Question 1: The function f : [0, 3] -> [1, 29] defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$ is
(a) one-one and onto
(b) onto but not one-one
(c) one-one but not onto
(d) neither one-one nor onto
Answer: (b)

Solution:
The function f : [0, 3] -> [1, 29] defined by
$f(x) = 2x^3 - 15x^2 + 36x + 1$
$f'(x) = 6x^2 - 30x + 36$
$= - 6(x^2 - 5x + 6)$
$= 6(x - 2)(x - 3)$

f(x) is increasing in [0, 2] and decreasing in [2, 3].

We know, if function is strictly increasing or decreasing in its domain, then it is one-one. But given function is increasing as well as decreasing.

So, f(x) is many one.
$f(0) = 1$
$f(2) = 29$ and
$f(3) = 28$
Range is [1, 29].
The function is onto.

Question 2: Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying (f o g o g o f)(x) = (g o g o f)(x), where (f o g)(x) = f(g(x)) is.
Solution:
$(g \circ f)(x) = g(f(x) = \sin x^2$

$g \circ (g \circ f)(x) = g(g(f(x)) = \sin(\sin x^2)$

$(f \circ g \circ g \circ f)(x) = [\sin(\sin x^2)]^2$ ...(i)

Again,

$(g \circ g \circ f)(x) = \sin(\sin^2 x)$ ...(ii)
Now,
$[\sin(\sin x^2)]^2 = \sin(\sin x^2)$
$\sin(\sin x^2)(\sin(\sin x^2 - 1) = 0$

=>$\sin x^2 = 0$, $\sin x^2 = \pi/2$

But, $\sin x^2 = \pi/2$ can not be a solution as $\sin x \in [-1, 1]$

=>$x^2 = n\pi$ or $x = \pm\sqrt{n\pi}$ where $n \in \{0, 1, 2, 3,.....\}$

Question 3: A function f from the set of natural numbers to integers defined by
$f(n) = (n - 1)/2$, when n is odd, and
$f(n) = -n/2$, when n is even
(a) one-one but not onto
(b) onto but not one-one
(c) one-one and onto both
(d) neither one-one nor onto
Answer: (c)

Solution:
If n = odd: clearly shown that function is one-one and onto.
For example, if f: N -> I
Let x, y ∈ N and both are even.
so, f(x) = f(y) => -x/2 = -y/2 => x = y

If n = even, values are set of all negative integers. Again, function is one-one and onto.

For example, if f: N -> I
Let x, y ∈ N and both are odd.
so, f(x) = f(y) => (x-1)/2 = (y-1)/2 => x = y

Therefore, option (c) is correct.

Question 4: The range of the function $f(x) = 3|\sin x| - 2|\cos x|$ is
(a) $[-2, \sqrt{13}]$
(b) $[-2, 3]$
(c) $[-3, 2]$
(d) $[3, \sqrt{13}]$
Answer: (c)

Solution:
$f(x) = 3|\sin x| - 2|\cos x|$

If f(x) is continuous function and $|\sin x|$ and $|\cos x|$ are always positive.

Find Minimum and Maximum value of f(x):

f(x) is minimum when |sin x| = 0 and |cos x| = 1

The minimum value will be = 0 - 3 = -3

f(x) is max when |sin x| = 1 and |cos x| = 0

The max value will be = 2 - 0 = 2

The required range is [-3, 2]

Question 5: If f:R-> R satisfy f(x + y) = f(x) + f(y), for all x, y ∈ R and f(1) = 7 then$\sum_{r=1}^{n} f(r)$ is
(a) 7n(n + 1)
(b) 7n/2
(c) 7(n+1)/2
(d) 7n(n+1)/2
Answer: (d)

Solution:
f(x + y) = f(x) + f(y) for all x, y ∈ R

f(1 + 1) = 2 f(1) = 2(7)

f(2) = 2(7)

Therefore, f(3) = f(1) + f(2) = 7 + 2(7) = 3(7)

=> f(x) = ax
=> a(1) = 7 => a = 7

f(x) = 7x

=> $\sum_{r=1}^{n} f(r) = 7(1 + 2r + \ldots + n) = \frac{7n(n+1)}{2}$

Question 6: The range of the function f(x) = ${}^{7-x}P_{x-3}$ is
(a) {1, 2, 3}
(b) {1, 2, 3, 4}
(c) {1, 2, 3, 4, 5}
(d) {1, 2, 3, 4, 5, 6}
Answer: (a)

Solution:
Given function is $f(x) = {}^{7-x}P_{x-3}$
When $7 - x \geq 0 \Rightarrow x \leq 7$
When $x - 3 \geq 0 \Rightarrow x \geq 3$
And $7 - x \geq x - 3 \Rightarrow x \leq 5$
This implies, $3 \leq x \leq 5$
$\Rightarrow x = 3, 4, 5$
So, the range is {1, 2, 3}

Question 7: If $f_x(x) = 1/k (\sin^k x + \cos^k x)$ where $x \in R$ and $k \geq 1$, then $f_4(x) - f_6(x)$ is

(a) 1/6 (b) 1/4 (c) 1/5 (d) 1/12
Answer: (d)
Solution:
$f_x(x) = 1/k (\sin^k x + \cos^k x)$
At x = 4 and at x = 6
$f_4(x) = 1/4 (\sin^4 x + \cos^4 x)$ and $f_6(x) = 1/6 (\sin^6 x + \cos^6 x)$

Now, $f_4(x) - f_6(x) = 1/4 (\sin^4 x + \cos^4 x) - 1/6 (\sin^6 x + \cos^6 x)$ …(1)

Using identity: $(a^4 + b^4) = (a^2 + b^2)^2 - 2a^2b^2$ and $(a^6 + b^6) = (a^2 + b^2)^3 - 3a^2b^2(a^2 + b^2)$ and also, we know, $\sin^2 x + \cos^2 x = 1$

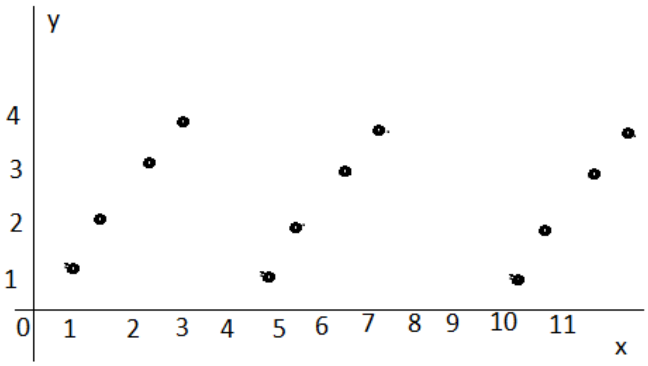
(1) $\Rightarrow f_4(x) - f_6(x) = 1/4(1 - 2 \sin^2 x \cos^2 x) - 1/6 (1 - 3 \sin^2 x \cos^2 x)$
$= 1/4 - 1/6$
$= 1/12$

Question 8: The function f: N -> N defined by f(x) = x - 5[x/5], where N is the set of natural numbers and [x] denotes the greatest integer less than or equal to x, is
(a) one-one and onto
(b) one-one but not onto
(c) onto but not one-one
(d) neither one-one nor onto
Answer: (d)

Solution:
The function f: N -> N defined by f(x) = x - 5[x/5], where N is the set of natural numbers and [x] denotes the greatest integer.
Consider x in intervals of 5 natural numbers.

$$f(x) = \begin{cases} x-5(0) & 0 \le x < 5 \\ x-5(1) & 5 \le x < 10 \\ x-5(2) & 10 \le x < 15 \\ x-5(3) & 15 \le x < 20 \end{cases}$$

Where $x \in N$.



From the above graph, we can conclude that f(x) is neither one-one nor onto.

Question 9: Let f : (–1, 1) -> B, be a function defined by $f(x) = \tan^{-1}(2x)/(1 - x^2)$, then f is both one-one and onto when B is the interval

(a) $[0, \pi/2]$ (b) $(0, \pi/2)$ (c) $(-\pi/2, \pi/2)$ (d) $[-\pi/2, \pi/2]$

Answer: (d)

Solution:

$f(x) = \tan^{-1}(2x)/(1 - x^2)$

For $x \in (-1, 1)$

$f(x) = \tan^{-1} [2x/(1-x^2)]$

Replace x by tan A

$$f(tanA) = tan^{-1}\left(\frac{2tanA}{1-tan^2A}\right)$$

$f(\tan A) = \tan^{-1} (\tan 2A) = 2 \tan^{-1} A$

$\Rightarrow -\pi/2 < \tan^{-1} (2x/(1-x^2)) < \pi/2$

Question 10: The function f: R -> [-1/2, 1/2] defined as $f(x) = x/(1+x^2)$, is
(a) invertible
(b) Surjective but not injective
(c) Neither injective nor surjective
(d) injective but not surjective

Answer: (b)

Solution:
Given: $f(x) = x/(1+x^2)$
Differentiate above function:

$$f'(x) = \frac{(1+x^2)-2x^2}{(1+x^2)^2} = \frac{1-x^2}{1+x^2)^2}$$

function is not injective as it changes sign in different intervals.

Let $y = x/(1+x^2)$
or $yx^2 - x + y = 0$
for y = 0 => x = 0
[for y ≠ 0]

And Range: [-1/2, 1/2]

Therefore, function is surjective but not injective.