Question 1: The locus of a point P(α, β) moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is

(a) an ellipse
(b) a circle
(c) a hyperbola
(d) a parabola
Answer: (c)

Solution:
Tangent to the hyperbola $x^2/a^2 - y^2/b^2 = 1$ is $y = mx \pm \sqrt{(a^2m^2 - b^2)}$
Given that $y = \alpha x + \beta$ is the tangent of hyperbola.
$m = \alpha$ and $a^2m^2 - b^2 = \beta^2$
Therefore, $a^2 \alpha^2 - b^2 = \beta^2$
Locus is $a^2 x^2 - y^2 = b^2$, which is parabola.

Question 2: If a hyperbola passes through the point P(10, 16) and it has vertices at (±6, 0), then the equation of the normal at P is:
(a) $3x + 4y = 94$
(b) $x + 2y = 42$
(c) $2x + 5y = 100$
(d) $x + 3y = 58$
Answer: (c)

Solution:
Vertex of hyperbola: $(\pm a, 0) = (\pm 6, 0)$ => $a = 6$
We know, equation of hyperbola, $x^2/a^2 - y^2/b^2 = 1$
=> $x^2/36 - y^2/b^2 = 1$
Point P(10,16) lies on parabola, so
$100/36 – 256/b^2 = 1$
=> $64/36 = 256/b^2$
=> $b^2 = 144$
Equation of hyperbola becomes, $x^2/36 – y^2/144 = 1$ and
Equation of normal : $a^2x/x_1 + b^2y/y_1 = a^2 + b^2$
=> $36x/10 + 144y/16 = 180$
=> $x/50 + y/20 = 1$
Or $2x + 5y = 100$

Question 3: The eccentricity of the hyperbola whose latus rectum is 8 and length of the conjugate axis is equal to half the distance between the foci, is
(a) $\sqrt{3}$ (b) 4/3 (c) $2/\sqrt{3}$ (d) $4/\sqrt{3}$
Answer: (c)

Solution:
We know, conjugate axis of hyperbola = 2b and
Latus rectum = $2b^2/a$
Given: The eccentricity of the hyperbola whose latus rectum is 8 and length of the conjugate axis is equal to half the distance between the foci.
$\Rightarrow 2b = 1/2(2ae)$ and $2b^2/a = 8$
$\Rightarrow 2/a(ae/2)^2 = 8$
$\Rightarrow ae^2 = 16$ …(i)
Also, we know, $b^2 = a^2 (e^2 – 1)$
From equation, $2b^2/a = 8 \Rightarrow b^2 = 4a$
So, $a^2 (e^2 – 1) = 4a$
$\Rightarrow ae^2 - a = 4$
Using (i)
$\Rightarrow 16 – a = 4$
Or a = 12
Again, (i)$\Rightarrow 12 e^2 = 16$
$\Rightarrow e = 2/\sqrt{3}$

Question 4: An ellipse passes through the foci of the hyperbola, $9x^2 -4y^2 =36$ and its major and minor axes lie along the transverse and conjugate axes of the hyperbola respectively. If the product of eccentricities of the two conics is 1/2. Find the equation of ellipse.

Solution:
Equation of hyperbola is $9x^2 -4y^2 =36$ or $x^2/4 - y^2/9 = 1$
(Here a < b)
Focus = $(0, \pm be)$
Eccentricity = e = $\sqrt{(1+4/9} = \sqrt{13}/3$
So, Foci of hyperbola: $(0, \pm\sqrt{13})$
Standard equation of the ellipse, $x^2/a^2 + y^2/b^2 = 1$ ...(i)
Eccentricity = $e' = \sqrt{(1-a^2/b^2)}$ ...(ii)
ee' = 1/2 (given)
Using eccentricity value of hyperbola, $e' = 1/2 \times 3/\sqrt{13} = 3/2\sqrt{13}$
(ii) $\Rightarrow e'^2 = (1-a^2/b^2)$

$9/52 = (1-a^2/b^2)$
Find the value of b2 form (i) using focii $13/b^2 = 1 \Rightarrow b^2 = 13$
$\Rightarrow 9/52 = (1-a^2/13)$
$\Rightarrow 9/4 = 13 - a^2$
$\Rightarrow a^2 = 43/4$
Now equation of ellipse is $4x^2/43 + y^2/13 = 1$

Question 5: If e_1 and e_2 are the eccentricities of the ellipse, $x^2/18 + y^2/4 = 1$ and the hyperbola, $x^2/9 - y^2/4 = 1$ respectively and (e_1, e_2) is a point on the ellipse, $15x^2 + 3y^2 = k$. Then k is equal to

(a) 14 (b) 15 (c) 17 (d) 16

Answer: (d)

Solution:
e_1 and e_2 are the eccentricities of the ellipse, $x^2/18 + y^2/4 = 1$ and the hyperbola, $x^2/9 - y^2/4 = 1$ respectively

$$e_1 = \sqrt{1 - \frac{4}{18}} = \frac{\sqrt{13}}{3}$$

As (e_1, e_2) lies on the ellipse $15x^2 + 3y^2 = k$
Because, $15e_1^2 + 3e_2^2 = k$
$\Rightarrow 15 \times 7/9 + 3 \times 13/9 = k$
$\Rightarrow k = 16$

Question 6: A tangent to the hyperbola $x^2/4 - y^2/2 = 1$ meets x -axis at P and y -axis at Q. Lines PR and QR are drawn such that OPRQ is a rectangle (where O is the origin). Then R lies on:

(a) $2/x^2 - 4/y^2 = 1$
(b) $4/x^2 - 2/y^2 = 1$
(c) $4/x^2 + 2/y^2 = 1$
(d) $2/x^2 + 4/y^2 = 1$

Answer: (b)

Solution:
Equation of tangent at point (a sec θ, b tanθ) is $x/a \sec\theta - y/b \tan\theta = 1$
Equation of tangent of hyperbola $x^2/4 - y^2/2 = 1$ is $(x \sec\theta)/2 - (y \tan\theta)/\sqrt{2} = 1$ at any parametric point Q.
The coordinate of P and Q are $(a \cos\theta, 0)$ and $(0, -b \cot\theta)$ respectively.
OPRQ is a rectangle, Let the coordinates of point R be (h, k).

then h = a cos θ and k = -b cotθ
Therefore, secθ = a/h and tanθ = -b/k
=> $a^2/h^2 - b^2/k^2 = 1$
So, the required locus is $4/x^2 - 2/y^2 = 1$

Question 7: If the eccentricity of the hyperbola

$$\frac{x^2}{cos^2\theta} - \frac{y^2}{sin^2\theta} = 1$$

is more than 2 when θ Є (0, π/2) then values of length of latus rectum lies in the interval
(a) (3, ∞) (b) (1, 3/2) (c) (2, 3) (d) (-3, -2)
Answer: (a)

Solution:
We know, for hyperbola, $e^2 = 1 + b^2/a^2$
$= 1 + \tan^2 \theta = \sec^2 \theta$
If e > 2 => sec θ > 2
=> θ Є (π/3, π/2)
Now, length of latus rectum of hyperbola = $2b^2/a$ = 2 tan θ sin θ
= 2 > √3 > √3/2 > 3
Therefore, the values of length of latus rectum lies in the interval (3, ∞).

Question 8: Let the eccentricity of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ be the reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then
(a) the equation of the hyperbola is $x^2/3 - y^2/2 = 1$
(b) the eccentricity of the hyperbola is √(5/3)
(c) a focus of the hyperbola is (2, 0)
(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$
Answer: (d)

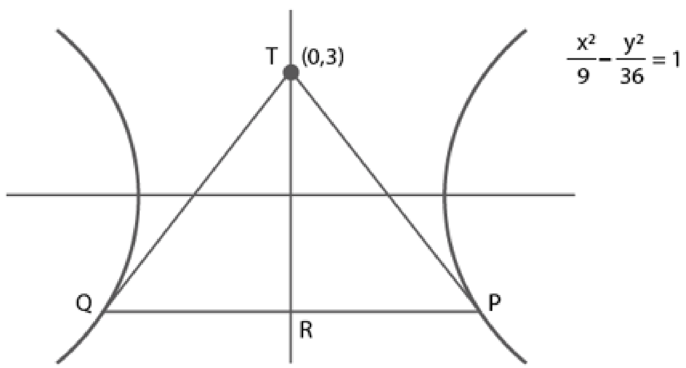
Solution:
The eccentricity of ellipse, e = $\sqrt{(1-b^2/a^2)}$
Given equation of ellipse $x^2 + 4y^2 = 4$ can be rewritten as $x^2/4 + y^2/1 = 1$.
eccentricity = √(1-1/4) = √3/2
Given: The eccentricity of the hyperbola $x^2/a^2 - y^2/b^2 = 1$ be the reciprocal to that of the ellipse $x^2 + 4y^2 = 4$.
=> eccentricity of hyperbola = 2/√3
Now,
=> $\sqrt{(1+b^2/a^2)}$ = 2/√3
=> $(1+b^2/a^2) = (2/\sqrt{3})^2$

=> b/a = 1/√3
Focus of ellipse = (±ae, 0) = (±√3, 0)
Hyperbola passes through Focus, so $3/a^2 = 1$ => a = √3
And b/a = 1/√3 = > b/√3 = 1/√3 => b = 1
The equation of hyperbola is $x^2/3 – y^2/1 = 1$
Or $x^2 – 3y^2 = 3$
Focus of hyperbola = (±ae, 0) = (±2, 0)

Question 9: Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q. If these tangents intersect at the point T(0, 3) then find the area of △PTQ. [Area in sq. units]

Solution:

Equation of hyperbola can be written as, $4x^2/36 - y^2/36 = 1$
or $x^2/9 - y^2/36 = 1$
Let the equation of tangent at any point $P(x_1, y_1)$ is $xx_1/a^2 - yy_1/b^2 = 1$
As tangent is passing through the point (0, 3) (Given)
So, $0/a^2 - 3y/36 = 1$
=> y = -12 and
Again, $4x^2 – y^2 = 36$
Using value of y, we have x = ±3√5



So, the coordinates of P and Q are (3√5, -12) and (-3√5, -12) respectively.
Now, area od triangle TPQ,

$$A = \frac{1}{2}\begin{vmatrix} 3\sqrt{5} & -12 & 1 \\ -3\sqrt{5} & -12 & 1 \\ 0 & 3 & 1 \end{vmatrix}$$

= 1/2|3√5(-12 – 3) + 12 (-3√5 – 0) + 1(-9√5 – 0)|
= 1/2|-45√5 - 36√5 - 9√5|
= 1/2|90√5|
= 45√5

Question 10: The circle $x^2 + y^2 = 8x$ and hyperbola $x^2/9 - y^2/4 = 1$ intersect at the points A and B. Find the equation of a common tangent with positive slope to the circle as well as to the hyperbola.
Solution:
The equation of circle $x^2 + y^2 = 8x$ can be rewritten as $(x - 4)^2 + y^2 = 16$
Tangent to hyperbola is $y = mx + \sqrt{(9m^2-4)}$, $m > 0$.
Distance from center to the tangent is:

$$\left|\frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1 + m^2}}\right| = 4$$

on solving above equation, we get
$m = 2/\sqrt{5}$
$\Rightarrow y = 2x/\sqrt{5} + 4/\sqrt{5}$
Or $2x - \sqrt{5}\, y + 4 = 0$
Therefore, $2x - \sqrt{5}\, y + 4 = 0$ is equation of a common tangent with positive slope to the circle as well as to the hyperbola.