Question 1: The projection of any line on co-ordinate axes be, respectively 3, 4, 5 then its length is

______.

Solution:

Let the line segment be AB, then as given AB cos α = 3, AB cos β = 4, AB cos γ = 5

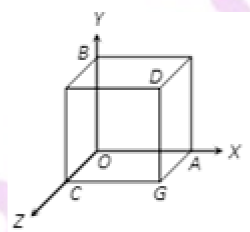
⇒ $AB^2 (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = 3^2 + 4^2 + 5^2$

AB = √[9 + 16 + 25] = 5√2,

where α, β and γ are the angles made by the line with the axes.

Question 2: The angle between two diagonals of a cube will be _____.

Solution:



Let the cube be of side 'a' and O (0, 0, 0), D (a, a, a), B (0, a, 0), G (a, 0, a).

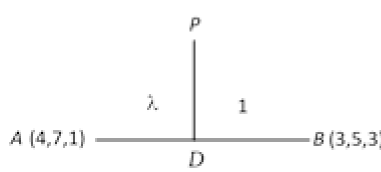
Then the equation of OD and BG are x/a = y/a = z/a and x/a = [y − a]/[−a] = z/a, respectively.

Hence, the angle between OD and BG is

$$\cos^{-1}\frac{a^2-a^2+a^2}{\sqrt{3}a^2\times\sqrt{3}a^2} = \cos^{-1}\left(\frac{1}{3}\right)$$

Question 3: The co-ordinates of the foot of perpendicular drawn from point P (1, 0, 3) to the line joining the points A (4, 7, 1) and B (3, 5, 3) is _______.

Solution:



Let D be the foot of perpendicular drawn from P (1, 0, 3) on the line AB joining (4, 7, 1) and (3, 5, 3).

If D divides AB in ratio λ : 1, then

$D = \ [(\frac{3\lambda+4}{\lambda+1}), (\frac{5\lambda+7}{\lambda+1}), (\frac{3\lambda+1}{\lambda+1})]$ .....(i)

Direction ratios of PD are 2λ + 3, 5λ + 7, −2.

Direction ratios of AB are −1, −2, 2 [Because PD⊥AB]

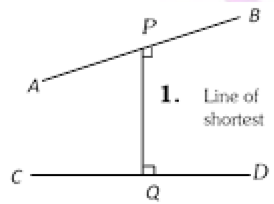
− (2λ + 3) −2 (5λ + 7) − 4 = 0

λ = −7 / 4

Putting the value of λ in (i), we get the point D (5/3, 7/3, 17/3).

Question 4: A line with direction cosines proportional to 2, 1, 2 meets each of the lines x = y + a = z and x + a = 2y = 2z. The coordinates of each of the points of intersection are given by __________.

Solution:



Let the two lines be AB and CD having equations

x / 1 = [y + a] / 1 = z / 1 = λ and [x + a] / [2] = y / 1 = z / 1 = μ then

P = (λ, λ − a, λ) and Q = (2μ − a, μ, μ)

So, according to question, [λ − 2μ + a] / [2] = [λ − a − μ] / [1] = [λ − μ] / [2]

μ = a and λ = 3a

Therefore, P = (3a, 2a, 3a) and $[(x-2)^2 + (y-3)^2 + (z-4)^2]$.

Question 5: The distance of the point (1, 2, 3) from the plane x − y + z = 5 measured parallel to the line x / 2 = y / 3 = z / −6, is __________.

Solution:

Direction cosines of line = (2 / 7, 3 / 7, −6 / 7)

Now, x′ = 1 + [2r /7], y′ = −2+ [3r / 7] and z′ = 3 − [6r / 7]

Therefore, (1 + [2r / 7]) − (−2 + [3r / 7]) + (3 − [6r / 7])

= 5

⇒ r = 1

Question 6: A point moves so that the sum of its distances from the points (4, 0, 0) and (4, 0, 0) remains 10. The locus of the point is _______.

Solution:

$$\sqrt{(x-4)^2 + y^2 + z^2} + \sqrt{(x+4)^2 + y^2 + z^2} = 10$$

$$2(x^2 + y^2 + z^2) + 2\sqrt{[(x-4)^2 + y^2 + z^2][(x+4)^2 + y^2 + z^2]} = 100 - 32 = 68$$

$\Rightarrow (x^2 + y^2 + z^2 - 34)^2 = [(x - 4)^2 + y^2 + z^2]\,[(x + 4)^2 + y^2 + z^2]$

$\Rightarrow (x^2 + y^2 + z^2)^2 - 68\,(x^2 + y^2 + z^2) + (34)^2$

$= [(x^2 + y^2 + z^2 + 16) - 8x]\,[(x^2 + y^2 + z^2 + 16) + 8x]$

$= (x^2 + y^2 + z^2 + 16)^2 - 64x^2$

$= (x^2 + y^2 + z^2) + 32\,(x^2 + y^2 + z^2) - 64x^2 + (16)^2$

$\Rightarrow 9x^2 + 25y^2 + 25z^2 - 225 = 0$

Question 7: The centre of the sphere passes through four points (0, 0, 0), (0, 2, 0), (1, 0, 0) and (0, 0, 4) is ________.

Solution:

Let the equation of sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

Because it passes through origin (0, 0, 0), d = 0 and also, it passes through (0, 2, 0)

$4 + 4v = 0 \Rightarrow v = -1$

Also, it passes through (1, 0, 0)

$1 + 2u = 0 \Rightarrow u = -1/2$

And it passes through (0, 0, 4)

$16 + 8w \Rightarrow w = -2$

Centre $(-u, -v, -w) = (1/2, 1, 2)$

Question 8: Co-ordinate of a point equidistant from the points (0,0,0), (a, 0, 0), (0, b, 0), (0, 0, c) is ___________.

Solution:

The required point is the centre of the sphere through the given points.

Let the equation of sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ .....(i)

Sphere (i) is passing through (0, 0, 0), (a, 0, 0), (0, b, 0) and (0, 0, c), hence, d = 0

$a^2 + 2ua = 0 \Rightarrow u = -a/2$

$b^2 + 2vb = 0 \Rightarrow v = -b/2$

$c^2 + 2wc = 0 \Rightarrow w = -c/2$

Therefore, centre of sphere is (a/2, b/2, c/2), which is also the required point.

Question 9: If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the midpoint of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals ____________.

Solution:

$S1 = x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$, $C1 = (-3, 4, 1)$

$S2 = x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, $C2 = (5, -2, 1)$

So, mid point of C1 C2 (say P) = P([5 − 3] / [2], [4 − 2] / [2], [1 + 1] / [2]) = P(1, 1, 1)

Now the plane $2ax - 3ay + 4az + 6 = 0$ passes through the point P,

So, $2a(1) - 3a(1) + 4a(1) + 6 = 0 = 2a - 3a + 4a + 6 = 0$

$3a + 6 = 0$

$3a = -6$

$\Rightarrow a = -2$

Question 10: The radius of the sphere $x + 2y + 2z = 15$ and $x^2 + y^2 + z^2 - 2y - 4z = 11$ is ____________.

Solution:

Equation of sphere is, $x^2 + y^2 + z^2 - 2y - 4z = 11$

Centre of sphere = (0, 1, 2) and radius of sphere = 4

Let the centre of the circle be (α, β, γ).

The direction ratios of a line joining from centre of the sphere to the centre of the circle are

(α − 0, β − 1, γ − 2) or (α, β − 1, γ − 2)

But, this line is normal at plane $x + 2y + 2z = 15$

$\alpha / 1 = [\beta - 1] / 2 = [\gamma - 2] / 2 = k$

$\alpha = k, \beta = 2k + 1, \gamma = 2k + 2$ [Because, centre of circle lies on $x + 2y + 2z = 15$]

$k + 2(2k + 1) + 2(2k + 2) = 15$

$\Rightarrow k = 1$

So, the centre of circle = (1, 3, 4)

Therefore, Radius of circle = $\sqrt{[(\text{Radius of sphere})^2 - (\text{Length of joining line of centre})^2]}$

$$\sqrt{4^2 - (1-0)^2 + (3-1)^2 + (4-2)^2} = \sqrt{16-9}$$

$= \sqrt{7}$

Question 11: The point at which the line joining the points (2, 3, 1) and (3, 4, 5) intersects the plane 2x + y + z = 7 is _________.

Solution:

$$\text{Ratio} = \frac{2(2)+(-3)(1)+(1)(1)-7}{2(3)+(-4)(1)+(-5)(1)-7} = \frac{-(-5)}{-10} = -(1/2)$$

$$\text{Hence, } x = \frac{2(2)-3(1)}{1} = 1,\ y = \frac{-3(2)-(-4)}{1} = -2 \text{ and } z = \frac{1(2)-(-5)}{1} = 7.$$

Therefore, P (1, −2, 7).

Trick : As (1, 2, 7) and (1, 2, 7) satisfy the equation 2x + y + z = 7, but the point (1, 2, 7) is collinear with (2, 3, 1) and (3, 4, 5).

Note: If a point divides the joining of two points in some particular ratio, then this point must be collinear with the given points.

Question 12: The point where the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ meets the plane $2x + 4y - z = 1$, is ___________.

Solution:

Let point be (a, b, c), then 2a + 4b − c = 1 .....(i) and a = 2k + 1,b = −3k + 2 and c = 4k − 3,

(where k is constant)

Substituting these values in (i), we get

2 (2k + 1) + 4 (−3k + 2) − (4k − 3) = 1

⇒ k = 1

Hence, the required point is (3, 1, 1).

Trick: The point must satisfy the lines and plane.

Question 13: The equation of the straight line passing through the point (a, b, c) and parallel to the z-axis, is ________.

Solution:

The line through (a,b,c) is

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad ....(i)$$

Since the line is parallel to z-axis, therefore any point on this line will be of the form (a, b, z_1).

Also, any point on line (i) is (lr + a, mr + b, nr + c).

Hence, lr + a = a & mr + b = b ⇒ l = m = 0

Hence, the line will be

$$\frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{1}$$

Question 14: The acute angle between the line joining the points (2, 1, 3), (3, 1, 7) and a line parallel to

$$\frac{x-1}{3} = \frac{y}{4} = \frac{z+3}{5}$$

through the point (1, 0, 4) is ____________.

Solution:

Direction ratio of the line joining the point (2, 1, −3), (−3, 1, 7) are (a_1, b_1, c_1) ⇒ (−3 −2, 1 − 1, 7 − (−3))

⇒ (−5, 0, 10)

Direction ratio of the line parallel to line $\frac{x-1}{3} = \frac{y}{4} = \frac{z+3}{5}$

are (a_2, b_2, c_2)

$\Rightarrow (3, 4, 5)$

Angle between two lines,

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)} \times \sqrt{(a_2^2 + b_2^2 + c_2^2)}}$$

$$\cos\theta = \frac{-5\times 3 + 0\times 4 + 10\times 5}{\sqrt{(25+0+100)} \times \sqrt{(9+16+25)}}$$

$$\cos\theta = \frac{35}{25\sqrt{10}}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{7}{5\sqrt{10}}\right)$$

Question 15: If l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of two perpendicular lines, then the direction cosine of the line which is perpendicular to both the lines, will be ______.

Solution:

Let lines are $l_1x + m_1y + n_1z + d = 0$ ..(i) and

$l_2x + m_2y + n_2z + d = 0$ .....(ii)

If $lx + my + nz + d = 0$ is perpendicular to (i) and (ii), then, $ll_1 + mm_1 + nn_1 = 0$, $ll_2 + mm_2 + nn_2 = 0$

$$\Rightarrow \frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - l_1 n_2} = \frac{n}{l_1 m_2 - l_2 m_1} = d$$

Therefore, direction cosines are $(m_1n_2 - m_2n_1)$, $(n_1l_2 - l_1n_2)$, $(l_1m_2 - l_2m_1)$.