Question 1: If n is any integer, then

$$\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x \, dx$$

Solution:

Since cos(2n + 1)(π - x) = cos[(2n + 1)π - (2n + 1)x] = -cos(2n + 1)x and

$\cos^2(\pi - x) = \cos^2 x$

So that f (2a − x) = −f (x), and hence by the property of definite integral

$$\int_0^{\pi} e^{\cos^2 x} \cos^3 (2n+1)x \, dx = 0.$$

Question 2: Let f be a positive function. Let

$$I_1 = \int_{1-k}^{k} x \, f\left\{x(1-x)\right\} dx,$$

$$I_2 = \int_{1-k}^{k} f\left\{x(1-x)\right\} dx$$

when 2k − 1 > 0. Then find I_1/I_2.

Solution:

$$I_1 = \int_{1-k}^{k} x f\{x(1-x)\} dx$$

$$= \int_{1-k}^{k} (1-k+k-x) f[(1-k+k-x)\{1-(1-k+k-x)\}] dx$$

(Because $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx)$

$$= \int_{1-k}^{k} (1-x) f\{x(1-x)\} \, dx$$

$$= \int_{1-k}^{k} f\{x(1-x)\} \, dx - \int_{1-k}^{k} x f\{x(1-x)\} \, dx = I_2 - I_1$$

$$therefore \; 2I_1 = I_2 \Rightarrow \frac{I_1}{I_2} = \frac{1}{2}$$

Question 3: Let a, b, c be non-zero real numbers such that

$$\int_0^1 (1+\cos^8 x)(ax^2+bx+c)\,dx = \int_0^2 (1+\cos^8 x)(ax^2+bx+c)\,dx$$

Then the quadratic equation $ax^2 + bx + c = 0$ has _____ root(s).

Solution:

$$\int_0^2 f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx, \text{ where } f(x) = (ax^2+bx+c)(1+\cos^8 x)$$

$$\text{If } f(x) > 0(<0)\, x \in (1, 2) \text{ then } \int_1^2 f(x)dx > 0(<0).$$

Thus $f(x) = (1 + \cos^8 x)(ax^2 + bx + c)$ must be positive for some value of x in [1, 2] and must be negative for some value of x in [1, 2].

As $(1 + \cos^8 x) \geq 1$ follows that if $g(x) = ax^2 + bx + c$ then there exist some $\alpha, \beta \in (1, 2)$ such that $g(\alpha) > 0$ and $g(\beta) < 0$. Since g is continuous on R, therefore there exists some c between $\alpha$and βsuch that $g(c) = 0$.

Thus $ax^2 + bx + c$ has at least one root in (1, 2) and hence in (0, 2).

Question 4: The value of

$$\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx,\ a > 0$$

Solution:

$$I = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx = \int_{\pi}^{-\pi} \frac{\cos^2 x}{1+a^{-x}} (-dx)$$

$$= \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^{-x}} dx$$

$$\Rightarrow I + I = \int_{-\pi}^{\pi} \cos^2 x \left(\frac{1}{1+a^x} + \frac{1}{1+a^{-x}}\right) dx$$

$$= \int_{-\pi}^{\pi} \cos^2 x \, dx$$

$$2I = 2\int_0^{\pi} \cos^2 x . dx = \int_0^{\pi} (1 + \cos 2x) dx$$

$$2I = [x]_0^{\pi} + \left[\frac{\sin 2x}{2}\right]_0^{\pi}$$

$$\Rightarrow 2I = \pi \Rightarrow I = \frac{\pi}{2}.$$

Question 5:

If $l(m, n) = \int_0^1 t^m (1+t)^n dt,$

then the expression for l (m, n) in terms of l (m + 1, n − 1) is

Solution:

$$l(m,n) = \int_0^1 t^m (1+t)^n dt$$

$$\left[(1+t)^n \frac{t^{m+1}}{m+1}\right]_0^1 - \int_0^1 n(1+t)^{n-1} \frac{t^{m+1}}{m+1} dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} l(m+1, n-1).$$

Question 6: The area bounded by the curves y = |x| − 1 and y = −|x| + 1 is

Solution: The lines are y = x − 1 , x ≥ 0

y = −x − 1, x < 0, y = −x + 1, x ≥ 0, y = x + 1, x < 0

Area = 4 x (1/2 x 1 x 1) = 2

Question 7: If for a real number y, [y] is the greatest integer less than or equal to y, then the value of the integral

$$\int_{\pi/2}^{3\pi/2} [2\sin x]\, dx \text{ is}$$

Solution:

We know, -1 ≤ sin x ≤ 1 => -2 ≤ 2 sin x ≤ 2

$$I = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} [2\sin x]dx$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} [2\sin x]dx + \int_{\frac{5\pi}{6}}^{\pi} [2\sin x]dx + \int_{\pi}^{\frac{7\pi}{6}} [2\sin x]dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} [2\sin x]dx$$

$$= \int_{\frac{\pi}{2}}^{\frac{5\pi}{6}} (1)dx + \int_{\frac{5\pi}{6}}^{\pi} (0)dx + \int_{\pi}^{\frac{7\pi}{6}} (-1)dx + \int_{\frac{7\pi}{6}}^{\frac{3\pi}{2}} (-2)dx$$

$$= \left(\frac{5\pi}{6} - \frac{\pi}{2}\right) + 0 - \left(\frac{7\pi}{6} - \pi\right) - 2\left(\frac{3\pi}{2} - \frac{7\pi}{6}\right)$$

$$= \frac{2\pi}{6} - \frac{\pi}{6} - \frac{4\pi}{6} = -\frac{\pi}{2}.$$

Question 8: If f(x) = A sin(πx/2) + B, f'(1/2) = √2 and $\int_0^1 f(x)dx = \frac{2A}{\pi}$, then find the constants A and B respectively.

Solution:

f(x) = A sin(πx/2) + B, f'(1/2) = √2 and $\int_0^1 f(x)dx = \frac{2A}{\pi}$

$$\int_0^1 \left\{A\sin\left(\frac{\pi x}{2}\right)+B\right\}dx=\frac{2A}{\pi}$$

$$\left|-\frac{2A}{\pi}\cos\frac{\pi x}{2}+Bx\right|_0^1=\frac{2A}{\pi}$$

$$B-\left(\frac{-2A}{\pi}\right)=\frac{2A}{\pi}\Rightarrow B=0$$

Therefore, f(x) = A sin (πx/2) => f’(x) = (πA/2) cos(πx/2)

And f’(1/2) = (πA/2)(1/√2) = √2 => πA = 4 => A = 4/π

A = 4/π and B = 0.

Question 9:

Evaluate $\int\limits_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin x}\,dx, (n\in N)$

Solution:

2 sin (x/2) (1/2 + cos x + cos 2x + ……+ cos nx)

= sin(x/2) + 2 sin(x/2) cos x + 2 sin(x/2) cos 2x + ….+ 2 sin(x/2) cos nx

= sin(x/2) + sin(3x/2) – sin(x/2) + sin (5x/2) – sin(3x/2) + ……+ sin(n + ½) x – sin(n – ½)x

We know, sin(n – ½)x = sin(n + ½)x

$$\frac{1}{2}+\cos x+\cos 2x+\ldots\ldots+\cos nx=\frac{\sin\left(n+\frac{1}{2}\right)x}{2\sin\left(\frac{x}{2}\right)}$$

$$\int_0^{\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)}dx=2\left(\int_0^{\pi}\frac{1}{2}dx+\int_0^{\pi}\cos x\,dx+\ldots\ldots+\int_0^{\pi}\cos nx\,dx\right)$$

$$=2\left(\frac{\pi}{2}+\sin x+\ldots\ldots+\frac{\sin nx}{n}\right)_0^{\pi}=\pi.$$

Question 10: The sine and cosine curves intersect infinitely many times giving bounded regions of equal areas. Then find the area of one such region.

Solution:

Point of intersection of y = sin x and y = cos x is π/4, π/4, 5π/4.

Since, sin x ≥ cos x on the interval [π/4, 5 π/4]

Therefore, area of one such region,

$$= \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = 2\sqrt{2}\ sq.\ unit.$$

Question 11: The integral below is equal to

$$\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8cos2x}{(tanx+cotx)^3}\ dx$$

(1) 15/18 (2) 13/32 (3) 13/256 (4) 15/64

Answer: (1)

Solution:

Let I = $\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8cos2x}{(tanx+cotx)^3}\ dx$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8cos2x}{\left(\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}\right)^3}\ dx$$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8cos2x}{\left(\frac{1}{\cos x\sin x}\right)^3} dx$$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} \frac{8cos2x}{\left(\frac{2}{\sin 2x}\right)^3} dx$$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 8\ cos2x \times \frac{sin^3(2x)}{8}\ dx$$

$$I = \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} (sin2x\ cos2x) \times\ sin^2(2x)\ \ dx$$

$$I = \frac{1}{2} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} sin4x\ (\frac{1-cos4x}{2})\ \ dx$$

$$I = \frac{1}{4} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} sin4x\ dx - \frac{1}{8} \int_{\frac{\pi}{12}}^{\frac{\pi}{4}} sin8x\ dx$$

On solving above integral and putting lower and upper limits, we have

I = 15/128 (Answer!)

Question 12: If $f(a + b + 1 - x) = f(x)\ \forall\ x$, where a and b are fixed positive real numbers, then below expression is equal to

$$\frac{1}{(a+b)} \int_a^b x(f(x) + f(x+1))\ dx$$

a. $\int_{a-1}^{b-1} f(x)\ dx$

b. $\int_{a+1}^{b+1} f(x+1)\ dx$

c. $\int_{a-1}^{b-1} f(x+1)\ dx$

d. $\int_{a+1}^{b+1}\ f(x)\ dx$

Solution:

f(a + b + 1 - x) = f(x) ...(1)

At x -> x + 1

f(a + b - x) = f(x+1) ...(2)

From (1) and (2)

I = $\frac{1}{a+b} \int_a^b (a + \ b-x)[f(x + \ 1) + \ f(x)]\ \ dx$ ...(4)

Adding equation (4) to given, we have

$$2I = \int_a^b f(a+b-x+1)dx + \int_a^b f(x)dx$$

$$2I = 2\int_a^b f(x)dx$$

$$I = \int_a^b f(x)dx$$

As, $x = t+1, dx = dt$

$$I = \int_{a-1}^{b-1} f(t+1)dt$$

$$I = \int_{a-1}^{b-1} f(x+1)dx$$

Question 13: Find the value of

$$\int_0^1 \frac{8\ log(1+x)}{1+x^2}\ dx$$

Solution:

$$\int_0^1 \frac{8\ log(1+x)}{1+x^2}\ dx$$

Put x = tanθ, So θ = $\tan^{-1}$ x

dθ = 1/(1+x^2) dx

Now,

Let I = $\int_0^{\frac{\pi}{4}} 8log(1+\ tan\theta)d\theta$ ...(1)

Lets say, θ = π/4 - θ because x + y = π/4

=> (1 + tan x)(1 + tan y) = 2

(1)=> I = π log 2