Question 1: If $(1 + ax)^n = 1 + 8x + 24x^2 + ...$, then the value of a and n are ________.

Solution:

As given $(1 + ax)^n = 1 + 8x + 24x^2 + ...$

$1 + \frac{n}{1} ax + \frac{n(n-1)}{1.2} a^2x^2 + ...$

$= 1 + 8x + 24x^2 + ...$

$\Rightarrow na = 8, \frac{n(n-1)}{1.2} a^2 = 24$

$\Rightarrow na (n - 1) a = 48$

$8 (8 - a) = 48$

$8 - a = 6$

$\Rightarrow a = 2$

$\Rightarrow n = 4$

Question 2: If the coefficient of x^7 in $(ax^2 + [1/bx])^{11}$ is equal to the coefficient of x^{-7} in $(ax - 1/bx^2)^{11}$, then ab = __________.

Solution:

In the expansion of $(ax^2 + [1 / bx])^{11}$, the general term is

$T_{r+1} = {}^{11}C_r (ax^2)^{11-r} (1/bx)^r$

$= {}^{11}C_r * a^{11-r} * [1/b^r] * x^{22-3r}$

For x^7, we must have $22 - 3r = 7$

r = 5, and the coefficient of $x^7 = {}^{11}C_5 * a^{11-5} * [1/b^5] = {}^{11}C_5 * a^6 * b^5$

Similarly, in the expansion of $(ax^{-1}bx^2)^{11}$, the general term is

$T_{r+1} = {}^{11}C_r * (-1)^r * [a^{11-r} / b^r] * x^{11-3r}$

For x^{-7} we must have, $11 - 3r = -7$, r = 6, and the coefficient of x^{-7} is ${}^{11}C_6 * [a^5 / b^6] = {}^{11}C_5 * a_5 * b_6$.

As given, ${}^{11}C_5 [a^6 / b^5] = {}^{11}C_5 * [a^5 / b^6]$

$\Rightarrow ab = 1$

Question 3: Find the value of

$$\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} + \ldots\ldots + \binom{30}{20}\binom{30}{30}$$

Solution:

$(1 - x)^{30} = {}^{30}C_0x_0 - {}^{30}C_1x_1 + {}^{30}C_2x^2 + \ldots\ldots + (-1)^{30}\,{}^{30}C_{30}x^{30}$ ....(i)

$(x + 1)^{30} = {}^{30}C_0\,x^{30} + {}^{30}C_1x^{29} + {}^{30}C_2x^{28} + \ldots\ldots + {}^{30}C_{10}x^{20} + \ldots. + {}^{30}C_{30}x^0$ ....(ii)

Multiplying (i) and (ii) and equating the coefficient of x^{20} on both sides, we get

required sum = coefficient of x^{20} in $(1 - x^2)^{30} = {}^{30}C_{10}$.

Question 4: If

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{\sqrt{4-x}}$$

is approximately equal to a + bx for small values of x, then (a, b) = ________.

Solution:

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{2\left[1-\frac{x}{4}\right]^{1/2}} = \frac{\left[1+\frac{1}{2}(-3x)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}(-3x)^2+\ldots\right]+\left[1+\frac{5}{3}(-x)+\frac{5}{3}\frac{2}{3}\frac{1}{2}(-x)^2+\ldots\right]}{2\left[1+\frac{1}{2}\left(-\frac{x}{4}\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left(-\frac{x}{4}\right)^2+\ldots\right]}$$

$$= \frac{\left[1-\frac{19}{12}x+\frac{53}{144}x^2-\ldots\right]}{\left[1-\frac{x}{2}-\frac{1}{8}x^2-\ldots\right]} = 1 - \frac{35}{24}x + \ldots$$ Neglecting higher powers of x, then

$$a + bx = 1 - \frac{35}{24}x$$

$$\Rightarrow a = 1, b = -\frac{35}{24}.$$

Question 5: In the polynomial (x − 1) (x − 2) (x − 3) . . . . . (x − 100), what is the coefficient of x^{99}?

Solution:

(x − 1) (x − 2) (x − 3) . . . . . (x − 100)

Number of terms = 100

Coefficient of x^{99} = (x − 1) (x − 2) (x − 3) . . . . . (x − 100)

= (−1 −2 −3 −. . . . . . −100)

= − (1 + 2 + ... +100)

= − [100 * 101 / 2

= - 5050

Question 6: In the expansion of $(x + a)^n$, the sum of odd terms is P and sum of even terms is Q, then the value of $(P^2 - Q^2)$ will be _________.

Solution:

$(x + a)^n = x^n + {}^nC_1x^{n-1} a + \ldots\ldots = (x^n + {}^nC_2x^{n-2}a^2 + \ldots\ldots\ldots + ({}^nC_1x^{n-1}a + {}^nC_3x^{n-3}a^3 + \ldots\ldots) = P+Q$

$(x - a)^n = P - Q$

As the terms are altered,

$P^2 - Q^2 = (P + Q) (P - Q) = (x + a)^n (x - a)^n$

$P^2 - Q^2 = (x^2 - a^2)^n$

Question 7: If the sum of the coefficients in the expansion of $(1 - 3x + 10x^2)^n$ is a and if the sum of the coefficients in the expansion of $(1 +x^2)^n$ is b, then what is the relation between a and b?

Solution:

We have a = sum of the coefficient in the expansion of $(1 - 3x + 10x^2)^n = (1 - 3 + 10)^n = (8)^n$

$(1 - 3x + 10x^2)^n = (2)^{3n}$, [Putting x = 1] .

Now, b = sum of the coefficients in the expansion of $(1 + x^2)^n = (1 + 1)^n = 2n$.

Clearly, $a = b^3$

Question 8: $(1 + x)^n - nx - 1$ is divisible by (where $n \in N$)

A) 2x

B) x^2

C) $2x^3$

D) All of these

Solution:

$(1 + x)^n = 1 + nx + ([n (n - 1)] / [2!]) * x^2 + ([n (n - 1) (n - 2)] / [3!]) * x^3 + \ldots\ldots$

$(1 + x)^n - nx - 1 = x^2 [([n (n - 1)] / [2!]) + ([n (n - 1) (n - 3)] / [3!]) * x + \ldots\ldots]$

From above it is clear that $(1 + x)^n - nx - 1$ is divisible by x^2.

Trick: $(1 + x)^n - nx - 1$, put n = 2 and x = 3;

Then $4^2 - 2 * 3 - 1 = 9$ is not divisible by 6, 54 but divisible by 9, which is given by option (b) i.e., $x^2 = 9$.

Question 9: If the three consecutive coefficients in the expansion of $(1 + x)^n$ are 28, 56 and 70, then the value of n is ______.

Solution:

Let the three consecutive coefficients be $^nC_{r-1} = 28$, $^nC_r = 56$ and $^nC_{r+1} = 70$, so that

^nC_r / $^nC_{r-1}$ = [n − r + 1] / [r] = 56 / 28 = 2 and $^nC_{r+1}$ / ^nC_r = [n − r] / [r + 1] = 70 / 56 = 5 / 4

This gives n + 1 = 3r and 4n − 5 = 9r

4n − 5 / n + 1 = 3

⇒ n = 8

Question 10: Let $R = (5\sqrt{5} + 11)^{2n+1}$ and f = R − [R], where [.] denotes the greatest integer function. The value of R * f is

Solution:

Since $(5\sqrt{5} - 11)(5\sqrt{5} + 11) = 4$

$5\sqrt{5} - 11 = 4 / (5\sqrt{5} + 11)$, Because $0 < 5\sqrt{5} - 11 < 1 \Rightarrow 0 < (5\sqrt{5} - 11)^{2n+1} < 1$, for positive integral n.

Again, $(5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1} = 2\{^{2n+1}C_1 (5\sqrt{5})^{2n} * 11 + {}^{2n+1}C_3(5\sqrt{5})^{2n-2} \times 11^3 + \ldots +$
${}^{2n+1}C_{2n+1}\, 11^{2n+1}\}$

$= 2\{^{2n+1}C_1(125)^n * 11 + {}^{2n+1}C_3(125)^{n-1}\, 11^3 + \ldots\ldots + {}^{2n+1}C_{2n+1}\, 11^{2n+1}\}$

= 2k, (for some positive integer k)

Let $f' = (5\sqrt{5} - 11)^{2n+1}$, then [R] + f − f′=2k

f − f′ = 2k − [R]

⇒ f − f′ is an integer.

But, 0 ≤ f < 1; 0 < f′<1

⇒ −1 < f − f′ < 1

f − f′ = 0 (integer)

f = f′

Therefore, $Rf = Rf' = (5\sqrt{5} + 11)^{2n+1} * (5\sqrt{5} - 11)^{2n+1}$

$= ([5\sqrt{5}]^2 + 11^2)^{2n+1}$

$= 4^{2n+1}$

Question 11: The digit in the unit place of the number (183!) + 3^{183} is _____.

Solution:

We know that n! terminates in 0 for n^3 at 5 and 3^{4n} terminator in 1, (because $3^4 = 81$)

Therefore, $3^{180} = (3^4)^{45}$ terminates in 1.

Also $3^3 = 27$ terminates in 7

$3^{183} = 3^{180} * 3^3$ terminates in 7.

183! + 3^{183} terminates in 7 i.e. the digit in the unit place = 7.

Question 12: If the coefficients of p^{th}, $(p + 1)^{th}$ and $(p + 2)^{th}$ terms in the expansion of $(1 + x)^n$ are in A.P., then find the equation in terms of n.

Solution:

Coefficient of p^{th}, $(p + 1)^{th}$ and $(p + 2)^{th}$ terms in expansion of $(1 + x)^n$ are ${}^nC_{p-1}$, ${}^nC_p$, ${}^nC_{p+1}$. Then $2\,{}^nC_p = {}^nC_{p-1} + {}^nC_{p+1}$

$\Rightarrow n^2 - n(4p + 1) + 4p^2 - 2 = 0$

Trick: Let p = 1, hence ${}^nC_0$, ${}^nC_1$ and ${}^nC_2$ are in A.P.

$\Rightarrow 2 * {}^nC_1 = {}^nC_0 + {}^nC_2$

$\Rightarrow 2n = 1 + [n(n - 1)] / [2]$

$\Rightarrow 4n = 2 + n^2 - n$

$\Rightarrow n^2 - 5n + 2 = 0$

Question 13: The interval in which x must lie so that the greatest term in the expansion of $(1 + x)^{2n}$ has the greatest coefficient, is _____.

Solution:

Here the greatest coefficient is ${}^{2n}C_n$.

Therefore, ${}^{2n}C_n x^n > {}^{2n}C_{n+1} x^{n-1} \Rightarrow x > n / [n + 1]$ and ${}^{2n}C_n x^n > {}^{2n}C_{n-1} x^{n+1}$

$\Rightarrow x < [n + 1] / n$

Hence, the required interval is (n / [n + 1], [n + 1] / n).

Question 14: The approximate value of (1.0002) 3000 is __________.

Solution:

(1.0002) 3000 = (1 + 0.0002) 3000

= 1 + (3000) (0.0002) + ([(3000) (2999)] / [1.2]) * $(0.0002)^2$

= 1 + (3000) (0.0002)

= 1.6

Question 15: The last digit in 7^{300} is **__________.**

Solution:

We have $7^2 = 49 = 50 - 1$

Now, $7^{300} = (7^2)^{150}$

$= (50 - 1)^{150}$

$= {}^{150}C_0 (50)^{150} (-1)^0 + {}^{150}C_1 (50)^{149} (-1)^1 + \ldots\ldots + {}^{150}C_{150} (50)^0 (-1)^{150}$

Thus, the last digits of 7^{300} are ${}^{150}C_{150} * 1 * 1$ i.e., 1.