Question 1: A value of x satisfying the equation $\sin[\cot^{-1}(1 + x)] = \cos[\tan^{-1}x]$, is:
(a) ½ (b) -1 (c) 0 (d) -1/2
Answer: (d)
Solution:
Given equation is: $\sin[\cot^{-1}(1 + x)] = \cos[\tan^{-1}x]$
Let $\cot^{-1}(1 + x)] = a \Rightarrow \cot a = 1 + x$ …(1)
And $\tan^{-1}x = b \Rightarrow x = \tan b$ …(2)
Solve (1) for a in terms of sin function:
$\cot a = 1 + x$
We know, $\operatorname{cosec} a = \sqrt{(1+\cot^2 a)} = \sqrt{(1 + (1 + x)^2} = \sqrt{(x^2 + 2x + 2)}$
Also, $\sin a = 1/\operatorname{cosec} a$
$\Rightarrow \sin a = 1/\sqrt{(x^2 + 2x + 2)}$
Or $a = \sin^{-1}[1/ \sqrt{(x^2 + 2x + 2)}]$
Solve (2) for b in terms of cos function:
$x = \tan b$
We know, $\sec b = \sqrt{(1+\tan^2 b)} = \sqrt{(1+x^2)}$
Also, $\cos b = 1/\sec b = 1/\sqrt{(1+x^2)}$
Or $b = \cos^{-1}[1/\sqrt{(1+x^2)}]$
Given equation =>
$\sin(\sin^{-1}[1/ \sqrt{(x^2 + 2x + 2)}]) = \cos[\cos^{-1}(1/\sqrt{(1+x^2)})]$
$\Rightarrow 1/ \sqrt{(x^2 + 2x + 2)} = 1/\sqrt{(1+x^2)}$
Squaring both sides and solving, we get
$2x = -1$
Or $x = -1/2$

Question 2: Let $f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$, where $|x| > 1$. If dy/dx = 1/2 d/dx(sin-1(f(x))) and $y(\sqrt{3}) = \pi/6$, then $y(-\sqrt{3})$ is equal to
(a) $\pi/3$ (b) $2\pi/3$ (c) $-\pi/6$ (d) $\pi/7$

Solution:
$f(x) = (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 - 1$
Let $\tan^{-1}(x) = A$ where $A \in (-\pi/2, -\pi/4) \cup (\pi/4, \pi/2)$
$\Rightarrow (\sin(\tan^{-1}x) + \sin(\cot^{-1}x))^2 – 1 = (\sin A + \cos B)^2 – 1$
$= 1 + 2 \sin A \cos A – 1$
$= \sin 2A$
$= 2x/(1+x^2)$
Given that, $dy/dx = (1/2)\, d/dx(\sin^{-1}(f(x)))$
$\Rightarrow dy/dx = -1/(1+x^2)$ for $|x| > 1$
($x > 1$ and $x< -1$)

To find the value of $y(-\sqrt{3})$, integrate dy/dx. To integrate the expression, interval should be continuous. So we have to integrate the expression in both the intervals.

=>y = $-\tan^{-1}x + c_1$ for x>1 and y = $-\tan^{-1}x + c_2$ for x < -1
For x > 1, $c_1 = \pi/2$ [because y(√3) = π/6]

But c_2 can't be determined as no other information is given for x < −1, so can't determine the value of c_2. Therefore, all the options can be true.

Question 3: If $\alpha = 3\sin^{-1}(6/11)$ and $\beta = 3\cos^{-1}(4/9)$ where the inverse trig functions take only the principal values, then the right option is
(a) cosβ > 0 (b) cos(α + β) > 0 (c) sin β < 0 (d) cosα < 0
Answer: (d)

Solution:
$\alpha = 3\sin^{-1}(6/11)$ and $\beta = 3\cos^{-1}(4/9)$
As 6/11 > 1/2 => $\sin^{-1}(6/11) > \sin^{-1}(1/2)$
=> $3\sin^{-1}(6/11) > 3\sin^{-1}(1/2) = \pi/2$
Therefore, α > π/2
and cosα < 0

Question 4: If $f'(x) = \tan^{-1}(\sec x + \tan x)$, $-\pi/2 < x < \pi/2$, and f(0) = 0 then f(1) is equal to
(a) (π+1)/4 (b) (π+2)/4 (c) ¼ (d) (π-1)/4
Answer: (a)

Solution:
$f'(x) = \tan^{-1}(\sec x + \tan x) = \tan^{-1}(1/\cos x + \sin x/\cos x) = \tan^{-1}[(1+\sin x)/\cos x]$

$$f'(x) = \tan^{-1}\left(\frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2} + 2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}\right)$$

$$f'(x) = \tan^{-1}\left[\frac{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)^2}{\left(\cos\frac{x}{2} + \sin\frac{x}{2}\right)\left(\cos\frac{x}{2} - \sin\frac{x}{2}\right)}\right]$$

$$f'(x) = \tan^{-1}\left[\frac{1 + \tan\frac{x}{2}}{1 - \tan\frac{x}{2}}\right]$$

$$f'(x) = \tan^{-1}\left[\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]$$

$f'(x) = \pi/4 + x/2$
=> $f(x) = (\pi/4)x + x^2/4 + c$
At f(0) = 0 => c = 0
At $f(1) = \pi/4 + 1/4 = (\pi+1)/4$

Question 5: The value of

$$\tan^{-1}\left[\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$$

$|x| < 1/2$, $x \neq 0$, is equal to
(a) $\pi/4 + (1/2)\cos^{-1}x^2$
(b) $\pi/4 + \cos^{-1}x^2$
(c) $\pi/4 - (1/2)\cos^{-1}x^2$
(d) $\pi/4 - \cos^{-1}x^2$

Answer: (a)

Solution: Let $x^2 = \cos 2A \Rightarrow A = (1/2)\cos^{-1}(x^2)$ ...(1)

$$tan^{-1}\left[\frac{\sqrt{1+cos\ 2A}+\sqrt{1-cos\ 2A}}{\sqrt{1+cos\ 2A}-\sqrt{1-cos\ 2A}}\right]$$

$$= tan^{-1}\left[\frac{\sqrt{2cos^2\ A}+\sqrt{2sin^2\ A}}{\sqrt{2cos^2 A}-\sqrt{2sin^2 A}}\right]$$

$$= tan^{-1}\left[\frac{\sqrt{2}cos A+\sqrt{2}sin A}{\sqrt{2}cos A-\sqrt{2}sin A}\right]$$

$$= tan^{-1}\left[\frac{1+tan\ A}{1-tan\ A}\right]$$

We know, $\tan \pi/4 = 1$, using in above equation, we get
$= \tan^{-1}(\tan(\pi/4 + A))$
$= \pi/4 + A$
Using (1)
$= \pi/4 + (1/2)\cos^{-1}(x^2)$

Question 6: Find the value of x satisfying the equation $\sin[\cot^{-1}(1+x)] = \cos[\tan^{-1}x]$, is
(a) -1/2 (b) -1 (c) 0 (d) ½

Answer: (a)

Solution:

We know, $\cot^{-1} x = \sin^{-1} [1/\sqrt{(1+x^2)}]$ ...(1)

Also, we know

$\tan^{-1} x = \cos^{-1} [1/\sqrt{(1+x^2)}]$

=> $sin[sin^{-1}\frac{1}{1+(1+x)^2}] = cos(tan^{-1}x)$ ...(2)

Using above result, given equation become,

(2)=> $sin[sin^{-1}\frac{1}{1+(1+x)^2}] = cos[cos^{-1}\frac{1}{\sqrt{1+x^2}}]$

or $\frac{1}{1+(1+x)^2} = \frac{1}{\sqrt{1+x^2}}$

[As $\sin^{-1}$ (sin A) = A ∈ (-π/2, π/2) and cos-1(cos A) = A ∈ [0, π]]

Solving above equation, we get

$1 + (1 + x)^2 = 1 + x^2$

or x = -1/2

Question 7: The value of

$cot[\sum_{n=1}^{23} cot^{-1}(1 + \sum_{k=1}^{n} 2k)]$ is

(a) 23/25 (b) 25/23 (c) 23/24 (d) 24/23

Answer: (b)

Solution:

$$cot[\sum_{n=1}^{23} cot^{-1}(1+ \sum_{k=1}^{n} 2k)]$$

$$= cot[\sum_{n=1}^{23} cot^{-1}(1+ 2\times \frac{n(n+1)}{2}]$$

$$= cot[\sum_{n=1}^{23} cot^{-1}(n^2 + n + 1)]$$

$$= cot[\sum_{n=1}^{23} tan^{-1}(\frac{n+1-n}{1+n(1+n)})]$$

$$= cot[\sum_{n=1}^{23} tan^{-1}(n+ 1)–tan^{-1}n]$$

$$= cot[tan^{-1}(24)–tan^{-1}(1)]$$

$$= cot[tan^{-1}\frac{23}{25}]$$

$$= cot[cot^{-1}\frac{25}{23}]$$

= 25/23

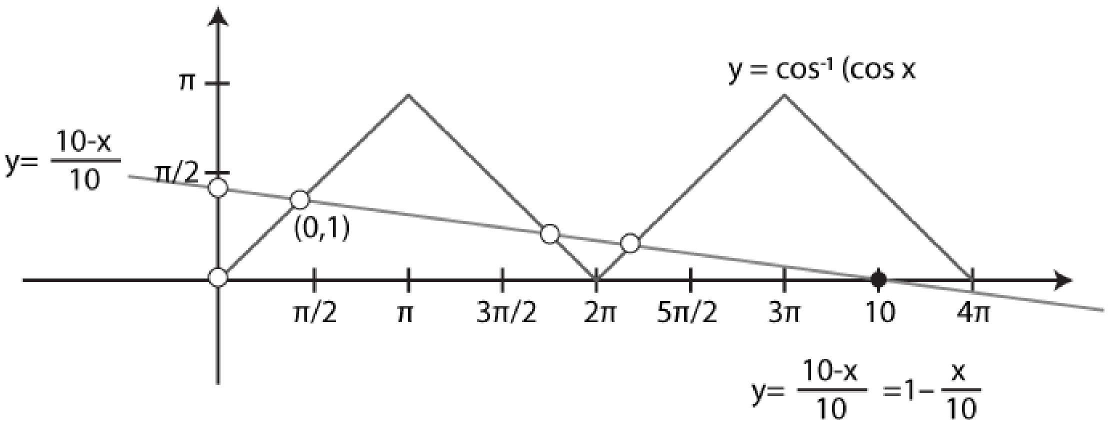
Question 8: Let f: [0, 4π] -> [0, π] be defined by f(x) = $\cos^{-1}$(cos x). The number of points x Є [0, 4π] satisfying the equation f(x) = (10-x)/10 is

(a) 1 (b) 2 (c) 3 (d) None of these

Answer: (c)

Solution:

Draw graph for f(x) = $\cos^{-1}$(cos x) and f(x) = (10-x)/10



Both the equations intersect at three different points, so the number of solutions be 3.

Question 9: Let f(x) = x $\cos^{-1}$(sin(-|x|)), x ∈ (-π/2, π/2) then which of the following is true?
(a) f′(0) = -π/2
(b) f' is decreasing in (-π/2, 0) and increasing in (0, π/2)
(c) f is not differentiable at x = 0
(d) f' is increasing in (-π/2, 0) and decreasing in (0, π/2)
Answer: (b)

Solution:
f(x) = x $\cos^{-1}$(sin(-|x|)) (Given)
=> f(x) = x $\cos^{-1}$(-sin(|x|))
[As sine is an odd function]
=> f(x) = x [π - $\cos^{-1}$(sin(|x|))
=> f(x) = x [π - (π/2 - $\sin^{-1}$(sin(|x|))
=> f(x) = x(π/2 + |x|)

$$=> f(x) = \begin{cases} x(\frac{\pi}{2}+x) & x \geq 0 \\ x(\frac{\pi}{2}-x) & x < 0 \end{cases}$$

$$=> f'(x) = \begin{cases} (\frac{\pi}{2}+x) & x \geq 0 \\ (\frac{\pi}{2}-x) & x < 0 \end{cases}$$

Therefore, f'(x) is decreasing (-π/2, 0) and increasing in (0, π/2).