

Miscellaneous Exercise

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Differentiate with respect to x the functions in Exercises 1 to 11.

1. $(3x^2 - 9x + 5)^9$

Solution: Consider $y = (3x^2 - 9x + 5)^9$

$$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 \frac{d}{dx}(3x^2 - 9x + 5)$$

$$\left[\because \frac{d}{dx} \{f(x)\}^n = n \{f(x)\}^{n-1} \frac{d}{dx} f(x) \right]$$

$$\frac{dy}{dx} = 9(3x^2 - 9x + 5)^8 [3(2x) - 9(1) + 0]$$

$$\frac{dy}{dx} = 27(3x^2 - 9x + 5)^8 [2x - 3]$$

2. $\sin^3 x + \cos^6 x$

Solution: Consider $y = \sin^3 x + \cos^6 x$

or $y = (\sin x)^3 + (\cos x)^6$

$$\frac{dy}{dx} = 3(\sin x)^2 \frac{d}{dx} \sin x + 6(\cos x)^5 \frac{d}{dx} \cos x$$

$$\frac{dy}{dx} = 3 \sin^2 x \cos x - 6 \cos^5 x \sin x$$

$$= 3 \sin x \cos x (\sin x - 2 \cos^4 x)$$

3. $(5x)^{3 \cos 2x}$

Solution: Consider $y = (5x)^{3 \cos 2x}$

Taking log both the sides, we get

$$\log y = \log (5x)^{3 \cos 2x}$$

$$\log y = 3 \cos 2x \log(5x)$$

Derivate above function:

$$\frac{d}{dx} \log y = 3 \frac{d}{dx} \{ \cos 2x \log(5x) \}$$

$$\frac{1}{y} \frac{dy}{dx} = 3 \left[\cos 2x \frac{d}{dx} \log(5x) + \log(5x) \frac{d}{dx} \cos 2x \right]$$

$$\frac{1}{y} \frac{dy}{dx} = 3 \left[\cos 2x \frac{1}{5x} \frac{d}{dx} 5x + \log(5x) (-\sin 2x) \frac{d}{dx} 2x \right]$$

$$\frac{1}{y} \frac{dy}{dx} = 3 \left[\cos 2x \frac{1}{5x} \cdot 5 - 2 \sin 2x \log(5x) \right]$$

$$\frac{dy}{dx} = 3y \left[\frac{\cos 2x}{x} - 2 \sin 2x \log(5x) \right]$$

$$\frac{dy}{dx} = 3(5x)^{3 \cos 2x} \left[\frac{\cos 2x}{x} - 2 \sin 2x \log(5x) \right] \text{ (using value of } y \text{)}$$

4. $\sin^{-1}(x\sqrt{x}), 0 \leq x \leq 1$

Solution: Consider $y = \sin^{-1}(x\sqrt{x})$

or $y = \sin^{-1}\left(x^{\frac{3}{2}}\right)$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 - \left(x^{\frac{3}{2}}\right)^2}} \frac{d}{dx} x^{\frac{3}{2}}$$

$$= \frac{1}{\sqrt{1 - x^3}} \cdot \frac{3}{2} x^{\frac{1}{2}}$$

$$= \frac{3}{2} \sqrt{\frac{x}{1-x^3}}$$

$$5. \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}, -2 < x < 2$$

Solution: Consider $y = \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}$

Apply derivation:

$$\frac{dy}{dx} = \frac{\sqrt{2x+7} \frac{d}{dx} \cos^{-1} \frac{x}{2} - \cos^{-1} \frac{x}{2} \frac{d}{dx} \sqrt{2x+7}}{(\sqrt{2x+7})^2}$$

[Using Quotient Rule]

$$\frac{dy}{dx} = \frac{\sqrt{2x+7} \left(\frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \right) \frac{d}{dx} \frac{x}{2} - \left(\cos^{-1} \frac{x}{2} \right) \frac{1}{2} (2x+7)^{-\frac{1}{2}} \frac{d}{dx} (2x+7)}{(\sqrt{2x+7})^2}$$

$$\frac{dy}{dx} = \frac{-\sqrt{2x+7} \cdot \frac{2}{\sqrt{4-x^2}} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}}}{(2x+7)}$$

$$= \frac{-\left[\frac{\sqrt{2x+7}}{\sqrt{4-x^2}} + \frac{\cos^{-1} \frac{x}{2}}{\sqrt{2x+7}} \right]}{(2x+7)}$$

$$\frac{dy}{dx} = - \left[\frac{2x+7 + \sqrt{4-x^2} \cos^{-1} \frac{x}{2}}{\sqrt{4-x^2} \sqrt{2x+7} (2x+7)} \right]$$

$$= - \left[\frac{2x+7 + \sqrt{4-x^2} \cos^{-1} \frac{x}{2}}{\sqrt{4-x^2} (2x+7)^{\frac{3}{2}}} \right]$$

6. $\cot^{-1} \left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right], 0 < x < \frac{\pi}{2}$

$$y = \cot^{-1} \left(\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \right), 0 < x < \frac{\pi}{2} \quad \dots\dots\dots(i)$$

Solution: Consider

Reduce the functions into simplest form,

$$\sqrt{1+\sin x} = \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} + 2 \sin \frac{x}{2} \cos \frac{x}{2}}$$

$$= \sqrt{\left(\cos \frac{x}{2} + \sin \frac{x}{2} \right)^2} = \cos \frac{x}{2} + \sin \frac{x}{2}$$

And $\sqrt{1-\sin x} = \sqrt{\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} - 2 \sin \frac{x}{2} \cos \frac{x}{2}}$

$$= \sqrt{\left(\cos \frac{x}{2} - \sin \frac{x}{2} \right)^2} = \cos \frac{x}{2} - \sin \frac{x}{2}$$

Now, we are available with the equation below:

$$y = \cot^{-1} \left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2} + \cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2} - \cos \frac{x}{2} + \sin \frac{x}{2}} \right)$$

$$= \cot^{-1} \left(\frac{2 \cos \frac{x}{2}}{2 \sin \frac{x}{2}} \right)$$

$$y = \cot^{-1} \left(\cot \frac{x}{2} \right)$$

$$= \frac{x}{2}$$

Apply derivation:

$$\frac{dy}{dx} = \frac{1}{2} (1) = \frac{1}{2}$$

7. $(\log x)^{\log x}, x > 1$

Solution: Consider $y = (\log x)^{\log x}, x > 1$ (i)

Taking log both sides:

$$\log y = \log (\log x)^{\log x} = \log x \log (\log x)$$

Apply derivation:

$$\frac{d}{dx} (\log y) = \frac{d}{dx} (\log x \log (\log x))$$

$$\frac{1}{y} \frac{dy}{dx} = \log x \frac{d}{dx} \log (\log x) + \log (\log x) \frac{d}{dx} \log x$$

$$\frac{1}{y} \frac{dy}{dx} = \log x \frac{1}{\log x} \frac{d}{dx} (\log x) + \log (\log x) \frac{1}{x}$$

$$= \frac{1}{x} + \frac{\log (\log x)}{x}$$

$$\frac{dy}{dx} = y \left(\frac{1 + \log (\log x)}{x} \right)$$

$$= (\log x)^{\log x} \left(\frac{1 + \log(\log x)}{x} \right)$$

8. $\cos(a \cos x + b \sin x)$ for some constants a and b.

Solution: Consider $y = \cos(a \cos x + b \sin x)$ for some constants a and b.
Apply derivation:

$$\frac{dy}{dx} = -\sin(a \cos x + b \sin x) \frac{d}{dx}(a \cos x + b \sin x)$$

$$\frac{dy}{dx} = -\sin(a \cos x + b \sin x)(-a \sin x + b \cos x)$$

$$\frac{dy}{dx} = -(-a \sin x + b \cos x) \sin(a \cos x + b \sin x)$$

$$\frac{dy}{dx} = (a \sin x - b \cos x) \sin(a \cos x + b \sin x)$$

9. $(\sin x - \cos x)^{\sin x - \cos x}$, $\frac{\pi}{4} < x < \frac{3\pi}{4}$

Solution: Consider $y = (\sin x - \cos x)^{\sin x - \cos x}$ (i)

Apply log both sides:

$$\log y = \log (\sin x - \cos x)^{\sin x - \cos x}$$

$$= (\sin x - \cos x) \log (\sin x - \cos x)$$

Apply derivation:

$$\frac{d}{dx} \log y = (\sin x - \cos x) \frac{d}{dx} (\sin x - \cos x) + \log (\sin x - \cos x) \frac{d}{dx} (\sin x - \cos x)$$

$$\frac{1}{y} \frac{dy}{dx} = (\sin x - \cos x) \frac{1}{(\sin x - \cos x)} \frac{d}{dx} (\sin x - \cos x) + \log (\sin x - \cos x) \cdot (\cos x + \sin x)$$

$$\frac{1}{y} \frac{dy}{dx} = (\cos x + \sin x) + (\cos x + \sin x) \log (\sin x - \cos x)$$

$$\frac{1}{y} \frac{dy}{dx} = (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

$$\frac{dy}{dx} = y (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

$$\frac{dy}{dx} = (\sin x - \cos x)^{\sin x - \cos x} (\cos x + \sin x) [1 + \log (\sin x - \cos x)]$$

10. $x^x + x^a + a^x + a^a$, for some fixed $a > 0$ and $x > 0$.

Solution: Consider $y = x^x + x^a + a^x + a^a$

Apply derivation:

$$\frac{dy}{dx} = \frac{d}{dx} x^x + \frac{d}{dx} x^a + \frac{d}{dx} a^x + \frac{d}{dx} a^a$$

$$= \frac{d}{dx} x^x + ax^{a-1} + a^x \log a + 0 \quad \dots\dots(i)$$

First term from equation (i) :

$$\frac{d}{dx} (x^x), \text{ Consider } u = x^x$$

$$\log u = \log x^x = x \log x$$

$$\frac{d}{dx} \log u = \frac{d}{dx} (x \log x)$$

$$\frac{1}{u} \frac{du}{dx} = x \frac{d}{dx} \log x + \log x \frac{d}{dx} x$$

$$\frac{1}{u} \frac{du}{dx} = x \frac{1}{x} + \log x \cdot 1$$

$$= 1 + \log x$$

This implies,

$$\frac{du}{dx} = u(1 + \log x)$$

Substitute value of u back:

$$\frac{d}{dx} x^x = x^x (1 + \log x) \quad \dots(ii)$$

Using equation (ii) in (i), we have

$$\frac{dy}{dx} = x^x (1 + \log x) ax^{a-1} + a^x \log a$$

11. $x^{x^2-3} + (x-3)^{x^2}$ for $x > 3$.

Solution: Consider $y = x^{x^2-3} + (x-3)^{x^2}$ for $x > 3$.

Put $u = x^{x^2-3}$ and $v = (x-3)^{x^2}$

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(i)$$

Now $u = x^{x^2-3}$

$$\log u = \log x^{x^2-3} = (x^2-3) \log x$$

$$\begin{aligned} \frac{1}{u} \frac{du}{dx} &= (x^2-3) \frac{d}{dx} \log x + \log x \frac{d}{dx} (x^2-3) \\ &= (x^2-3) \frac{1}{x} + \log x (2x-0) \end{aligned}$$

$$\frac{1}{u} \frac{du}{dx} = \frac{x^2-3}{x} + 2x \log x$$

$$\frac{du}{dx} = u \left(\frac{x^2-3}{x} + 2x \log x \right)$$

$$\frac{du}{dx} = x^{x^2-3} \left(\frac{x^2-3}{x} + 2x \log x \right) \quad \dots(ii)$$

Again $v = (x-3)^{x^2}$

$$\log v = \log (x-3)^{x^2}$$

$$= x^2 \log (x-3)$$

$$\frac{1}{v} \frac{dv}{dx} = x^2 \frac{d}{dx} \log (x-3) + \log (x-3) \frac{d}{dx} x^2$$

$$= x^2 \frac{1}{x-3} \frac{d}{dx} (x-3) + \log (x-3) 2x$$

$$\frac{1}{v} \frac{dv}{dx} = \frac{x^2}{x-3} + 2x \log (x-3)$$

$$\frac{dv}{dx} = v \left[\frac{x^2}{x-3} + 2x \log (x-3) \right]$$

$$\frac{dv}{dx} = (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log (x-3) \right] \dots\dots\dots (iii)$$

Using equation (ii) and (iii) in eq. (i), we have

$$\frac{dy}{dx} = x^{x^2-3} \left(\frac{x^2-3}{x} + 2x \log x \right) + (x-3)^{x^2} \left[\frac{x^2}{x-3} + 2x \log (x-3) \right]$$

12. Find $\frac{dy}{dx}$ if $y = 12(1 - \cos t)$ and $x = 10(t - \sin t), -\frac{\pi}{2} < t < \frac{\pi}{2}$.

Solution: Given expressions are $y = 12(1 - \cos t)$ and $x = 10(t - \sin t)$

$$\frac{dy}{dt} = 12 \frac{d}{dt} (1 - \cos t) = 12(0 + \sin t) = 12 \sin t$$

and $\frac{dx}{dt} = 10 \frac{d}{dt} (t - \sin t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{12 \sin t}{10(1 - \cos t)}$$

$$= \frac{6}{5} \cdot \frac{2 \sin \frac{t}{2} \cos \frac{t}{2}}{2 \sin^2 \frac{t}{2}}$$

$$= \frac{6}{5} \cdot \frac{\cos \frac{t}{2}}{\sin \frac{t}{2}} = \frac{6}{5} \cot \frac{t}{2}$$

13. Find $\frac{dy}{dx}$ if $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$, $-1 \leq x \leq 1$.

Solution: Given expression is $y = \sin^{-1} x + \sin^{-1} \sqrt{1-x^2}$
Apply derivation:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{d}{dx} \sqrt{1-x^2}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{1}{2} (1-x^2)^{-\frac{1}{2}} \cdot \frac{d}{dx} (1-x^2)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-1+x^2}} \cdot \frac{1}{2\sqrt{1-x^2}} (-2x)$$

$$= \frac{1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{x^2}} \left(\frac{-x}{\sqrt{1-x^2}} \right)$$

Which implies:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}}$$

$$= \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0$$

Therefore, $dy/dx = 0$

14.

If $x\sqrt{1+y} + y\sqrt{1+x} = 0$, for $-1 < x < 1$,

Prove that $\frac{dy}{dx} = \frac{-1}{(1+x)^2}$.

Solution: Given expression is $x\sqrt{1+y} + y\sqrt{1+x} = 0$
 $x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides:

$$x^2(1+y) = y^2(1+x)$$

$$x^2 + x^2y = y^2 + y^2x$$

$$x^2 - y^2 = -x^2y + y^2x$$

$$(x-y)(x+y) = -xy(x-y)$$

$$x+y = -xy$$

$$\Rightarrow y(1+x) = -x$$

$$\Rightarrow y = \frac{-x}{1+x}$$

Apply derivation:

$$\frac{dy}{dx} = -\frac{(1+x)\frac{d}{dx}x - x\frac{d}{dx}(1+x)}{(1+x)^2}$$

$$= -\frac{(1+x).1 - x.1}{(1+x)^2}$$

$$= -\frac{1}{(1+x)^2}$$

Hence Proved.

15.

If $(x-a)^2 + (y-b)^2 = c^2$, for some $c > 0$, prove that

$$\frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}$$

is a constant independent of a and b .

Solution: Given expression is $(x-a)^2 + (y-b)^2 = c^2$ (1)

Apply derivation:

$$2(x-a) + 2(y-b) \frac{dy}{dx} = 0$$

$$2(x-a) = -2(y-b) \frac{dy}{dx}$$

$$\frac{dy}{dx} = -\left(\frac{x-a}{y-b}\right) \text{(2)}$$

Again
$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b) \cdot 1 - (x-a) \frac{dy}{dx}\right]}{(y-b)^2}$$

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b) \cdot 1 - (x-a) \left(\frac{-(x-a)}{y-b}\right)\right]}{(y-b)^2}$$

[Using equation (2)]

$$\frac{d^2y}{dx^2} = \frac{-\left[(y-b) + \frac{(x-a)^2}{y-b}\right]}{(y-b)^2}$$

$$= \frac{-\left[(y-b)^2 + (x-a)^2\right]}{(y-b)^3}$$

$$= \frac{-c^2}{(y-b)^3} \text{(3)}$$

Put values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given, we get

$$\begin{aligned} & \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} \\ &= \frac{\left[1 + \frac{(x-a)^2}{(y-b)^2}\right]^{\frac{3}{2}}}{\frac{-c^2}{(y-b)^3}} \\ &= \frac{\left[(y-b)^2 + (x-a)^2\right]^{\frac{3}{2}}}{(y-b)^3} \times \frac{(y-b)^3}{-c^2} = \frac{(c^2)^{\frac{3}{2}}}{-c^2} = -c \text{ (Constant value)} \end{aligned}$$

Which is a constant and is independent of a and b.

16. If $\cos y = x \cos(a+y)$ **with** $\cos a \neq \pm 1$, **prove that** $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

Solution: Given expression is $\cos y = x \cos(a+y)$

$$x = \frac{\cos y}{\cos(a+y)}$$

Apply derivative w.r.t. y

$$\frac{dx}{dy} = \frac{d}{dy} \left(\frac{\cos y}{\cos(a+y)} \right)$$

$$\frac{dx}{dy} = \frac{\cos(a+y) \frac{d}{dy} \cos y - \cos y \frac{d}{dy} \cos(a+y)}{\cos^2(a+y)}$$

$$\frac{dx}{dy} = \frac{\cos(a+y)(-\sin y) - \cos y \{-\sin(a+y)\}}{\cos^2(a+y)}$$

$$= \frac{-\cos(a+y)\sin y + \sin(a+y)\cos y}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin(a+y-y)}{\cos^2(a+y)}$$

$$= \frac{\sin a}{\cos^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a} \quad [\text{Take reciprocal}]$$

17. If $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$, find $\frac{d^2y}{dx^2}$.

Solution: Given expressions are $x = a(\cos t + t \sin t)$ and $y = a(\sin t - t \cos t)$
 $x = a(\cos t + t \sin t)$

Differentiating both sides w.r.t. t

$$\frac{dx}{dt} = a \left(-\sin t + \frac{d}{dt} t \sin t \right)$$

$$\frac{dx}{dt} = a \left(-\sin t + t \frac{d}{dt} \sin t + \sin t \frac{d}{dt} t \right)$$

$$\frac{dx}{dt} = a(-\sin t + t \cos t + \sin t)$$

$$\Rightarrow \frac{dx}{dt} = at \cos t$$

And:

$$y = a(\sin t - t \cos t)$$

Differentiating both sides w.r.t. t

$$\frac{dy}{dt} = a \left(\cos t - \frac{d}{dt} t \cos t \right)$$

$$\frac{dy}{dt} = a \left(\cos t - \left(t \frac{d}{dt} \cos t + \cos t \frac{d}{dt} t \right) \right)$$

$$\frac{dy}{dt} = a(\cos t - (-t \sin t + \cos t))$$

$$\frac{dy}{dt} = at \sin t$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{at \sin t}{at \cos t} = \frac{\sin t}{\cos t} = \tan t$$

Now

$$\text{Again } \frac{d^2y}{dx^2} = \frac{d}{dx} \tan t = \sec^2 t \frac{d}{dx} t$$

$$= \sec^2 t \frac{dt}{dx} = \sec^2 t \frac{1}{at \cos t}$$

$$= \sec^2 t \frac{\sec t}{at} = \frac{\sec^3 t}{at}$$

18. If $f(x) = |x|^3$, show that $f''(x)$ exists for all real x and find it.

$$\text{Solution: Given expression is } f(x) = |x|^3 = \begin{cases} x^3, & \text{if } x \geq 0 \\ (-x^3), & \text{if } x < 0 \end{cases}$$

Step 1: when $x < 0$

$$f(x) = -x^3$$

Differentiate w.r.t. to x ,

$$f'(x) = -3x^2$$

Differentiate w.r.t. to x ,

$f'(x) = -6x$, exist for all values of $x < 0$.

Step 2: When $x \geq 0$

$$f(x) = x^3$$

Differentiate w.r.t. to x ,

$$f'(x) = 3x^2$$

Differentiate w.r.t. to x ,

$f'(x) = 6x$, exist for all values of $x > 0$.

Step 3: When $x = 0$

$$\lim_{h \rightarrow 0^-} \frac{f(0) - f(0+h)}{h} = \lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = f'(c)$$

$$f'(x) = \begin{cases} 3x^2, & \text{if } x \geq 0 \\ -3x^2, & \text{if } x < 0 \end{cases}$$

Now, Check differentiability at $x = 0$

L.H.D. at $x = 0$

$$\begin{aligned} \lim_{h \rightarrow 0^-} \frac{f'(0) - f'(0+h)}{h} \\ = \lim_{h \rightarrow 0^-} \frac{3(0) - (-3(-h)^2)}{h} \\ = \lim_{h \rightarrow 0^-} \frac{3h^2}{h} \end{aligned}$$

As $h = 0$,

$$= 0$$

And R.H.D. at $x = 0$

$$\begin{aligned} \lim_{h \rightarrow 0^+} \frac{f'(0+h) - f'(0)}{h} \\ = \lim_{h \rightarrow 0^+} \frac{f'(h) - f'(0)}{h} \end{aligned}$$

$$= \lim_{h \rightarrow 0^+} \frac{3(h)^2 - 3(0)^2}{h}$$

$$= \lim_{h \rightarrow 0^+} 3h = 0 \text{ (at } h = 0)$$

Again L.H.D. at $x = 0$ = R.H.D. at $x = 0$.

This implies, $f'(x)$ exists and differentiable at all real values of x .

19. Using mathematical induction, prove that $\frac{d}{dx}(x^n) = nx^{n-1}$ for all positive integers n .
Solution: Consider $p(n)$ be the given statement.

$$p(n) = \frac{d}{dx}(x^n) = nx^{n-1} \dots\dots(1)$$

Step 1: Result is true at $n = 1$

$$p(1) = \frac{d}{dx}(x^1) = (1)x^{1-1} = (1)x^0 = 1,$$

which is true as $\frac{d}{dx}(x) = 1$

Step 2: Suppose $p(m)$ is true.

$$p(m) = \frac{d}{dx}(x^m) = mx^{m-1} \dots\dots(2)$$

Step 3: Prove that result is true for $n = m+1$.

$$p(m+1) = \frac{d}{dx}(x^{m+1}) = (m+1)x^{m+1-1}$$

$$x^{m+1} = x^1 + x^m$$

$$\frac{d}{dx}x^{m+1} = \frac{d}{dx}(x.x^m)$$

$$= x \cdot \frac{d}{dx}x^m + x^m \frac{d}{dx}x$$

$$= x.mx^{m-1} + x^m(1)$$

Therefore, $mx^m + x^m = x^m(m+1)$

$$(m+1)x^m = (m+1)x^m$$

$$(m+1)x^{(m+1)-1}$$

Therefore, $p(m+1)$ is true if $p(m)$ is true but $p(1)$ is true.

Thus, by Principal of Induction $p(n)$ is true for all $n \in \mathbb{N}$.

20. Using the fact that $\sin(A+B) = \sin A \cos B + \cos A \sin B$ and the differentiation, obtain the sum formula for cosines.

Solution: Given expression is $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Consider A and B as function of t and differentiating both sides w.r.t. x,

$$\cos(A+B) \left(\frac{dA}{dt} + \frac{dB}{dt} \right) = \sin A (-\sin B) \frac{dB}{dt} + \cos B \left(\cos A \frac{dA}{dt} \right) + \cos A \cos B \frac{dB}{dt} + \sin B (-\sin A) \frac{dA}{dt}$$

$$\Rightarrow \cos(A+B) \left(\frac{dA}{dt} + \frac{dB}{dt} \right) = (\cos A \cos B - \sin A \sin B) \left(\frac{dA}{dt} + \frac{dB}{dt} \right)$$

$$\Rightarrow \cos(A+B) = (\cos A \cos B - \sin A \sin B)$$

21. Does there exist a function which is continuous everywhere but not differentiable at exactly two points?

Solution: Consider us consider the function $f(x) = |x| + |x-1|$

f is continuous everywhere but it is not differentiable at $x = 0$ and $x = 1$.

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix},$$

22. If

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}.$$

prove that

Solution: Given expression is

$$y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

Apply derivative:

$$\frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx} f(x) & \frac{d}{dx} g(x) & \frac{d}{dx} h(x) \\ l & m & n \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \\ a & b & c \end{vmatrix} + y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$$

23. If $y = e^{a \cos^{-1} x}$, $-1 \leq x \leq 1$, **show that** $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$.

Solution: Given expression is $y = e^{a \cos^{-1} x}$

$$\frac{dy}{dx} = e^{a \cos^{-1} x} \cdot \frac{d}{dx} a \cos^{-1} x$$

$$= e^{a \cos^{-1} x} \cdot a \left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$= \frac{-ay}{\sqrt{1-x^2}}$$

This implies,

$$\left(\frac{dy}{dx} \right)^2 = \frac{a^2 y^2}{1-x^2}$$

$$(1-x^2) \left(\frac{dy}{dx} \right)^2 = a^2 y^2$$

Differentiating both sides with respect to x , we have

$$(1-x^2) 2 \cdot \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 (-2x) = 2a^2 y \frac{dy}{dx}$$

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} = a^2 y$$

$$(1-x^2) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} - a^2 y = 0$$

Hence Proved.

