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Find the second order derivatives of the functions given in Exercises 1 to 10.

1. x^2+3x+2

Solution: Let $y=x^2+3x+2$

First derivative:

$$\frac{dy}{dx}=2x+3\times1+0=2x+3$$

Second derivative:

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=2\times1+0=2$$

2. x^{20}

Solution: Let $y=x^{20}$

Derivate y with respect to x, we get

$$\frac{dy}{dx}=20x^{19}$$

Derivate dy/dx with respect to x, we get

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=20\times19x^{18}=380x^{18}$$

3. $x\cos x$

Solution: Let $y=x\cos x$

$$\frac{dy}{dx}=x\frac{d}{dx}\cos x+\cos x\frac{d}{dx}x$$

$= -x\sin x+\cos x$

Now,

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=-\frac{d}{dx}(x\sin x)+\frac{d}{dx}\cos x$$

$= -\left[x\frac{d}{dx}\sin x+\sin x\frac{d}{dx}x\right]-\sin x$

$= -(x\cos x+\sin x)-\sin x$

$= -x\cos x-\sin x-\sin x$

$= -x\cos x-2\sin x$

$= -(x\cos x+2\sin x)$

4. $\log x$

Solution: Let $y=\log x$

$\frac{dy}{dx}=\frac{1}{x}$

$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}x^{-1}$

$\frac{d^2y}{dx^2}=(-1)x^{-2}$ $=\frac{-1}{x^2}$

5. $x^3\log x$

Solution: Let $y=x^3\log x$

$\frac{dy}{dx}=x^3\frac{d}{dx}\log x+\log x\frac{d}{dx}x^3$

$= x^3.\frac{1}{x}+\log x(3x^2)$

$= x^2+3x^2\log x$

Now,

$\frac{d^2y}{dx^2}=\frac{d}{dx}(x^2+3x^2\log x)$

$= \frac{d}{dx}x^2 + 3\frac{d}{dx}\left(x^2 \log x\right)$

$= 2x + 3\left[x^2 \frac{d}{dx}\log x + \log x \frac{d}{dx}x^2\right]$

$= 2x + 3\left(x^2 . \frac{1}{x} + (\log x)2x\right)$

$= 2x + 3(x + 2x \log x)$

$= 2x + 3x + 6x \log x$

$= 5x + 6x \log x$

$= x(5 + 6\log x)$

6. $e^x \sin 5x$

Solution: Let $y = e^x \sin 5x$

$\frac{dy}{dx} = e^x \frac{d}{dx}\sin 5x + \sin 5x \frac{d}{dx}e^x$

$= e^x \cos 5x \frac{d}{dx}5x + \sin 5x e^x$

$= e^x \cos 5x \times 5 + e^x \sin 5x$

$= e^x (5\cos 5x + \sin 5x)$

Now,

$\frac{d^2y}{dx^2} = e^x \frac{d}{dx}(5\cos 5x + \sin 5x) + (5\cos 5x + \sin 5x)\frac{d}{dx}e^x$

$= e^x \{5(-\sin x) \times 5 + (\cos 5x) \times 5\} + (5\cos 5x + \sin 5x)e^x$

$= e^x (-25\sin 5x + 5\cos 5x + 5\cos 5x + \sin 5x)$

$= e^x(10\cos 5x - 24\sin 5x)$

$= 2e^x(5\cos 5x - 12\sin 5x)$

7. $e^{6x}\cos 3x$

Solution: Let $y = e^{6x}\cos 3x$

$$\frac{dy}{dx} = e^{6x}\frac{d}{dx}\cos 3x + \cos 3x\frac{d}{dx}e^{6x}$$

$$= e^{6x}(-\sin 3x)\frac{d}{dx}(3x) + \cos 3x.e^{6x}\frac{d}{dx}(6x)$$

$$= -e^{6x}\sin 3x \times 3 + \cos 3x.e^{6x} \times 6$$

$$= e^{6x}(-3\sin 3x + 6\cos 6x)$$

Now,

$$\frac{d^2y}{dx^2} = e^{6x}\frac{d}{dx}(-3\sin 3x + 6\cos 3x) + (-3\sin 3x + 6\cos 3x)\frac{d}{dx}e^{6x}$$

$$= e^{6x}(-3\cos 3x \times 3 - 6\sin 3x \times 3) + (-3\sin 3x + 6\cos 3x)e^{6x} \times 6$$

$$= e^{6x}(-9\cos 3x - 18\sin 3x - 18\sin 3x + 36\cos 3x)$$

$$= 9e^{6x}(3\cos 3x - 4\sin 3x)$$

8. $\tan^{-1}x$

Solution: Let $y = \tan^{-1}x$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{1+x^2}\right)$$

$= \dfrac{(1+x^2)\dfrac{d}{dx}(1) - 1\dfrac{d}{dx}(1+x^2)}{(1+x^2)^2}$

$= \dfrac{(1+x^2)\times 0 - 2x}{(1+x^2)^2}$

$= \dfrac{-2x}{(1+x^2)^2}$

9. $\log(\log x)$

Solution: Let $y = \log(\log x)$

$\dfrac{dy}{dx} = \dfrac{1}{\log x}\dfrac{d}{dx}\log x$

$\left[\because \dfrac{d}{dx}\log f(x) = \dfrac{1}{f(x)}\dfrac{d}{dx}f(x)\right]$

$= \dfrac{1}{\log x}.\dfrac{1}{x} = \dfrac{1}{x\log x}$

$\Rightarrow \dfrac{d^2y}{dx^2} = \dfrac{(x\log x)\dfrac{d}{dx}(1) - 1\dfrac{d}{dx}(x\log x)}{(x\log x)^2}$

$= \dfrac{(x\log x)(0) - \left[x\dfrac{d}{dx}\log x + \log x\dfrac{d}{dx}x\right]}{(x\log x)^2}$

$= \dfrac{\left[x\dfrac{1}{x} + \log x \times 1\right]}{(x\log x)^2}$

$= \dfrac{[1+\log x]}{(x\log x)^2}$

10. $\sin(\log x)$

Solution: Let $y = \sin(\log x)$

$$\frac{dy}{dx} = \cos(\log x)\frac{d}{dx}(\log x)$$

$$= \cos(\log x).\frac{1}{x}$$

$$= \frac{\cos(\log x)}{x}$$

Now,

$$\frac{d^2y}{dx^2} = \frac{x\frac{d}{dx}\cos(\log x) - \cos(\log x)\frac{d}{dx}x}{x^2}$$

$$= \frac{x[-\sin(\log x)]\frac{d}{dx}(\log x) - \cos(\log x)\times 1}{x^2}$$

$$= \frac{-x\sin(\log x)\frac{1}{x} - \cos(\log x)}{x^2}$$

$$= \frac{-[\sin(\log x) + \cos(\log x)]}{x^2}$$

11. If $y = 5\cos x - 3\sin x$, **prove that** $\frac{d^2y}{dx^2} + y = 0.$

Solution: Let $y = 5\cos x - 3\sin x$ …….(1)

$$\frac{dy}{dx} = -5\sin x - 3\cos x$$

Now,

$$\frac{d^2y}{dx^2} = -5\cos x + 3\sin x$$

= $-(5\cos x - 3\sin x)$ = $-y$ [From (1)]

$$\frac{d^2y}{dx^2} = -y$$

$$\frac{d^2y}{dx^2} + y = 0$$

12. If $y = \cos^{-1} x.$ Find $\frac{d^2y}{dx^2}$ in terms of y alone.

Solution: Given: $y = \cos^{-1} x$

or x = cos y ……….(1)

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-x^2}}$$

= $\frac{-1}{\sqrt{1-\cos^2 y}}$ [From (1)]

= $\frac{-1}{\sqrt{\sin^2 y}} = \frac{-1}{\sin y} = -\operatorname{cosec} y$ ……….(2)

Now,

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}(\operatorname{cosec} y)$$

= $-\left[-\operatorname{cosec} y \cot y \frac{dy}{dx}\right]$

= $\operatorname{cosec} y \cot y(-\operatorname{cosec} y)$

= $-\operatorname{cosec}^2 y \cot y$ [From (2)]

13. If $y = 3\cos(\log x) + 4\sin(\log x)$, **show that**
$x^2 y_2 + x y_1 + y = 0.$

Solution: Given function is
$y = 3\cos(\log x) + 4\sin(\log x)$ ….(1)
Derivate with respect to x, we get

$$\frac{dy}{dx} = y_1 = -3\sin(\log x)\frac{d}{dx}\log x + 4\cos(\log x)\frac{d}{dx}\log x$$

$$y_1 = -3\sin(\log x)\frac{1}{x} + 4\cos(\log x)\frac{1}{x}$$

$$= \frac{1}{x}\left[-3\sin(\log x) + 4\cos(\log x)\right]$$

$$x y_1 = -3\sin(\log x) + 4\cos(\log x)$$

Now, derivate above equation once again

$$\frac{d}{dx}(x y_1) = -3\cos(\log x)\frac{d}{dx}\log x - 4\sin(\log x)\frac{d}{dx}\log x$$

$$x\frac{d}{dx}(y_1) + y_1\frac{d}{dx}x = -3\cos(\log x)\frac{1}{x} - 4\sin(\log x)\frac{1}{x}$$

$$x y_2 + y_1 = -\frac{\left[3\cos(\log x) + 4\sin(\log x)\right]}{x}$$

$$x(x y_2 + y_1) = -\left[3\cos(\log x) + 4\sin(\log x)\right]$$

$x(x y_2 + y_1) = -y$ [using equation (1)]

This implies, $x^2 y_2 + x y_1 + y = 0$

Hence proved.

14. If $y = Ae^{mx} + Be^{nx}$, **show that**
$$\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0.$$

Solution:

To Prove: $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny = 0$

$y = Ae^{mx} + Be^{nx}$ ….(1)

$\frac{dy}{dx} = Ae^{mx}\frac{d}{dx}(mx) + Be^{nx}\frac{d}{dx}(nx)$ $\left[\because \frac{d}{dx}e^{f(x)} = e^{f(x)}\frac{d}{dx}f(x)\right]$

$\frac{dy}{dx} = Ame^{mx} + Bne^{nx}$ ….(2)

Find the derivate of equation (2)

$\frac{d^2y}{dx^2} = Ame^{mx}.m + Bne^{nx}.n$

$= Am^2e^{mx} + Bn^2e^{nx}$ ….(3)

Now, L.H.S.= $\frac{d^2y}{dx^2} - (m+n)\frac{dy}{dx} + mny$

(Using equations (1), (2) and (3))

$= Am^2e^{mx} + Bn^2e^{nx} - (m+n)Ame^{mx} + Bne^{nx} + mn(Ae^{mx} + Be^{nx})$

$= Am^2e^{mx} + Bn^2e^{nx} - Am^2e^{mx} - Bmne^{nx} + Amne^{mx} - Bn^2e^{nx} + Amne^{mx} + Bmne^{nx}$

= 0

= R.H.S.

Hence proved.

15. If $y = 500e^{7x}$, **show that**
$\frac{d^2y}{dx^2} = 49y.$

Solution:

$y = 500e^{7x} + 600e^{-7x}$ ……….(1)

$\frac{dy}{dx} = 500e^{7x}(7) + 600e^{-7x}(-7)$

$= 500(7)e^{7x} - 600(7)e^{7x}$

Now,

$\frac{d^2y}{dx^2} = 500(7)e^{7x}(7) - 600(7)e^{7x}(-7)$

$= 500(49)e^{7x} + 600(49)e^{7x}$

=>

$\frac{d^2y}{dx^2} = 49\left[500e^{7x}(7) + 600e^{7x}\right]$

$= 49y$ [Uing equation (1)]

$\Rightarrow$ $\frac{d^2y}{dx^2} = 49y$

$\Rightarrow$ Hence proved.

16. If $e^y(x+1) = 1$, **show that**
$\frac{d^2y}{dx^2} = \left(\frac{dy}{dx}\right)^2.$

Solution: Given: $e^y(x+1) = 1$

So, $e^y = \frac{1}{x+1}$

Taking log on both the sides, we have

$\log e^y = \log \frac{1}{x+1}$

$y \log e = \log 1 - \log(x+1)$

$y = -\log(x+1)$

$\frac{dy}{dx} = -\frac{1}{x+1}\frac{d}{dx}(x+1)$

$= \frac{-1}{x+1} = (x+1)^{-1}$

Again,

$\frac{d^2y}{dx^2} = -(-1)(x+1)^{-2}\frac{d}{dx}(x+1)$

$\left[\because \frac{d}{dx}\{f(x)\}^n = n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$

So, $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

Now, L.H.S. = $\frac{d^2y}{dx^2} = \frac{1}{(x+1)^2}$

And R.H.S. = $\left(\frac{dy}{dx}\right)^2 = \left(\frac{-1}{x+1}\right)^2 = \frac{1}{(x+1)^2}$

L.H.S. = R.H.S.

Hence proved.

17. If $y=\left(\tan^{-1}x\right)^2$, **show that**

$$\left(x^2+1\right)^2 y_2+2x\left(x^2+1\right)y_1=2.$$

Solution: Given: $y=\left(\tan^{-1}x\right)^2$ ……….(1)

Represent y_1 as first derivative and y_2 as second derivative of the function.

$$y_1=2\left(\tan^{-1}x\right)\frac{d}{dx}\tan^{-1}x$$

$$\left[\because \frac{d}{dx}\{f(x)\}^n=n\{f(x)\}^{n-1}\frac{d}{dx}f(x)\right]$$

and $y_1=2\left(\tan^{-1}x\right)\frac{1}{1+x^2}$

$=\dfrac{2\tan^{-1}x}{1+x^2}$

So, $\left(1+x^2\right)y_1=2\tan^{-1}x$

Again differentiating both sides with respect to x.

$$\left(1+x^2\right)\frac{d}{dx}y_1+y_1\frac{d}{dx}\left(1+x^2\right)=2.\frac{1}{1+x^2}$$

$$\left(1+x^2\right)y_2+y_1.2x=\frac{2}{1+x^2}$$

$$\left(1+x^2\right)^2 y_2+2xy_1\left(1+x^2\right)=2$$

Hence proved.