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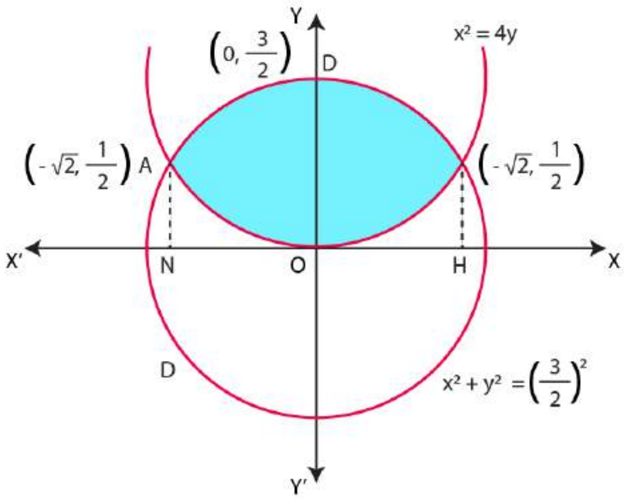
1. Find the area of the circle $4x^2 + 4y^2 = 9$ which is interior to the parabola $x^2 = 4y$.

Solution:

Step 1: Equation of the circle is $4x^2 + 4y^2 = 9$

$x^2 + y^2 = \frac{9}{4}$ ……….(1)



Here, centre of circle is (0, 0) and radius is 3/2

Equation of parabola is $x^2 = 4y$ …..(2)

Step 2: To find values of x and y

Put $x^2 = 4y$ in equation (1), $4y + y^2 = \frac{9}{4}$

$16y + 4y^2 = 9$

$4y^2 + 16y - 9 = 0$

$4y^2 + 18y - 2y - 9 = 0$

$2y(2y+9) - 1(2y+9) = 0$

$(2y+9)(2y-1)=0$

$2y+9=0$ or $2y-1=0$

$\Rightarrow$ $y=\frac{-9}{2}$ or $y=\frac{1}{2}$

Find the value of x:

Put $y=\frac{-9}{2}$ in $x^2=4y$,

$\Rightarrow$ $x^2=4\left(\frac{-9}{2}\right)=-18$

Put $y=\frac{1}{2}$ in $x^2=4y$,

$\Rightarrow$ $x^2=4\left(\frac{1}{2}\right)=2$

=> $x=\pm 2$

Therefore, Points of intersections of circle (1) and parabola (2) are

A $\left(-\sqrt{2},\frac{1}{2}\right)$ and B $\left(\sqrt{2},\frac{1}{2}\right)$.

Step 3: Area OBM = Area between parabola (2) and y-axis

= $\int_0^{\frac{1}{2}} x\,dy$

$\left[\because \text{At O, } y=0 \text{ and at B, } y=\frac{1}{2}\right]$

= $\int_0^{\frac{1}{2}} 2y^{\frac{1}{2}}\,dy$

$$\left[\because x^2 = 4y \Rightarrow x = 2\sqrt{y} = 2y^{\frac{1}{2}}\right]$$

$$= 2.\frac{\left(y^{\frac{3}{2}}\right)_0^{\frac{1}{2}}}{\frac{3}{2}} = 2.\frac{2}{3}\left[\left(\frac{1}{2}\right)^{\frac{3}{2}} - 0\right]$$

$$= \frac{4}{3}.\frac{1}{2}\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{3}$$ ……….(3)

Step 4: Now area BDM = Area between circle (1) and y-axis

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} x\,dy$$

$$\left[\because \text{At B}, y = \frac{1}{2} \text{ and at D}, y = \frac{3}{2}\right]$$

$$= \int_{\frac{1}{2}}^{\frac{3}{2}} \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\,dy$$

$$\left[\because x^2 = \left(\frac{3}{2}\right)^2 - y^2 \Rightarrow x = \sqrt{\left(\frac{3}{2}\right)^2 - y^2}\right]$$

$$= \left[\frac{y}{2}\sqrt{\left(\frac{3}{2}\right)^2 - y^2} + \frac{\left(\frac{3}{2}\right)^2}{2}\sin^{-1}\frac{y}{\frac{3}{2}}\right]_{\frac{1}{2}}^{\frac{3}{2}}$$

$$= \frac{3}{4}\sqrt{\left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2} + \frac{9}{8}\sin^{-1}\frac{\frac{3}{2}}{\frac{3}{2}} - \left[\frac{1}{4}\sqrt{\frac{9}{4} - \frac{1}{4}} + \frac{9}{8}\sin^{-1}\frac{\frac{1}{2}}{\frac{3}{2}}\right]$$

$= \left(\frac{3}{4}\times 0\right)+\frac{9}{8}\sin^{-1}1-\left[\frac{1}{4}\sqrt{\frac{8}{4}}+\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$

$= \frac{9}{8}\times\frac{\pi}{2}-\frac{1}{4}\sqrt{2}-\frac{9}{8}\sin^{-1}\frac{1}{3}$

$= \frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}$ …….(4)

Step 5.
Required shaded area = Area AOBDA = 2 (Area OBD) = 2 (Area OBM + Area MBD)

$= 2\left[\frac{\sqrt{2}}{3}+\left(\frac{9\pi}{16}-\frac{\sqrt{2}}{4}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right)\right] = 2\left[\sqrt{2}\left(\frac{1}{3}-\frac{1}{4}\right)+\frac{9\pi}{16}-\frac{9}{8}\sin^{-1}\frac{1}{3}\right]$

$= 2\sqrt{2}\left(\frac{4-1}{12}\right)+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3}$

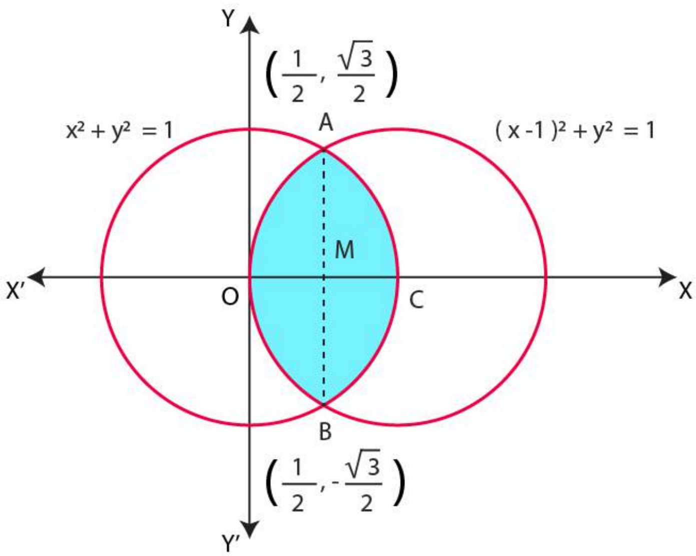
$= \frac{\sqrt{2}}{6}+\frac{9\pi}{8}-\frac{9}{4}\sin^{-1}\frac{1}{3} = \frac{\sqrt{2}}{6}+\frac{9}{4}\left(\frac{\pi}{2}-\sin^{-1}\frac{1}{3}\right)$

$= \frac{\sqrt{2}}{6}+\frac{9}{4}\cos^{-1}\frac{1}{3}$ $\left[\because \sin\left(\frac{\pi}{2}-\theta\right)=\cos\theta\right]$

2. Find the area bounded by the curves $(x-1)^2+y^2=1$ and $x^2+y^2=1$.

Solution:
Equations of two circles are

$x^2+y^2=1$ ……….(1)



And $(x-1)^2 + y^2 = 1$ ……….(2)

From equation (1), $y^2 = 1 - x^2$

Put this value in equation (2),

$(x-1)^2 + 1 - x^2 = 1$

$\Rightarrow x^2 + 1 - 2x + 1 - x^2 = 1$

$\Rightarrow -2x + 1 = 0$

$\Rightarrow x = \frac{1}{2}$

Put $x = \frac{1}{2}$ in $y^2 = 1 - x^2$,

$y^2 = 1 - \left(\frac{1}{2}\right)^2 = 1 - \frac{1}{4} = \frac{3}{4}$ $\Rightarrow$ $y = \pm\frac{\sqrt{3}}{4}$

The two points of intersections of circles (1) and (2) are $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, \frac{-\sqrt{3}}{2}\right)$.

Now, from equation (1), $y=\sqrt{1-x^2}$ in first quadrant and from equation (2), $y=\sqrt{1-(x-1)^2}$ in first quadrant.

Required area OACBO = 2 x Area OAC = 2 (Area OAD + Area DAC)

$$= 2\left[\int_0^{\frac{1}{2}} y \text{ of circle (ii) } dx + \int_{\frac{1}{2}}^{1} y \text{ of circle (i) } dx\right]$$

$$= 2\left[\int_0^{\frac{1}{2}} \sqrt{1-(x-1)^2}\, dx + \int_{\frac{1}{2}}^{1} \sqrt{1-x^2}\, dx\right]$$

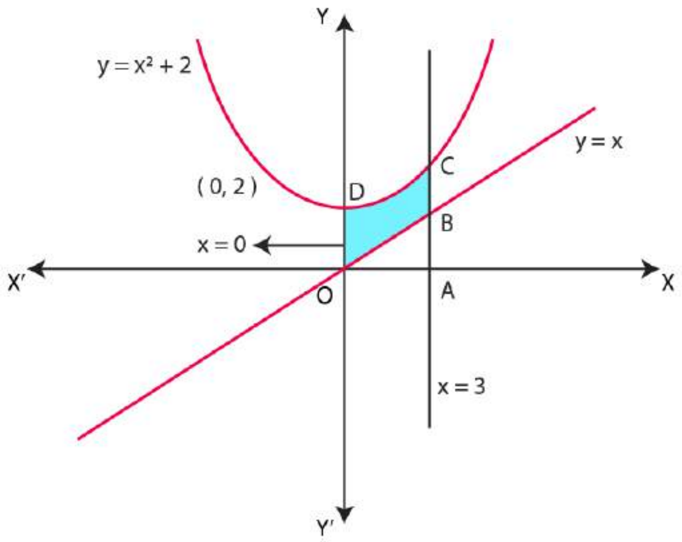
$$= 2\left[\left\{\frac{(x-1)\sqrt{1-(x-1)^2}}{2}+\frac{1}{2}\sin^{-1}(x-1)\right\}_0^{\frac{1}{2}}+\left\{\frac{x\sqrt{1-x^2}}{2}+\frac{1}{2}\sin^{-1}x\right\}_{\frac{1}{2}}^{1}\right]$$

$$= \left\{-\frac{1}{2}\sqrt{\frac{3}{4}}+\sin^{-1}\left(-\frac{1}{2}\right)\right\}-\sin^{-1}(-1)-\left\{\frac{1}{2}\sqrt{\frac{3}{4}}+\sin^{-1}\frac{1}{2}\right\}$$

$$= -\frac{\sqrt{3}}{4}-\frac{\pi}{6}+\frac{\pi}{2}+\frac{\pi}{2}-\frac{\sqrt{3}}{4}-\frac{\pi}{6} = \left(\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right) \text{ sq. units}$$

3. Find the area of the region bounded by the curves $y=x^2+2, y=x, x=0$ and x=3.

Solution: Equation of the given curve is



(Point D is (0,2)

$y = x^2 + 2$ ……….(1)

$x^2 = y - 2$

Here Vertex of the parabola is (0, 2).

Equation of the given line is y=x ……..(2)

x	0	1	2
y	0	1	2

We know that, slope of straight line passing through the origin is always 1, that means, making an angle of 45 degrees with x- axis.

Here also, Limits of integration area given to be x=0 to x=3.

Area bounded by parabola (1) namely $y = x^2 + 2$, the x-axis and the ordinates x=0 to x=3 is the area OACD and $\int_0^3 y\,dx = \int_0^3 \left(x^2 + 2\right) dx$

$= \left(\frac{x^3}{3}+2x\right)_0^3$

= (9 + 6) – 0 = 15 ……….(3)

Again Area bounded by parabola (2) namely y=x the x-axis and the ordinates x=0 to x=3 is the area OAB and

$\int_0^3 y\,dx = \int_0^3 x\,dx$

$= \left(\frac{x^2}{2}\right)_0^3 = \frac{9}{2} - 0 = \frac{9}{2}$ ……….(3)

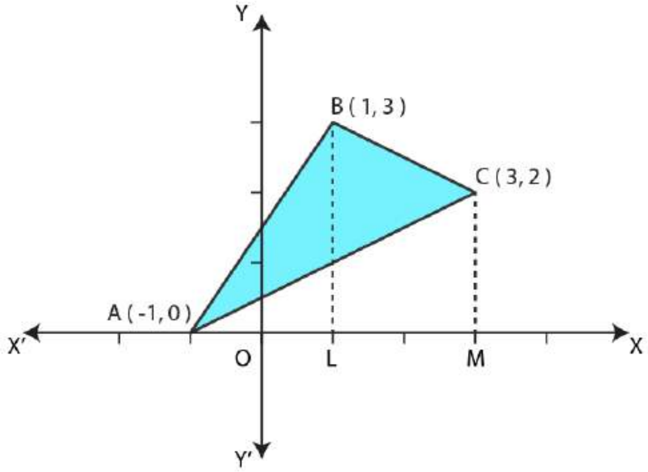
Required area = Area OBCD = Area OACD – Area OAB

= Area given by equation (3) – Area given by equation (4)

$= 15 - \frac{9}{2} = \frac{21}{2}$ sq. units

4. Using integration, find the area of the region bounded by the triangle whose vertices are (-1, 0), (1, 3) and (3, 2).

Solution:
Vertices of triangle are A(-1, 0), B (1, 3) and C (3, 2).



Therefore, eequation of the line is

$$y-0=\frac{3-0}{1-(-1)}\left(x-(-1)\right)$$

$$\left[\because y-y_1=\frac{y_2-y_1}{x_2-x_1}(x_2-x_1)\right]$$

$$y=\frac{3}{2}(x+1)$$

Area of ΔABL = Area bounded by line AB and x-axis

$$=\int_{-1}^{1} y\, dx$$

$$[\because \text{At A, } x=-1 \text{ and at B, } x=1]$$

$$=\int_{-1}^{1} \frac{3}{2}(x+1)\, dx$$

$$=\frac{3}{2}\left(\frac{x^2}{2}+x\right)_{-1}^{1}$$

$= \frac{3}{2}\left[\left(\frac{1}{2}+1\right)-\left(\frac{1}{2}-1\right)\right]$

$= \frac{3}{2}\left(\frac{3}{2}+\frac{1}{2}\right) = \frac{3}{2}.\frac{4}{2}=3$ ……….(1)

Again equation of line BC is $y-3=\frac{2-3}{3-1}(x-1) \Rightarrow y=\frac{1}{2}(7-x)$

Area of trapezium BLMC = Area bounded by line BC and x-axis

$\Rightarrow \int_1^3 y\,dx = \int_1^3 \frac{1}{2}(7-x)\,dx$

$= \frac{1}{2}\left(7x-\frac{x^2}{2}\right)_1^3$

$= \frac{1}{2}\left[\left(21-\frac{9}{2}\right)-\left(7-\frac{1}{2}\right)\right]$

$= \frac{1}{2}\left(21-\frac{9}{2}-7+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{42-9-14+1}{2}\right)$

$= \frac{1}{4}\times 20$ = 5 ……….(2)

Again equation of line AC is $y-0=\frac{2-0}{3-(-1)}(x-(-1)) \Rightarrow y=\frac{1}{2}(x+1)$

Area of triangle ACM = Area bounded by line AC and x-axis

=> $\int_{-1}^3 y\,dx = \int_{-1}^3 \frac{1}{2}(x+1)\,dx$

$= \frac{1}{2}\left[\left(\frac{x^2}{2}+x\right)_{-1}^{3}\right]$

$= \frac{1}{2}\left(\frac{9}{2}+3-\frac{1}{2}+1\right)$

$= \frac{1}{2}\left(\frac{9+6-1+2}{2}\right)$

$= \frac{1}{4}\times 16$ = 4 ……….(3)

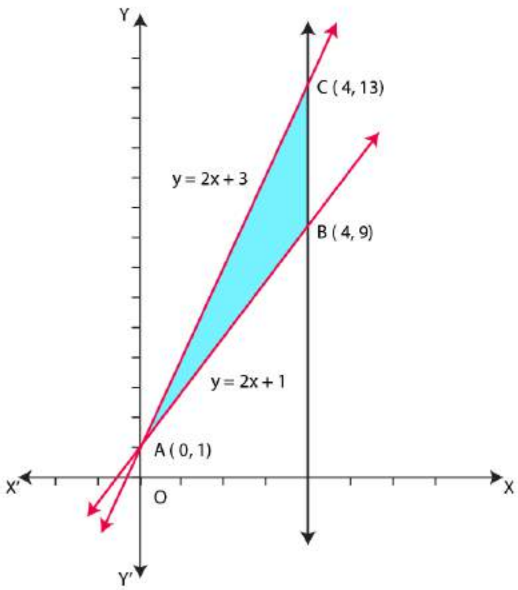
Therefore,

Required area = Area of ΔABL + Area of Trapezium BLMC – Area of ΔACM

= 3 + 5 – 4 = 4 sq. units

5. Using integration, find the area of the triangular region whose sides have the equations y = 2x + 1, y = 3x + 1 and x=4.

Solution: Equations of one side of triangle is



$y = 2x + 1$ ………..(1)

$y = 3x + 1$ ………..(2) and

x = 4 ……….(3)

Solving equation (1) and (2), we get x=0 and y=1

So, Point of intersection of lines (1) and (2) is A (0, 1)

Put x=4 in equation (1), we get y=9

So, Point of intersection of lines (1) and (3) is B (4, 9)

Put x=4 in equation (1), we get y=13

Point of intersection of lines (2) and (3) is C (4, 13)

Area between line (2), that is AC and x-axis

$= \int_0^4 y\,dx = \int_0^4 (3x+1)\,dx = \left(\frac{3x^2}{2} + x\right)_0^4$

= 24 + 4 = 28 sq. units ……….(iv)

Again Area between line (1) , that is AB and x-axis

$= \int_0^4 y\,dx = \int_0^4 (2x+1)\,dx$

$= \left(x^2 + x\right)_0^4$

= 16 + 4 = 20 sq. units ……….(v)

Therefore, Required area of ΔABC

= Area given by (4) – Area given by (5)

= 28 – 20 = 8 sq. units

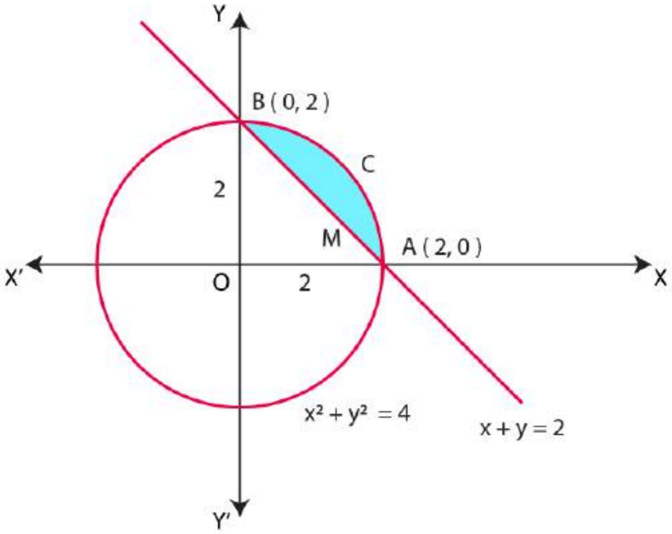
6. Choose the correct answer:

Smaller area enclosed by the circle $x^2+y^2=4$ **and the line** $x+y=2$ **is:**

(A) $2(\pi-2)$ **(B)** $\pi-2$ **(C)** $2\pi-1$ **(D)** $2(\pi+2)$

Solution:
Option (B) is correct.

Explanation:
Equation of circle is $x^2+y^2=2^2$ ……….(1)



$\Rightarrow\ y=\sqrt{2^2-x^2}$ ………(2)

Also, equation of the line is $x+y=2$ ……….(3)

or y = 2-x

x	0	2
y	2	0

Therefore graph of equation (3) is the straight line joining the points (0, 2) and (2, 0).

From the graph of circle (1) and straight line (3), it is clear that points of intersections of circle (1) and straight line (3) are A (2, 0) and B (0, 2).

Area OACB, bounded by circle (1) and coordinate axes in first quadrant

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 \sqrt{2^2-x^2}\,dx\right|$

$= \left(\frac{x}{2}\sqrt{2^2-x^2}+\frac{2^2}{2}\sin^{-1}\frac{x}{2}\right)_0^2$

$= \left(\frac{2}{2}\sqrt{4-4}+2\sin^{-1}1\right)-\left(0+2\sin^{-1}0\right)$

$= 0+2\left(\frac{\pi}{2}\right)-2(0)=\pi$ sq. units ……….(iv)

Area of triangle OAB, bounded by straight line (3) and coordinate axes

$= \left|\int_0^2 y\,dx\right| = \left|\int_0^2 (2-x)\,dx\right|$

$= \left(2x-\frac{x^2}{2}\right)_0^2$

$= (4-2)-(0-0)$ = 2 sq. units ……….(v)

Required shaded area = Area OACB given by (iv) – Area of triangle OAB by (v)

$= (\pi-2)$ sq. units

7. Choose the correct answer:

Area lying between the curves $y^2=4x$ and $y=2x$ is:

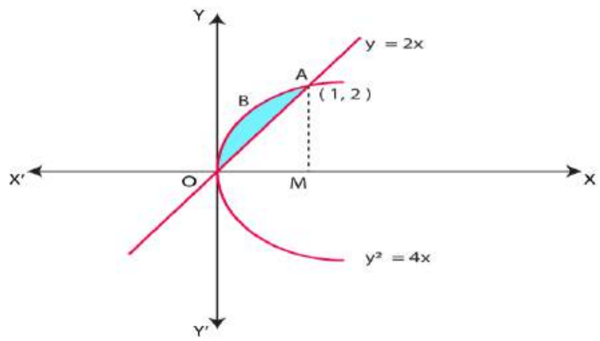
(A) $\frac{2}{3}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{1}{4}$ **(D)** $\frac{3}{4}$

Solution:

Option (B) is correct.

Explanation:

Equation of curve (parabola) is $y^2=4x$ …..(1)



$\Rightarrow y = 2\sqrt{x} = 2x^{\frac{1}{2}}$ …..(2)

Equation of another curve (line) is y=2x …..(3)

Solving equation (1) and (3), we get x=0 or x=1 and y=0 or y=2

Therefore, Points of intersections of circle (1) and line (2) are O (0, 0) and A (1, 2).

Now Area OBAM = Area bounded by parabola (1) and x-axis = $\left|\int_0^1 y\,dx\right|$

$= \left|\int_0^1 2x^{\frac{1}{2}}\,dx\right| = 2\frac{\left(x^{\frac{3}{2}}\right)_0^1}{\frac{3}{2}}$

$= \frac{4}{3}(1-0) = \frac{4}{3}$ ………(4)

Also, Area ΔOAM = Area bounded by parabola (3) and x-axis

$= \left|\int_0^1 y\,dx\right| = \left|\int_0^1 2x\,dx\right| = 2\left(\frac{x^2}{2}\right)_0^1$

$= (1-0) = 1$ ………(5)

Now required shaded area OBA = Area OBAM – Area of ΔOAM

$= \frac{4}{3} - 1 = \frac{4-3}{3} = \frac{1}{3}$ sq. units