt{1-\sin^2\theta}}\right)$$

$$= \tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta}\right)$$

= tan^{-1}(tan θ)

= θ

= sin^{-1}(x/a)

10. $\tan^{-1}\left(\frac{3a^2x-x^3}{a^3-3ax^2}\right), a > 0; \frac{-a}{\sqrt{3}} < x < \frac{a}{\sqrt{3}}$

Solution:

After dividing numerator and denominator by a^3 we have

$$\tan^{-1}\left(\frac{3\left(\frac{x}{a}\right)-\left(\frac{x}{a}\right)^3}{1-3\left(\frac{x}{a}\right)^2}\right)$$

Put x/a = tan θ and θ = $\tan^{-1}$(x/a)

$$= \tan^{-1}\left(\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}\right)$$

= $\tan^{-1}$ (tan 3 θ)

= 3 θ

= 3 $\tan^{-1}$(x/a)

Find the values of each of the following:

11. $\tan^{-1}\left[2\cos\left(2\sin^{-1}\frac{1}{2}\right)\right]$

Solution:

$$= \tan^{-1}\left[2\cos\left(2\sin^{-1}\sin\frac{\pi}{6}\right)\right]$$

$$= \tan^{-1}\left[2\cos\left(2\times\frac{\pi}{6}\right)\right]$$

= $\tan^{-1}$ (2 cos π/3)

= $\tan^{-1}$(2 x ½)

= $\tan^{-1}$ (1)

= $\tan^{-1}$ (tan (π/4))

= π/4

12. cot ($\tan^{-1}$a + $\cot^{-1}$a)

Solution:

cot ($\tan^{-1}$a + $\cot^{-1}$a) = cot π/2 = 0

Using identity: $\tan^{-1}$a + $\cot^{-1}$a = π/2

13.

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-y^2}{1+y^2}\right], |x|<1, y>0 \text{ and } xy<1$$

Solution:

Put x = tan θ and y = tan Φ, we have

$$\tan\frac{1}{2}\left[\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}+\cos^{-1}\frac{1-\tan^2\phi}{1+\tan^2\phi}\right]$$

= tan1/2[$\sin^{-1}$ sin 2 θ + $\cos^{-1}$ cos 2 Φ]

= tan (1/2) [2 θ + 2 Φ]

= tan (θ + Φ)

$$= \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$$

= (x+y) / (1-xy)

14. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, **then find the value of x.**

Solution:

We know that, sin 90 degrees = sin π/2 = 1

So, given equation turned as,

$$\sin^{-1}\frac{1}{5}+\cos^{-1}x=\frac{\pi}{2}$$

$$\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}\frac{1}{5}$$

Using identity: $\sin^{-1} t + \cos^{-1} t$ = π/2

We have, $\cos^{-1} x = \cos^{-1}\frac{1}{5}$

Which implies, the value of x is 1/5.

15. If $\tan^{-1}\frac{x-1}{x-2} + \tan^{-1}\frac{x+1}{x+2} = \frac{\pi}{4}$, then find the value of x.

Solution:

We have reduced the given equation using below identity:

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\frac{x+y}{1-xy}$$

$$\tan^{-1}\frac{\frac{x-1}{x-2}+\frac{x+1}{x+2}}{1-\left(\frac{x-1}{x-2}\right)\left(\frac{x+1}{x+2}\right)} = \frac{\pi}{4}$$

or

$$\tan^{-1}\frac{(x-1)(x+2)+(x+1)(x-2)}{(x-2)(x+2)-(x-1)(x+1)} = \frac{\pi}{4}$$

or

$$\tan^{-1}\frac{x^2+2x-x-2+x^2-2x+x-2}{x^2-4-(x^2-1)} = \frac{\pi}{4}$$

$$or\ \frac{2x^2-4}{x^2-4-x^2+1} = \tan\left(\frac{\pi}{4}\right)$$

or (2x^2 - 4)/-3 = 1

or 2x^2 = 1

or x = $\pm\frac{1}{\sqrt{2}}$

The value of x is either $\frac{1}{\sqrt{2}}\ or -\frac{1}{\sqrt{2}}$

Find the values of each of the expressions in Exercises 16 to 18.

16. $\sin^{-1}(\sin\left(\frac{2\pi}{3}\right))$

Solution:

Given expression is $\sin^{-1}(\sin\left(\frac{2\pi}{3}\right))$

First split $\frac{2\pi}{3}$ as $\frac{(3\pi-\pi)}{3}$ $or\ \pi-\frac{\pi}{3}$

After substituting in given we get,

$$\sin^{-1}(\sin\left(\frac{2\pi}{3}\right)) = \sin^{-1}(\sin(\pi-\frac{\pi}{3})) = \frac{\pi}{3}$$

Therefore, the value of $\sin^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is $\frac{\pi}{3}$

17. $tan^{-1}(\tan\left(\frac{3\pi}{4}\right))$

Solution:

Given expression is $tan^{-1}(\tan\left(\frac{3\pi}{4}\right))$

First split $\frac{3\pi}{4}$ as $\frac{(4\pi-\pi)}{4}$ $or\ \pi-\frac{\pi}{4}$

After substituting in given we get,

$$tan^{-1}(\tan\left(\frac{3\pi}{4}\right)) = \tan^{-1}(\tan(\pi-\frac{\pi}{4})) = -\frac{\pi}{4}$$

The value of $tan^{-1}(\tan\left(\frac{3\pi}{4}\right))$ is $\frac{-\pi}{4}$.

18. $\tan(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2})$

Solution:

Given expression is $\tan(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2})$

Putting, $sin^{-1}\left(\frac{3}{5}\right) = x\ and\ \cot^{-1}(\frac{3}{2}) = y$

Or sin(x) = 3/5 and cot y = 3/2

Now, sin(x) = 3/5 => cos x = $\sqrt{1-\sin^2 x}$ = 4/5 and sec x = 5/4

(using identities: cos x = $\sqrt{1-\sin^2 x}$ and sec x = 1/cos x)

Again, tan x = $\sqrt{\sec^2 x - 1}$ = $\sqrt{\frac{25}{16} - 1}$ = ¾ and tan y = 1/cot(y) = 2/3

Now, we can write given expression as,

$\tan(sin^{-1}\left(\frac{3}{5}\right) + \cot^{-1}\frac{3}{2})$ = tan(x + y)

$= \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\times\frac{2}{3}}$

= 17/6

19. $\cos^{-1}(\cos\frac{7\pi}{6})$ is equal to

(A) 7π/6 **(B) 5 π/6** **(C) π/3** **(D) π/6**

Solution:

Option (B) is correct.

Explanation:

$\cos^{-1}(\cos\frac{7\pi}{6}) = \cos^{-1}(\cos(2\pi - \frac{7\pi}{6})$

(As cos (2 π − A) = cos A)

Now $2\pi - \frac{7\pi}{6} = \frac{12\pi - 7\pi}{6} = \frac{5\pi}{6}$

20. $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ **is equal to**

(A) ½ (B) 1/3 (C) ¼ (D) 1

Solution:

Option (D) is correct

Explanation:

First solve for: $\sin^{-1}\left(-\frac{1}{2}\right)$

$$\sin^{-1}\left(-\frac{1}{2}\right)=\sin^{-1}\left(-\sin\frac{\pi}{6}\right)=\sin^{-1}\left[\sin\left(-\frac{\pi}{6}\right)\right]$$

$= -\pi/6$

Again,

$$\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right]$$

$$=\sin\left[\frac{\pi}{3}+\frac{\pi}{6}\right]$$

$= \sin(\pi/2)$

$= 1$

21. $\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ is equal to

(A) π (B) -π/2 (C) 0 (D) $2\sqrt{3}$

Solution:

Option (B) is correct.

Explanation:

$\tan^{-1}\sqrt{3} - \cot^{-1}(-\sqrt{3})$ can be written as

$$= \tan^{-1}\tan\frac{\pi}{3} - \cot^{-1}\left(-\cot\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3} - \cot^{-1}\left[\cot\left(\pi - \frac{\pi}{6}\right)\right]$$

$$= \frac{\pi}{3} - (\pi - \frac{\pi}{6})$$

$$= \frac{\pi}{3} - \frac{5\pi}{6}$$

$$= \frac{-3\pi}{6}$$

$= -\pi/2$