Miscellaneous Examples

1. Find the area under the given curves and given lines:

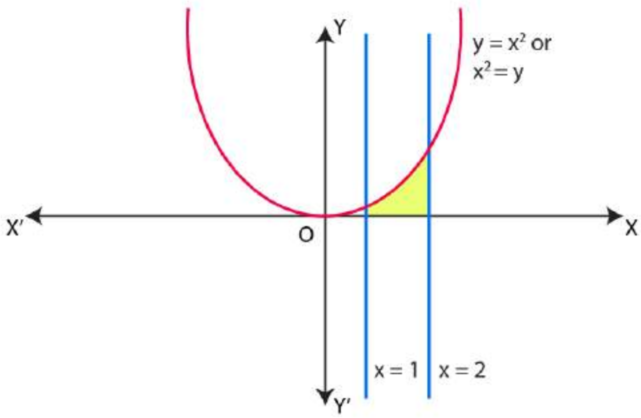
(i) $y = x^2, x = 1, x = 2$ **and x-axis.**

(ii) $y = x^4, x = 1, x = 5$ **and x-axis.**

Solution:

(i) Equation of the curve is

y= x^2 ……..(1)



Required area bounded by curve (1), vertical line x=1, x=2 and x-axis

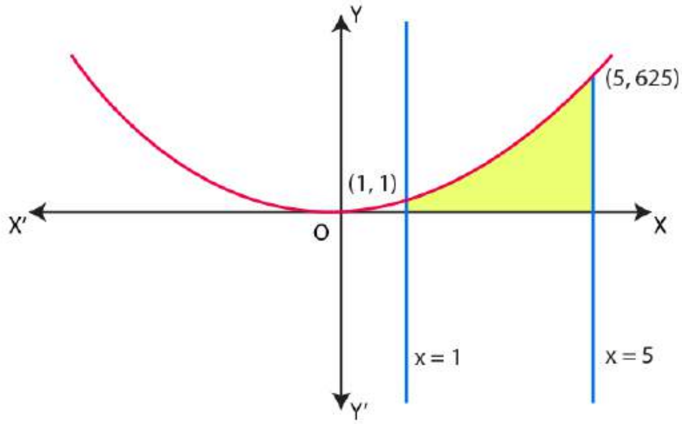
$= \int_1^2 y\, dx$

$= \left(\frac{x^3}{3}\right)_1^2$

$= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$ sq. units

(ii) Equation of the curve

y= x^4 …..(1)



It is clear that curve (1) passes through the origin because x=0 from (1) y=0.

Table of values for curve $y = x^4$

x	1	2	3	4	5
y	1	16	81	256	625

Required shaded area between the curve $y = x^4$, vertical lines $x = 1, x = 5$ and $x-$ axis

$= \int_1^5 y\,dx = \int_1^5 x^4\,dx$

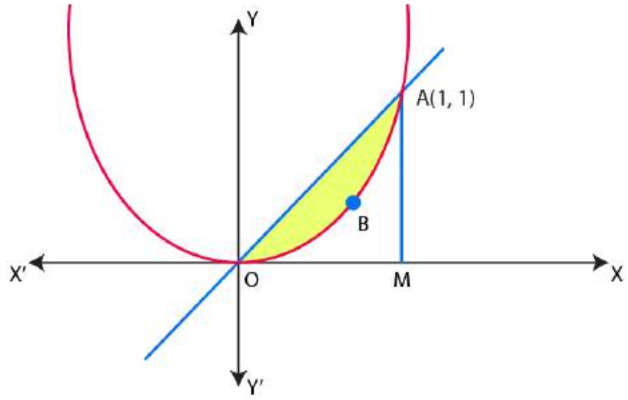
$= \left(\frac{x^5}{5}\right)_1^5 = \frac{5^5}{5} - \frac{1^5}{5}$

$= \frac{3125-1}{5} = \frac{3124}{5}$

= 624.8 sq. units

2. Find the area between the curves y=x and y=x^2

Solution: Equation of one curve (straight line) is y=x …..(i)



Equation of second curve (parabola) is $y = x^2$ …..(ii)

Solving equation (i) and (ii), we get $x=0$ or $x=1$ and $y=0$ or $y=1$

Points of intersection of line (i) and parabola (ii) are O (0, 0) and A (1, 1).

Now Area of triangle OAM

= Area bounded by line (i) and x-axis

$= \int_0^1 y\,dx = \int_0^1 x\,dx$

$= \left(\frac{x^2}{2}\right)_0^1$

$= \frac{1}{2} - 0 = \frac{1}{2}$ sq. units

Also Area OBAM = Area bounded by parabola (ii) and x-axis

$= \int_0^1 y\,dx = \int_0^1 x^2\,dx$

$= \left(\frac{x^3}{3}\right)_0^1$

= $\frac{1}{3} - 0 = \frac{1}{3}$ sq. units

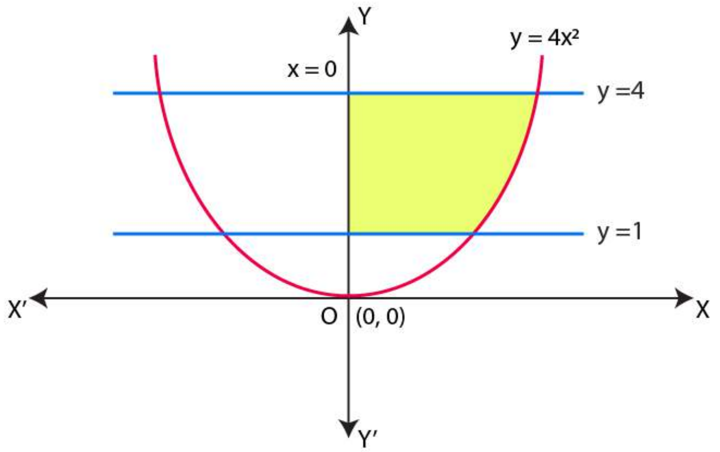
Required area OBA between line (i) and parabola (ii)

= Area of triangle OAM – Area of OBAM

= $\frac{1}{2} - \frac{1}{3}$ = $\frac{3-2}{6} = \frac{1}{6}$ sq. units

3. Find the area of the region lying in the first quadrant and bounded by $y = 4x^2, x = 0, y = 1$ and y=4.

Solution: Equation of the curve is $y = 4x^2$



$x^2 = \frac{y}{4}$ ……….(i)

or $x = \frac{\sqrt{y}}{2}$ ……….(ii)

Here required shaded area of the region lying in first quadrant bounded by parabola (i), $x = 0$

and the horizontal lines $y = 1$ and $y = 4$ is

$\int_1^4 x\,dy$ = $\int_1^4 \frac{\sqrt{y}}{2}\,dy$ $= \frac{1}{2}\int_1^4 y^{\frac{1}{2}}$

$= \frac{1}{2}\left|\frac{\left(y^{\frac{3}{2}}\right)_1^4}{\frac{3}{2}}\right|$

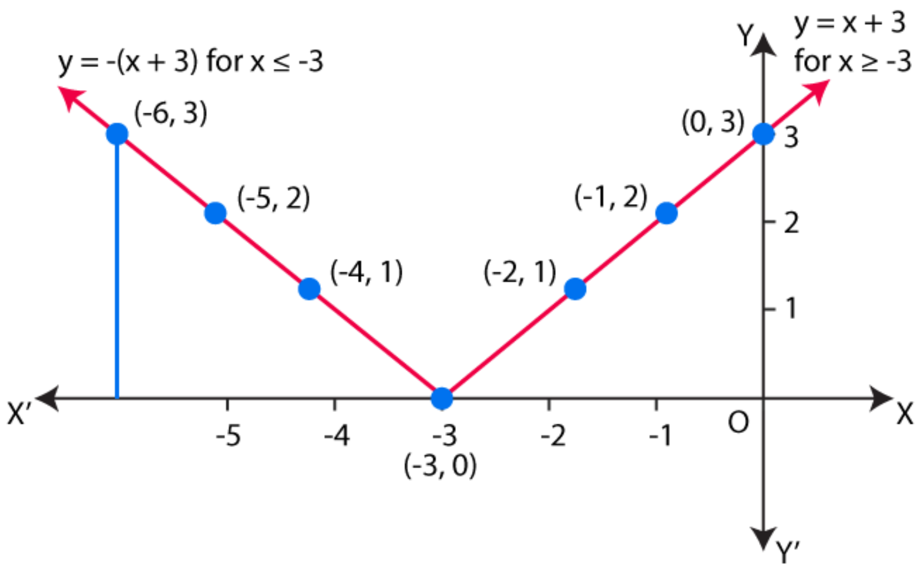
$= \frac{1}{2}\cdot\frac{2}{3}\left(4^{\frac{3}{2}}-1^{\frac{3}{2}}\right)$

$= \frac{1}{3}\left(4\sqrt{4}-1\right)$

$= \frac{1}{3}(8-1) = \frac{7}{3}$ sq. units

4. Sketch the graph of $y=|x+3|$ and evaluate $\int_{-6}^{0}|x+3|\,dx.$

Solution: Equation of the given curve is $y=|x+3|$ ……….(i)



$y = |x+3| \geq 0$ for all real x.

Graph of curve is only above the x-axis i.e., in first and second quadrant only.

$y = |x+3|$

= $x+3$

If $x+3 \geq 0$

$x \geq -3$ ….(ii)

And $y = |x+3|$

= $-(x+3)$

If $x+3 \leq 0$

$x \leq -3$ …….(iii)

Table of values for $y = x+3$ for $x \geq -3$

x	y
-3	0
-2	1
-1	2
0	3

Table of values for $y = x+3$ for $x \leq -3$

x	y
-3	0
-4	1
-5	2
-6	3

Now, $\int_{-6}^{0} |x+3| \, dx$

$= \int_{-6}^{-3} |x+3| \, dx + \int_{-3}^{0} |x+3| \, dx$

$= \int_{-6}^{-3} -(x+3) \, dx + \int_{-3}^{0} (x+3) \, dx$

$= \left(\frac{x^2}{2}+3x\right)_{-6}^{-3} + \left(\frac{x^2}{2}+3x\right)_{-3}^{0}$

$= \left[\frac{9}{2}-9-(18-18)\right]+\left[0-\left(\frac{9}{2}-9\right)\right]$

$= \frac{9}{2}+9+0+0-\frac{9}{2}+9$

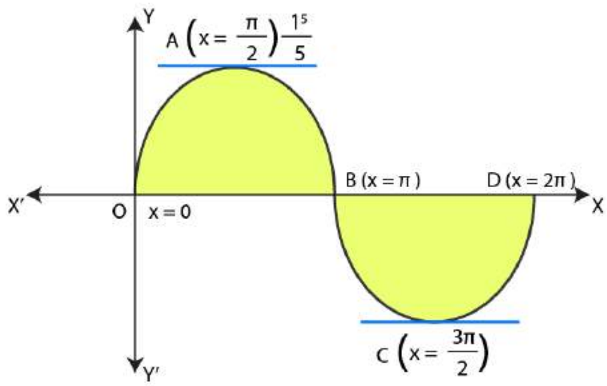
$= 18-\frac{18}{2}=18-9=9$ sq. units

5. Find the area bounded by the curve y = sin x between x=0 and x=2π.

Solution: Equation of the curve is y = sin x ……….(i)

$y = \sin x \geq 0$ for $0 \leq x \leq \pi$: as graph is in I and II quadrant.

And $y = \sin x \leq 0$ for $\pi \leq x \leq 2\pi$: as graph is in III and IV quadrant.



If tangent is parallel to x-axis, then

$$\frac{dy}{dx} = 0$$

$$\Rightarrow \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

Table of values for curve y= sin x between x =0 and $x = 2\pi$

x	y
0	0
$\frac{\pi}{2}$	1
π	0
$\frac{3\pi}{2}$	-1
2π	0

Now Required shaded area = Area OAB + Area BCD

$= \int_0^{\pi} y\,dx + \int_{\pi}^{2\pi} y\,dx$

$= \int_0^{\pi} \sin x\,dx + \int_{\pi}^{2\pi} \sin x\,dx$

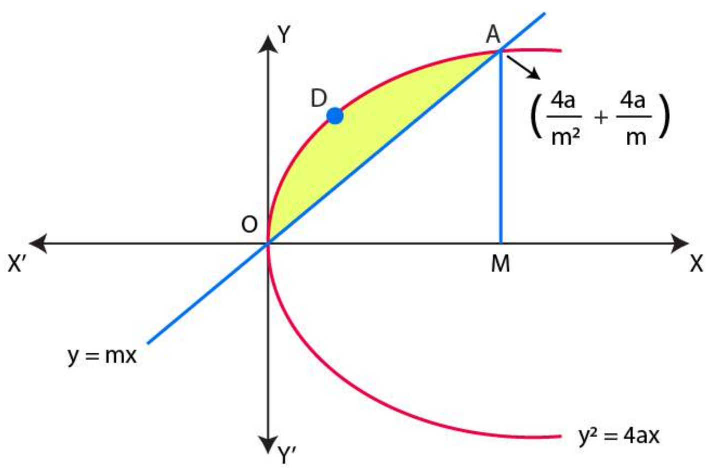
$= -(\cos x)_0^{\pi} + (\cos x)_{\pi}^{2\pi}$

$= -1(-1-1) + -(1+1)$

= 2 + 2 = 4 sq. units

6. Find the area enclosed by the parabola $y^2 = 4ax$ and the line y=mx.

Solution: Equation of parabola is $y^2 = 4ax$ ……..(i)



The area enclosed between the parabola and line is the shaded area OADO.

Form figure: And the points of intersection of curve and line are

O (0, 0) and A $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

Now Area ODAM = Area of parabola and x-axis

$= \int_0^{\frac{4a}{m^2}} 2\sqrt{a}\,x^{\frac{1}{2}}\,dx$

$= \dfrac{2\sqrt{a}\left(x^{\frac{3}{2}}\right)_0^{\frac{4a}{m^2}}}{\frac{3}{2}}$

$= \dfrac{4\sqrt{a}}{3}\left(\dfrac{4a}{m^2}\right)^{\frac{3}{2}}$

$= \dfrac{32a^2}{3m^3}$ ……….(ii)

Again Area of $\Delta\mathrm{OAM}$ = Area between line and x-axis

$= \left|\int_0^{\frac{4a}{m^2}} mx\,dx\right| = m\left(\dfrac{x^2}{2}\right)_0^{\frac{4a}{m^2}}$

$= \dfrac{m}{2}\left(\left(\dfrac{4a}{m^2}\right)^2 - 0\right)$

$= \dfrac{m}{2}\cdot\dfrac{16a^2}{m^4} = \dfrac{8a^2}{m^3}$ ……….(ii)

Requires shaded area = Area ODAM – Area of $\Delta\mathrm{OAM}$

$= \dfrac{32a^2}{3m^3} - \dfrac{8a^2}{m^3}$

$= \frac{a^2}{m^3}\left(\frac{32}{3}-8\right)$

$= \frac{8a^2}{3m^3}$

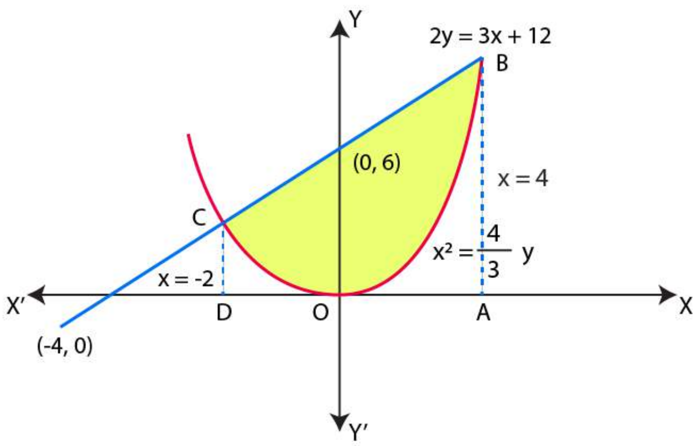
7. Find the area enclosed by the parabola $4y=3x^2$ and the line $2y=3x+12$.

Solution:

Equation of the parabola is

$4y=3x^2$ ……….(i)

or $x^2=\frac{4}{3}y$



Equation of the line is $2y=3x+12$ ……(ii)

From graph, points of intersection are B (4, 12) and C(-2, 3).

Now, Area ABCD = $\left|\int_{-2}^{4}\left(\frac{3}{2}x+6\right)dx\right|$

$= \left[\frac{3}{4}x^2+6x\right]_{-2}^{4}$

$= (12+24)-(3-12)$

= 45 sq. units

Again, Area CDO + Area OAB = $\int_{-2}^{4}\left(\frac{3}{4}x^2\right)dx$

= $\frac{1}{4}\left[64-(-8)\right]$ = 18 sq. units

Therefore,

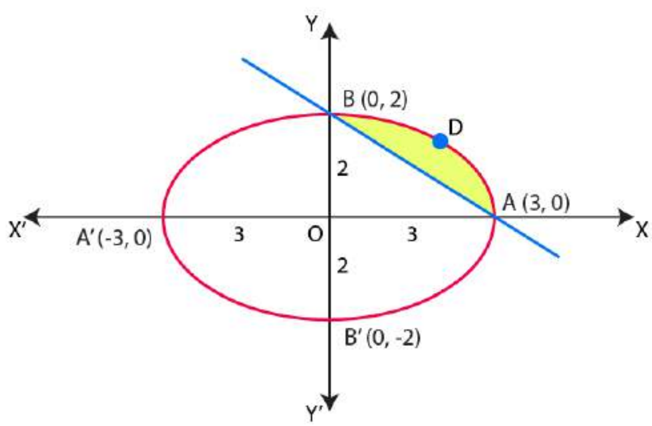
Required area = Area ABCD – (Area CDO + Area OAB)

= 45 – 18 = 27 sq. units

8. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the line $\frac{x}{3}+\frac{y}{2}=1$.

Solution: Equation of the ellipse is

$\frac{x^2}{9}+\frac{y^2}{4}=1$ ………(i)



Here points of intersection of ellipse (i) with x-axis are

A (3, 0) and A'(-3, 0) and intersection of ellipse (i) with y- axis are B (0, 2) and B'(0, -2).

Also, the points of intersections of ellipse (i) and line $\frac{x}{3}+\frac{y}{2}=1$ are A (3, 0) and B (0, 2).

Therefore,

Area OADB = Area between ellipse (i) (arc AB of it) and x-axis

$= \int_0^3 \frac{2}{3}\sqrt{9-x^2}\,dx$

$= \frac{2}{3}\left[\frac{x}{2}\sqrt{3^2-x^2}+\frac{3^2}{2}\sin^{-1}\frac{x}{3}\right]$

$= \frac{2}{3}\left[\frac{3}{2}\sqrt{9-9}+\frac{9}{2}\sin^{-1}1-\left(0+\frac{9}{2}\sin^{-1}0\right)\right]$

$= \frac{2}{3}.\frac{9\pi}{4} = \frac{3\pi}{2}$ sq. units……….(ii)

Again Area of triangle OAB = Area bounded by line AB and x-axis

$= \int_0^3 \frac{2}{3}\sqrt{3-x}\,dx$

$= \frac{2}{3}\left\{\left(9-\frac{9}{2}\right)-0\right\}$

$= \frac{2}{3}.\frac{9}{2}=3$ sq. units ….(iii)

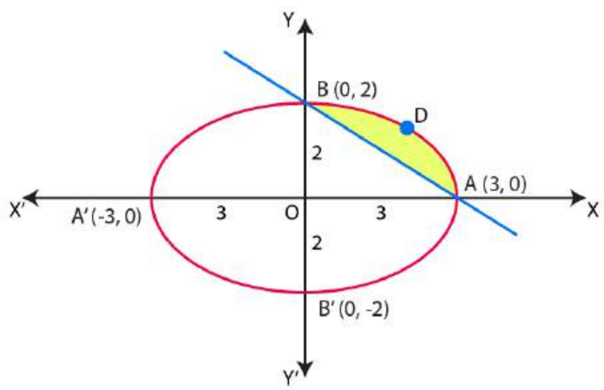
Now Required shaded area = Area OADB – Area OAB

$= \frac{3\pi}{2}-3$

$= 3\left(\frac{\pi}{2}-1\right)=\frac{3}{2}(\pi-2)$ sq. units

9. Find the area of the smaller region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the line $\frac{x}{a}+\frac{y}{b}=1$.

Solution: Equation of ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ ……..(i)



Area between arc AB of the ellipse and x-axis

$= \int_0^a \frac{b}{a}\sqrt{a^2-x^2}\,dx$

$= \frac{b}{a}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$

$= \frac{b}{a}\left[0+\frac{a^2}{2}\sin^{-1}1-(0+0)\right]$

$= \frac{b}{a}.\frac{a^2}{2}.\frac{\pi}{2}=\frac{\pi ab}{4}$ ………..(ii)

Also Area between chord AB and x-axis

= $\int_0^a \frac{b}{a}(a-x)\, dx$

= $\frac{b}{a}\left[ax - \frac{x^2}{2}\right]_0^a$

= $\frac{b}{a}\left(a^2 - \frac{a^2}{2}\right)$

= $\frac{b}{a}.\frac{a^2}{2} = \frac{1}{2}ab$

Now, Required area = (Area between arc AB of the ellipse and x-axis) – (Area between chord AB and x-axis)

= $\frac{\pi ab}{4} = \frac{ab}{2} = \frac{ab}{4}(\pi - 2)$ sq. units

10. Find the area of the region enclosed by the parabola $x^2 = y$, the line $y = x+2$ and x- axis.

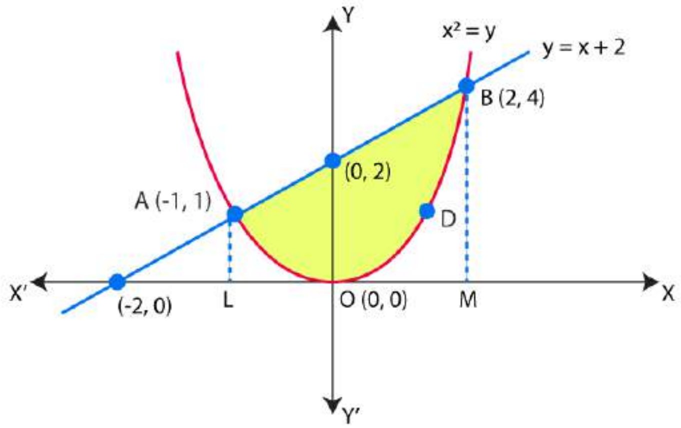
Solution: Equation of parabola is $x^2 = y$ ………..(i)
Equation of line is $y = x+2$ ………..(ii)

Here the two points of intersections of parabola (i) and line (ii) are A(-1, 1) and B (2, 4).

Area ALODBM = Area bounded by parabola (i) and x-axis

= $\int_{-1}^{2} x^2\, dx$ = $\left(\frac{x^3}{3}\right)_{-1}^{2}$

= $\frac{8}{3} + \frac{1}{3}$ = $\frac{9}{3}$ = 3 sq. units



Also, Area of trapezium ALMB = Area bounded by line (ii) and x-axis

$= \int_{-1}^{2} (x+2)\, dx = \left(\frac{x^2}{2} + 2x\right)_{-1}^{2}$

$= 2 + 4 - \left(\frac{1}{2} - 2\right)$

$= \frac{15}{2}$ sq. units

Now, required area = Area of trapezium ALMB – Area ALODBM

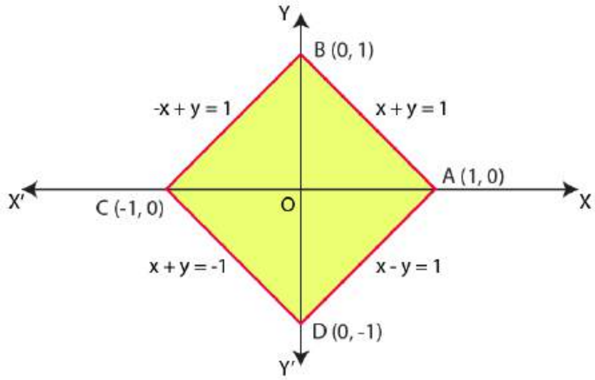
$= \frac{15}{2} - 3 = \frac{9}{2}$ sq. units

11. Using the method of integration, find the area enclosed by the curve $|x| + |y| = 1$.

[Hint: The required region is bounded by lines x + y = 1, x– y = 1, – x + y = 1 and – x – y = 1].

Solution: Equation of the curve is

$|x| + |y| = 1$ …(i)



The area bounded by the curve (i) is represented by the shaded region ABCD.

The curve intersects the axes at points A (1, 0), B (0, 1), C(-1, 0) and D(0, -1)

As, given curve is symmetrical about x-axis and y-axis.

Area bounded by the curve = Area of square ABCD = 4 x Δ OAB

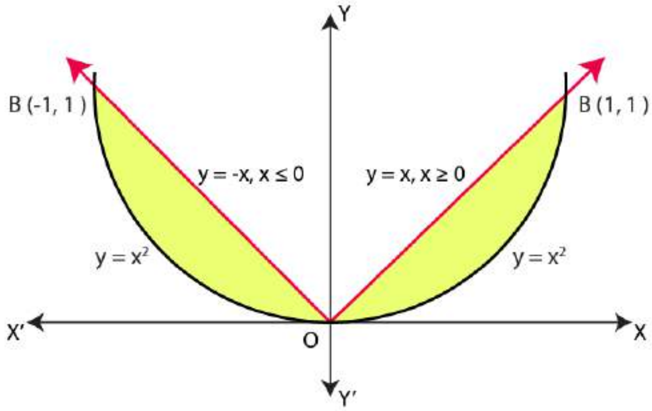
$= 4\int_0^1 (1-x)\, dx$

$= 4\left(x - \frac{x^2}{2}\right)_0^1$

$= 4 \times \frac{1}{2}$ = 2 sq. units

12. Find the area bounded by the curves $\{(x, y) : y \geq x^2 \text{ and } y = |x|\}$.

Solution: The area bounded by the curves $\{(x, y) : y \geq x^2 \text{ and } y = |x|\}$ is represented by the shaded region.



Since, area is symmetrical about y-axis.

Therefore, Required area = Area between parabola and x-axis between limits x=0 and x=1

$= \int_0^1 y\ dx = \int_0^1 x^2\ dx$

$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$ ………..(i)

And Area of ray y=x and x-axis,

$= \int_0^1 y\ dx = \int_0^1 x\ dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}$ ………..(ii)

Required shaded area in first quadrant

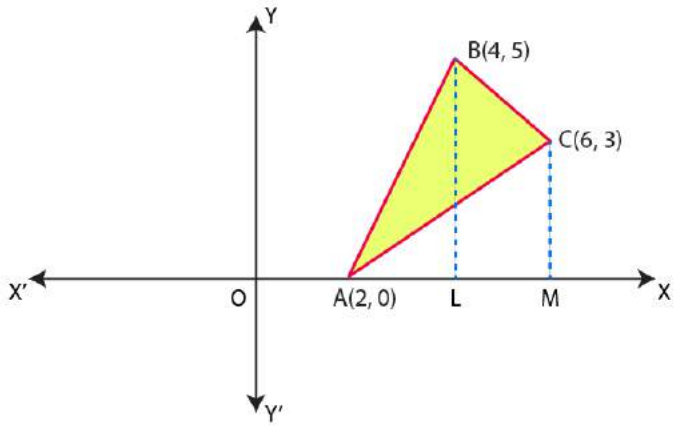
= (Area between ray $y = x$ for $x \geq 0$ and x-axis) – (Area between parabola $y = x^2$ and x-axis in first quadrant)

= Area given by equation (ii) – Area given by equation (i)

$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ sq. units

13.Using the method of integration, find the area of the triangle whose vertices are A (2, 0), B (4, 5) and C (6, 3).

Solution: Vertices of the given triangle are A (2, 0), B (4, 5) and C (6, 3).



Equation of side AB is $y-0=\frac{5-0}{4-2}(x-2)$

= $y=\frac{5}{2}(x-2)$

Equation of side BC is $y-5=\frac{3-5}{6-4}(x-4)$

= $y=9-x$

Equation of side AC is $y-0=\frac{3-0}{6-2}(x-2)$

= $y=\frac{3}{4}(x-2)$

Now, Required shaded area = Area ΔALB + Area of trapezium BLMC – Area ΔAMC

= $\int_2^4 \frac{5}{2}(x-2)\,dx + \int_4^6 (9-x)\,dx - \int_2^6 \frac{3}{4}(x-2)\,dx$

$= \left[\frac{5}{2}(8-8)-(2-4)\right]+\left|54-18-(36-8)\right|-\left[\frac{3}{4}\{18-12-(2-4)\}\right]$

$= \frac{5}{2}(0+2)+\left|36-36+8\right|-\frac{3}{4}(6+2)$

= 5 + 8 – 6 = 7 sq. units

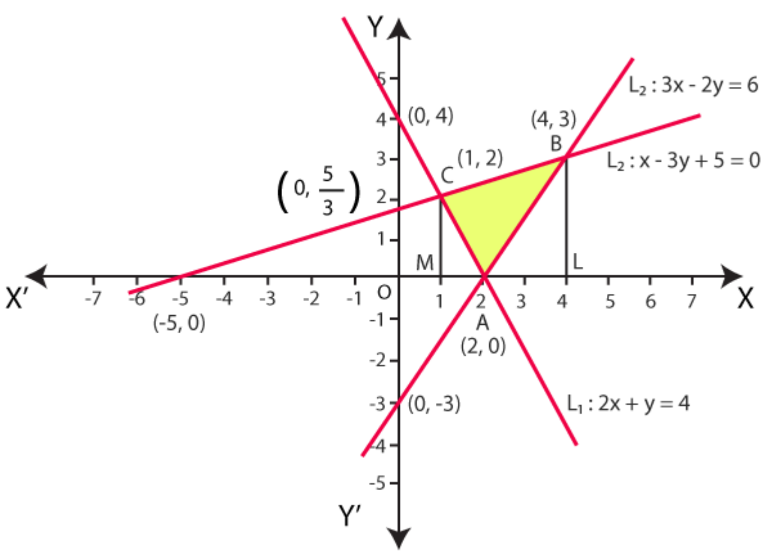
14. Using the method of integration, find the area of the region bounded by the lines: $2x+y=4$, $3x-2y=6$ and $x-3y+5=0$.

Solution:

Lets say, equation of one line l_1 is $2x+y=4$,
equation of second line l_2 is $3x-2y=6$

And equation of third line l_3 is $x-3y+5=0$.

Draw all the lines on the coordinate plane, we get



Here, vertices of triangle ABC are A (2, 0), B (4, 3) and C (1, 2).

Now, Required area of triangle = Area of trapezium CLMB – Area ΔACM – Area ΔABL

= $\int_1^4 \frac{1}{3}(x+5)\,dx - \int_1^2 (4-2x)\,dx - \int_2^4 \frac{3}{2}(x-2)\,dx$

= $\frac{1}{3}\left[8+20-\left(\frac{1}{2}+5\right)\right]-\{(8-4)-(4-1)\}-\frac{3}{2}|(8-8)-(2-4)|$

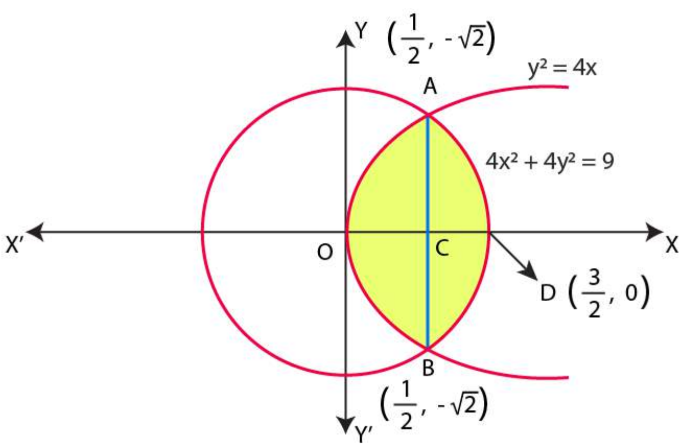
= $\frac{1}{3}\left(28-\frac{11}{2}\right)-(4-3)-\frac{3}{2}\times 2$

= $\frac{1}{3}\times\frac{45}{2}-1-3$

= $\frac{15}{2}-1-3=\frac{7}{2}$ sq. units

15.Find the area of the region $\{(x,y): y^2 \le 4x \text{ and } 4x^2+4y^2 \le 9\}$.

Solution: Equation of parabola is $y^2 = 4x$ ……(i)
And equation of circle is $4x^2+4y^2=9$ ……(ii)



From figures, points of intersection of parabola (i) and circle (ii) are

A$\left(\frac{1}{2},\sqrt{2}\right)$ and B$\left(\frac{1}{2},-\sqrt{2}\right)$

Required shaded area OADBO (Area of the circle which is interior to the parabola)

= 2 x Area OADO = 2 [Area OAC + Area CAD]

$$= 2\left[\int_0^{\frac{1}{2}} 2\sqrt{x}\,dx + \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\frac{9}{4}-x^2}\,dx\right]$$

$$= \left[\left\{2.\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right\}_0^{\frac{1}{2}} + \left\{\frac{x\sqrt{\frac{9}{4}x^2}}{2}+\frac{\frac{9}{4}}{2}\sin^{-1}\frac{x}{3/2}\right\}_{\frac{1}{2}}^{\frac{3}{2}}\right]$$

$$= 2\left[\frac{4}{3}\times\frac{1}{2\sqrt{2}}+\frac{9}{8}\sin^{-1}1-\frac{\frac{1}{2}\sqrt{2}}{2}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= 2\left[\frac{\sqrt{2}}{3}+\frac{9}{8}.\frac{\pi}{2}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$$

$$= \left(\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}+\frac{\sqrt{2}}{6}\right)$$ sq. units

16. Choose the correct answer:

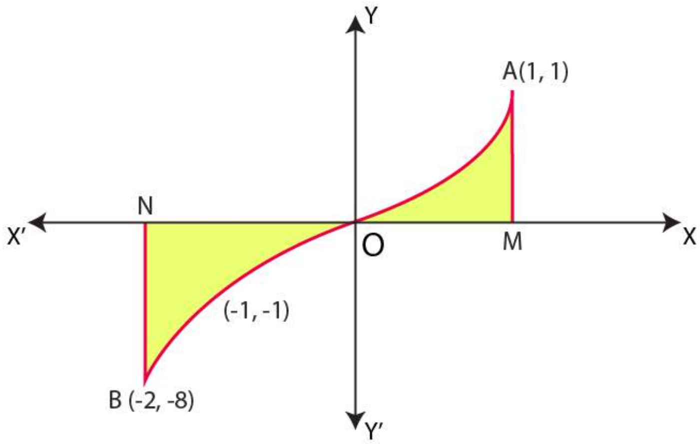
Area bounded by the curve y=x^3 the x-axis and the ordinate x=-2 and x=1 is:

(A) -9 (B) -15/4 (C) 15/4 (D) 17/4

Solution:
Option (D) is correct.

Explanation:

Equation of the curve is $y = x^3$



Now, Area OBN ($y = x^3$ for $-2 \le x \le 0$) and Area OAM ($y = x^3$ for $0 \le x \le 1$)

Therefore, Required area = Area OBN + Area OAM

$= \int_{-2}^{0} x^3 \, dx + \int_{0}^{1} x^3 \, dx$

$= \frac{17}{4}$ sq. units

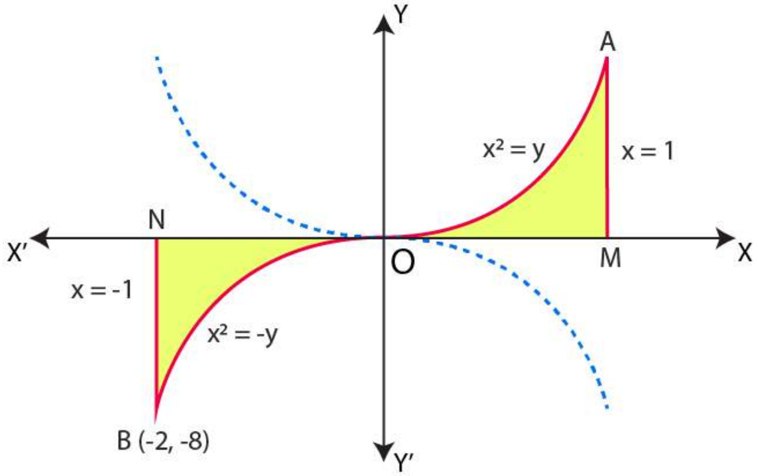
17. Choose the correct answer:

The area bounded by the curve y = x|x|, x- axis and the ordinates x = -1 and x = 1 is given by:
(A) 0 (B) 1/3 (C) 2/3 (D) 4/3

Solution:
Option (C) is correct.

Explanation:

Equation of the curve is



$y = x|x| = x(x) = x^2$ if $x \geq 0$ ……….(1)

And $y = x|x| = x(-x) = -x^2$ if $x \leq 0$ ………..(2)

Required area = Area ONBO + Area OAMO

$= \int_{-1}^{0} -x^2\, dx + \int_{0}^{1} x^2\, dx$

= 2/3 sq. units

18. Choose the correct answer:

The area of the circle $x^2 + y^2 = 16$ **exterior to the parabola** $y^2 = 6x.$

(A) $\frac{4}{3}\left(4\pi - \sqrt{3}\right)$ **(B)** $\frac{4}{3}\left(4\pi + \sqrt{3}\right)$

(C) $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$ **(D)** $\frac{4}{3}\left(8\pi + \sqrt{3}\right)$

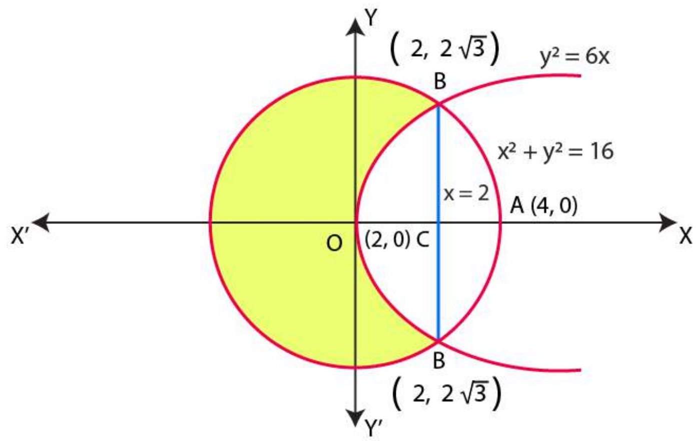
Solution:

Option (C) is correct.

Explanation:

Equation of the circle is $x^2 + y^2 = 16$ …..(1)
Thus, radius of circle is 4
This circle is symmetrical about x-axis and y- axis.

Here two points of intersection are B$\left(2, 2\sqrt{3}\right)$ and B’$\left(2, -2\sqrt{3}\right)$.



Required area = Area of circle – Area of circle interior to the parabola

= πr^2 - Area OBAB’O

= 16π - 2 x Area OBACO

= 16π - 2[Area OBCO + Area BACB]

= $16\pi - 2\left[\int_0^2 \sqrt{6x}\, dx + \int_2^4 \sqrt{16-x^2}\, dx\right]$

= $16\pi - 2\left[\frac{2}{3}\sqrt{6}\left(2\sqrt{2}\right) + 8\sin^{-1}1 - \sqrt{12} - 8\sin^{-1}\frac{1}{2}\right]$

= $16\pi - 2\left[\frac{8}{\sqrt{3}} + 8.\frac{\pi}{2} - 2\sqrt{3} - 8.\frac{\pi}{6}\right]$

= $16\pi - 2\left[\frac{8}{\sqrt{3}} - 2\sqrt{3} + 8\pi\left(\frac{1}{2} - \frac{1}{6}\right)\right]$

= $16\pi - 2\left[\frac{2}{\sqrt{3}} + \frac{8\pi}{3}\right]$

= $16\pi\left(1 - \frac{1}{3}\right) - \frac{4}{\sqrt{3}}$

= $\frac{4}{3}\left(8\pi - \sqrt{3}\right)$ sq. units

19. Choose the correct answer:

The area bounded by the y-axis, y = cos x and y= sin x when $0 \le x \le \frac{\pi}{2}$ **is:**

(A) $2\left(\sqrt{2} - 1\right)$ **(B)** $\sqrt{2} - 1$ **(C)** $\sqrt{2} + 1$ **(D)** $\sqrt{2}$

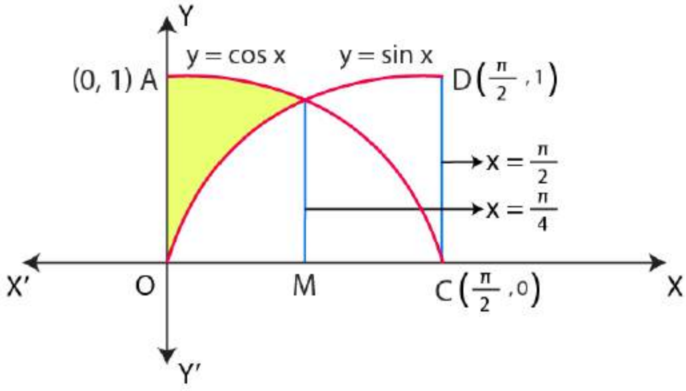
Solution:

Option (B) is correct.

Explanation:

Graph of both the functions are intersect at the point

B $\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$.



Required Shaded Area = Area OABC – Area OBC

= Area OABC – (Area OBM + Area BCM)

$$= \int_0^{\pi/2} \cos x\, dx - \left(\int_0^{\pi/4} \sin x\, dx + \int_{\pi/4}^{\pi/2} \cos x\, dx \right)$$

$$= \left(\sin\frac{\pi}{2} - \sin 0° \right) - \left(-\cos\frac{\pi}{4} + \cos 0° + \sin\frac{\pi}{2} - \sin\frac{\pi}{4} \right)$$

$$= 1 + \frac{1}{\sqrt{2}} - 1 - 1 + \frac{1}{\sqrt{2}} = \left(\sqrt{2} - 1\right)$$ sq. units