ISC Class 11 Maths Specimen Question Paper 2018 with Solutions

SECTION - A

Question 1

(i) Let f : R → R be a function defined by f(x) = (x - m)/ (x - n), where m ≠ n. Then show that f is one-one but not onto.

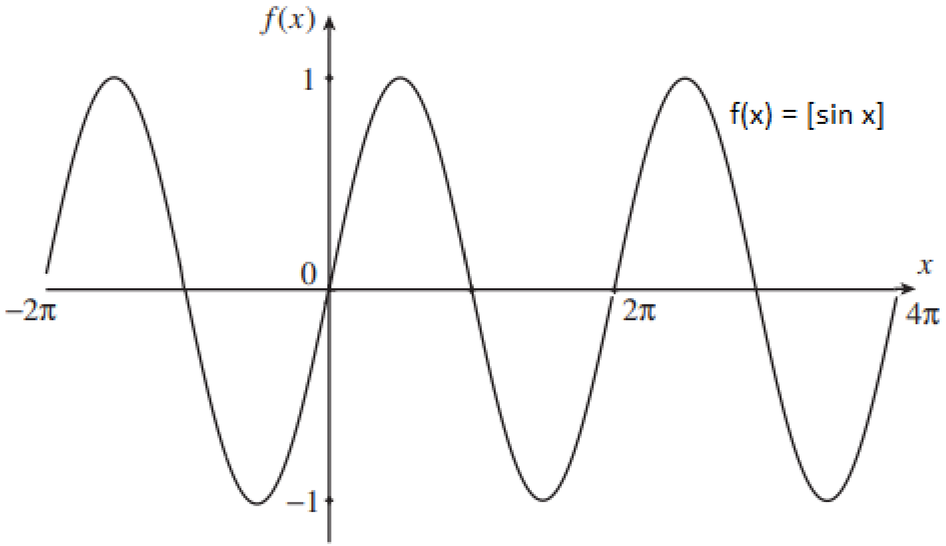
Solution:
Given,
$f(x) = (x - m)/ (x - n), m \neq n$
$f'(x) = [(x - n)(1) - (x - m)(1)]/ (x - n)^2$
$= (m - n)/ (x - n)^2$
Thus, $f'(x) < 0$ or $f'(x) > 0$
Therefore, f is one one.
Let $y = (x - m)/ (x - n)$
$xy - ny = x - m$
$xy - x = ny - m$
$x(y - 1) = ny - m$
$x = (ny - m)/ (y - 1)$
Thus, y should not equal 1.
$y \in R - \{1\}$
Therefore, f is one-one but not onto.

Alternative method:

Let x_1, x_2 be the two elements in the domain R such that $f(x_1) = f(x_2)$.
$\Rightarrow (x_1 - m)/ (x_1 - n) = (x_2 - m)/ (x_2 - n)$
$\Rightarrow (x_1 - m)(x_2 - n) = (x_1 - n)(x_2 - m)$
$\Rightarrow x_1x_2 - nx_1 - mx_2 + mn = x_1x_2 - mx_1 - nx_2 + mn$
$\Rightarrow mx_1 - nx_1 = mx_2 - nx_2$
$\Rightarrow (m - n)x_1 = (m - n)x_2$
$\Rightarrow x_1 = x_2$
Therefore, f is one one.
Let $f(x) = y$
$y = (x - m)/ (x - n)$
$xy - ny = x - m$
$xy - x = ny - m$
$x(y - 1) = ny - m$
$x = (ny - m)/ (y - 1)$
Thus, y should not equal 1.
$y \in R - \{1\}$
Therefore, f is one-one but not onto.

(ii) Find the domain and range of the function f(x) = [sin x].

Solution:
Given,
f(x) = [sin x]



Domain = $(-\infty, \infty)$
or
Domain = $\{x: x \in R\}$
And
$-1 \leq \sin x \leq 1$
Range = [-1, 1]

(iii) Find the square root of the complex number 11 - 60i.

Solution:
Given,
11 - 60i
Comparing with a + ib,
a = 11, b = -60 < 0

$$\sqrt{11-60i} = \pm\left[\left(\frac{11+\sqrt{(\ 11)^2+(60)^2}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-11+\sqrt{(\ 11)^2+(-60)^2}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{11+\sqrt{121+3600}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-11+\sqrt{121+3600}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{11+\sqrt{3721}}{2}\right)^{\frac{1}{2}} - i\left(\frac{-11+\sqrt{3721}}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{11+61}{2}\right)^{\frac{1}{2}} - i\left(\frac{-11+61}{2}\right)^{\frac{1}{2}}\right]$$

$$= \pm\left[\left(\frac{72}{2}\right)^{\frac{1}{2}} - i\left(\frac{50}{2}\right)^{\frac{1}{2}}\right]$$

$= \pm [(36)^{½} - i (25)^{½}]$
$= \pm [6 - i5]$
Therefore, the square root of 11 - 6i is ± (6 - i5).

(iv) For what value of k will the equations $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ have one common root.

Solution:
Given,
$x^2 - kx - 21 = 0$....(i)
$x^2 - 3kx + 35 = 0$....(ii)
Let α be the common root of (i) and (ii),
$\Rightarrow \alpha^2 - k\alpha - 21 = 0$....(iii)
And
$\alpha^2 - 3k\alpha + 35 = 0$....(iv)
Subtracting (iv) from (iii),
$\alpha^2 - k\alpha - 21 - (\alpha^2 - 3k\alpha + 35) = 0$
$(3\alpha - \alpha)k - 56 = 0$
$2\alpha k = 56$
$\alpha = 56/2k$
$\alpha = 28/k$
Substituting α = 28/k in (iii),
$(28/k)^2 - k(28/k) - 21 = 0$
$(28/k)^2 - 28 - 21 = 0$
$(28/k)^2 = 49$
$\Rightarrow 28/k = \pm 7$
$\Rightarrow k = \pm 28/7$
$\Rightarrow k = \pm 4$

(v) In a ΔABC, show that ∑(b + c) cos A = 2s where, s = (a + b + c)/2

Solution:
∑(b + c) cos A = (b + c) cos A + (c + a) cos B + (a + b) cos C
= b cos A + c cos A + c cos B + a cos B + a cos C + b cos C

= (b cos A + a cos B) + (c cos A + a cos C) + (c cos B + b cos C)
= c + b + a…(i)
We know that,
In a ΔABC, s = (a + b + c)/2
2s = a + b + c….(ii)
From (i) and (ii),
$\sum(b + c) \cos A = 2s$

(vi) Find the number of ways in which 6 men and 5 women can dine at a round table if no two women are to sit together.

Solution:
Given,
Number of men = 6
Number of women = 5
The number of ways of 6 men can sit at a round table = (n – 1)! = (6 – 1)!= 5!
There will be six places between the 6 men.
Thus, 5 women can sit in $6P_5$ ways.
Required number of ways = 5! × $6P_5$
= 5! × 6!

(vii) Prove that sin 20° sin 40° sin 80° = √3/8.

Solution:
LHS = sin 20° sin 40° sin 80°
= (1/2) (2 sin 20° sin 40°) sin 80°
= (1/2) [cos (20° - 40°) - cos (20° + 40°)] sin 80°
= (1/2) [cos (-20°) - cos 60°] sin 80°
= (1/2) [cos 20° - (1/2)] sin 80
= (1/2) sin 80° cos 20° - (1/4) sin 80°
= (1/2) sin (90° - 10°) cos 20° - (1/4) sin 80°
= (1/4) [2 cos 10° cos 20°] - (1/4) sin 80°
= (1/4) [cos (10° + 20°) + cos (10° - 20°)] - (1/4) sin 80°]
= (1/4) [cos 30° + cos (-10°)] - (1/4) sin 80°
= (1/4) [cos 30° + cos 10°] - (1/4) sin 80°
= (1/4) cos 30° + (1/4) cos (90° - 80°) - (1/4) sin 80°
= (1/4) (√3/2) + (1/4) sin 80° - (1/4) sin 80°
= √3/8
= RHS
Hence proved.

(viii) If two dice are thrown simultaneously, find the probability of getting a sum of 7 or 11.

Solution:
Given,
Two dice are thrown simultaneously.
Thus, the total number of outcomes = n(S) = 36
Let A be the event of getting a sum of 7.
A = {(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)}
n(A) = 6

P(A) = n(A)/n(S) = 6/36
Let B be the event of getting a sum of 11.
B = {(5, 6), (6, 5)}
n(B) = 2
P(B) = n(B)/n(S) = 2/36
P(A U B) = P(A) + P(B)
= (6/36) + (2/36)
= 8/36
= 2/9
Hence, the probability of getting a sum of 7 or 11 is 2/9.

(ix)

Show that $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

Solution:

Left hand limit:

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2} = \lim_{x \to 2^-} \frac{2-x}{x-2} = \lim_{x \to 2^-} (-1) = -1$$

Right hand limit:

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2} = \lim_{x \to 2^+} \frac{x-2}{x-2} = \lim_{x \to 2^+} 1 = 1$$

Left hand limit ≠ Right hand limit

Therefore, $\lim_{x \to 2} \frac{|x-2|}{x-2}$ does not exist.

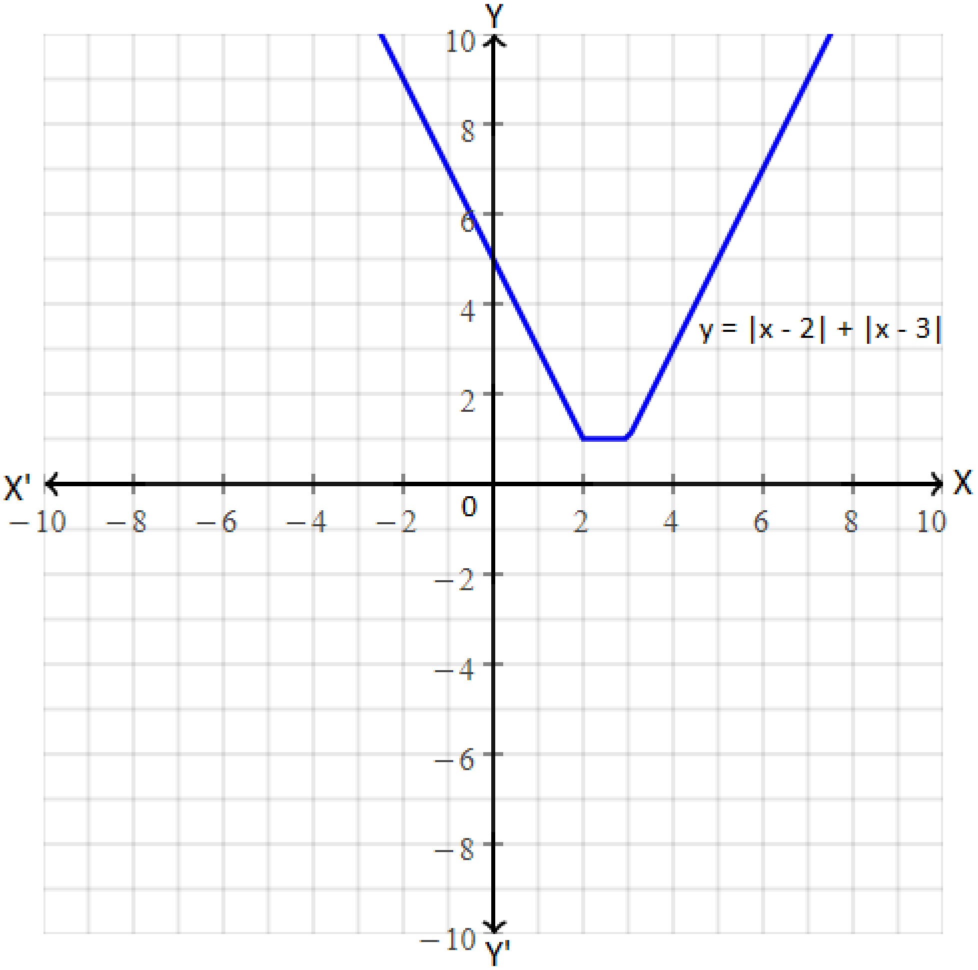
(x) Find the point on the curve $y^2 = 4x$, the tangent at which is parallel to the straight line y = 2x + 4.

Solution:
Given curve is:
$y^2 = 4x$….(i)
Equation of the straight line is y = 2x + 4
This is of the form y = mx + c,
Slope = m = 2
Differentiating the equation of given curve,
2y (dy/dx) = 4
dy/dx = (4/2y)
dy/dx = 2/y
We know that the slope of the tangent at a point is given by the value of the derivative of the equation of the curve at that point.

2/y = 2
⇒ y = 1
Substituting y = 1 in (i),
$(1)^2 = 4x$
⇒ x = 1/4
Hence, the required point is (¼, 1).

Question 2
Draw the graph of the function y = |x - 2| + |x - 3|.

Solution:
|x - 2| = x - 2 if x > 2, and |x - 2| = -x + 2 if x < 2
Similarly,
|x - 3| = x - 3 if x > 3, and |x - 3| = -x + 3 if x < 3.
If x < 2, then |x - 2| + |x - 3| = -2x + 5.
If x > 3, then |x - 2| + |x - 3| = 2x - 5.
Now consider another case:
When x is between the values of 2 and 3 then |x - 2| = x - 2 and |x - 3| = -x + 3.
⇒ |x - 2| + |x - 3| = (x - 2) + (-x + 3) = 1
Thus, for x values between 2 and 3, we get the equation y = 1, which is just a horizontal line.



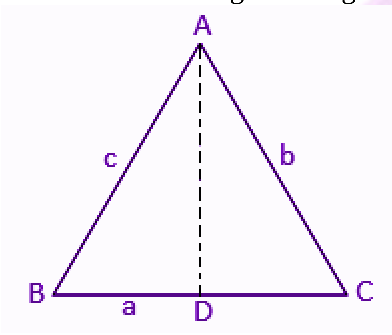
Question 3
Prove that cot A + cot(60 + A) + cot(120 + A) = 3 cot 3A.

Solution:
LHS = cot A + cot (60 + A) + cot (120 + A)
= cot A + cot (60 + A) + cot [180 - (60 - A)]
= cot A + cot (60 + A) - cot (60 - A)
= cot A + [(cot 60 cot A - 1)/ (cot 60 + cot A)] - [(cot 60 cot A + 1)/ (cot A - cot 60)]
= cot A + {[(1/√3)cot A - 1]/ [(1/√3) + cot A]} - {[(1/√3) cot A + 1]/ [cot A - (1/√3)]}
= cot A + [(cot A - √3)/ (√3 cot A + 1)] - [(cot A + √3)/ (√3 cot A - 1)]
= cot A + [(cot A - √3)(√3 cot A - 1) - (cot A + √3)(√3 cot A + 1)]/ [(√3 cot A + 1)(√3 cot A - 1)]
= cot A + [(√3 $\cot^2$A - cot A - 3 cot A + √3 - √3 $\cot^2$A - cot A - 3 cot A - √3)/ (3 $\cot^2$A - 1)
= cot A + [(-8 cot A)/ (3 $\cot^2$A - 1)
= (3 $\cot^3$A - 9 cot A)/ (3 $\cot^2$A - 1)
= 3[($\cot^3$A - 3 cot A)/ (3 $\cot^2$A - 1)]
= 3 cot 3A
= RHS
Hence proved.

OR

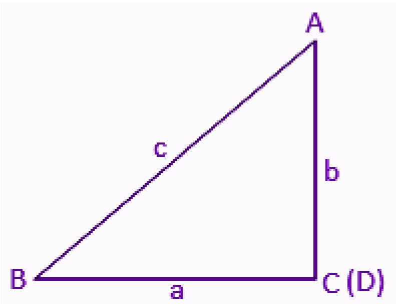
In a ∆ABC prove that b cos C + c cos B = a.

Solution:
Let ABC be a triangle.
Case 1:
ABC is an acute-angled triangle.



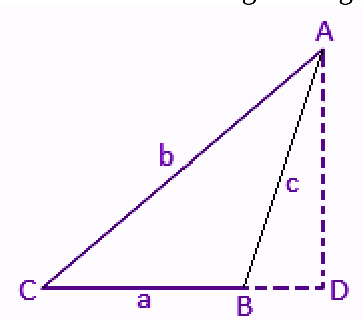
a = BC = BD + CD....(i)
cos B = BD/AB
⇒ BD = AB cos B
⇒ BD = c cos B (AB = c)
And
cos C = CD/AC
⇒ CD = AC cos C
⇒ CD = b cos C (AC = b)
From the above,
a = c cos B + b cos C

Case 2:
ABC is a right-angled triangle.



a = BC
cos B = BC/AB
⇒ BC = AB cos B
⇒ BC = c cos B, [since, AB = c]
a = c cos B
⇒ a = c cos B + 0
b = c cos A + a cos C
⇒ a = c cos B + b cos C (C = 90° ⇒ cos C = cos 90 = 0)

Case 3:
ABC is an obtuse angle triangle.



a = BC = CD - BD
cos C = CD/AC
⇒ CD = AC cos C
⇒ CD = b cos C (since AC = b)
And,
cos (π - B) = BD/AB
⇒ BD = AB cos (π - B)
a = b cos C + c cos B
⇒ BD = -c cos B [since, AB = c and cos (π - θ) = -cos θ]
From the above,
a = b cos C - (-c cos B)
⇒ a = b cos C + c cos B

Question 4
Find the locus of a complex number, Z = x + iy, satisfying the relation |(z - 3i)/ (z + 3i)| ≤ √2. Illustrate the locus of Z in the argand plane.

Solution:
Given,
z = x + iy

$$\left|\frac{z-3i}{z+3i}\right| \le \sqrt{2}$$

$$\Rightarrow \left|\frac{x+iy-3i}{x+iy+3i}\right| \le \sqrt{2}$$

$$\Rightarrow |x+i(y-3)| \le \sqrt{2}\ |x+i(y+3)|$$

$$\Rightarrow \sqrt{x^2+(y-3)^2} \le \sqrt{2}\sqrt{x^2+(y+3)^2}$$

Squaring on both sides,

$x^2 + (y - 3)^2 \le 2 [x^2 + (y + 3)^2]$

$x^2 + y^2 + 9 - 6y \le 2(x^2 + y^2 + 9 + 6y)$

$x^2 + y^2 + 9 - 6y \le 2x^2 + 2y^2 + 18 + 12y$

$\Rightarrow 2x^2 + 2y^2 + 18 + 12y - x^2 - y^2 - 9 + 6y \ge 9$

$\Rightarrow x^2 + y^2 + 18y + 9 \ge 0$

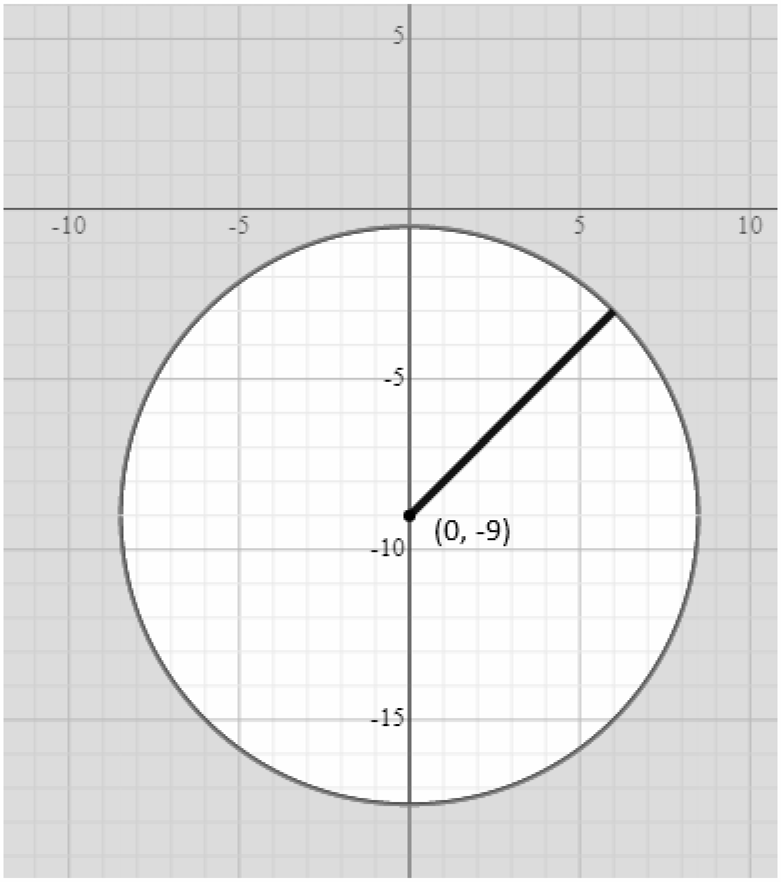
$\Rightarrow x^2 + (y^2 + 18y + 81) + 9 \ge 81$

$\Rightarrow x^2 + (y + 9)^2 \ge 81 - 9$

$\Rightarrow x^2 + [y - (-9)]^2 \ge 72$

$\Rightarrow (x - 0)^2 + [y - (-9)]^2 \ge (6\sqrt{2})^2$

Therefore, the locus of z represents a circle with centre (0, -9) and radius 6√2 units.



Question 5
Find the number of words which can be formed by taking four letters at a time from the word "COMBINATION".

Solution:
Given word is:
COMBINATION
Number of letters = 11
O - 2 times
I - 2 times
N - 2 times
Thus, 8 distinct letters and 3 letters repeated twice.
Case 1: Selecting 4 different letters as one word
In this case, the number of words can be formed = $8C_4 \times 4! = 70 \times 24 = 1680$
Case 2: 2 pairs of repeated letters (for example IIOO)
In this case, the number of words can be formed = $3C_2 \times (4!/ 2!\, 2!) = 3 \times 6 = 18$
Case 3: one pair of repeated letters and 2 different letters (for example NNCB)
In this case, the number of words can be formed = $3C_1 \times 7C_2 \times (4!/2!)$
$= 3 \times 21 \times 12 = 756$
Therefore, the total number of words = 1680 + 18 + 756 = 2454

OR

A committee of 7 members has to be formed from 9 boys and 4 girls. In how many ways can this be done when the committee consists of:
(i) exactly 3 girls
(ii) at least 3 girls and
(iii) at most three girls.

Solution:
Given,
Number of boys = 9
Number of girls = 4
The number of members should be in the committee = 7
(i) Exactly 3 girls
Total number of ways of forming a committee in which exactly 3 girls included = $9C_4 \times 4C_3$
$= [(9 \times 8 \times 7 \times 6)/ (4 \times 3 \times 2 \times 1)] \times 4$
= 504
(ii) At least 3 girls
The committee consists of 3 girls + 4 boys and 4 girls + 3 boys
Total number of ways in this case = $4C_3 \times 9C_4 + 4C_4 \times 9C_3$
$= 504 + 1 \times [(9 \times 8 \times 7)/ (3 \times 2 \times 1)]$
= 504 + 84
= 588
(iii) At most 3 girls
(No girl + 7 boys) or (1 girl + 6 boys) or (2 girls + 5 boys) or (3 girls + 4 boys)
No girl + 7 boys = $4C_0 \times 9C_7 = 1 \times 9C_2 = (9 \times 8)/ (2 \times 1) = 36$
1 girls + 6 boys = $4C_1 \times 9C_6 = 4 \times 9C_3 = 4 \times [(9 \times 8 \times 7)/ (3 \times 2 \times 1)] = 336$
2 girls + 5 boys = $4C_2 \times 9C_5 = [(4 \times 3)/ (2 \times 1)] \times 9C_4 = 6 \times 126 = 756$

3 girls + 4 boys = $4C_3 \times 9C_4$ = 504
Total number of ways = 36 + 336 + 756 + 504 = 1632

Question 6
Prove by the method of induction.
(1/1.2) + (1/2.3) + (1/3.4) + ... up to n terms = n/(n + 1) where n ∈ N.

Solution:
Using principle of mathematical induction,

When $n = 1$,

$$\frac{1}{1\cdot 2} = \frac{1}{2}$$

When $n = 2$,

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3} = \frac{2}{2+1}.$$

Hence, the given statement is true for $n = 1$ and $n = 2$

Let as assume that the given statement is true for $n = k$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

Also, let $n = k + 1$

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + ... + \frac{1}{k(k+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

= [k(k + 2) + 1]/ [(k + 1)(k + 2)]
= $(k^2 + 2k + 1)$/ (k + 1)(k + 2)
= $(k + 1)^2$/ (k + 1)(k + 2)
= (k + 1)/ (k + 2)
Therefore, the given statement is also true for n = k + 1.
Hence proved that (1/1.2) + (1/2.3) + (1/3.4) + ... up to n terms = n/(n + 1)

Question 7

Find the term independent of x and its value in the expansion of $\left(\sqrt{\frac{x}{3}} - \frac{\sqrt{3}}{2x}\right)^{12}$.

Solution:

$[(\sqrt{x}/\sqrt{3}) - (\sqrt{3}/2x)]^{12}$

We know that the general term of $(a + b)^n$ is:

$T_{r+1} = nCr\,(a)^{n-r}\,(a)^r$

$T_{r+1} = 12C_r\,(\sqrt{x}/\sqrt{3})^{12-r}\,.\,(-\sqrt{3}/2x)^r$

$= 12C_r\,(x/3)^{(12-r)/2}\,.\,(-3)^{r/2}\,[1/2^r x^r]$

$= 12C_r\,(3)^{-(12-r)/2}\,(-3)^{r/2}\,2^{-r}\,x^{[(12-r)/2-r]}$

To get the term which is independent of x, assume the value of power of x as 0.

$[(12 - r)/2 - r] = 0$

$12 - r - 2r = 0$

$3r = 12$

$r = 4$

Thus, the fourth term will be the required term.

$T_{4+1} = 12C_4\,(3)^{-(12-4)/2}\,(-3)^{4/2}\,2^{-4}$

$= 12C_4\,[3^{-4}\,(9)]/16$

$= 495 \times 9/(81 \times 16)$

$= 55/16$

OR

Find the sum of the terms of the binomial expansion to infinity:

$$1 + \frac{2}{4} + \frac{2.5}{4.8} + \frac{2.5.8}{4.8.12} + \cdots \ldots \ldots .\, to\ \infty.$$

Solution:

$$1+\frac{2}{4}+\frac{2.5}{4.8}+\frac{2.5.8}{4.8.12}+....$$

Let $s=1+\frac{2}{4}+\frac{2.5}{4.8}+....$

Comparing with $(1+x)^n=1+nx+\frac{n(n-1)}{2}x^2+.....$

$$nx=\frac{2}{4},\frac{nx(nx-x)}{2}=\frac{2.5}{4.8}$$

$$=\frac{\frac{2}{4}\left(\frac{2}{4}-x\right)}{2}=\frac{2.5}{4.8}$$

$$\Rightarrow x=\frac{-3}{4}$$

$$n\left(-\frac{3}{4}\right)=\frac{2}{4}$$

$$\Rightarrow n=-\frac{2}{3}$$

$$\therefore s=(1+x)^n=\left(1-\frac{3}{4}\right)^{-2/3}=\left(\frac{1}{4}\right)^{-2/3}=\sqrt[3]{16}$$

Question 8

Differentiate from first principle: f(x) = √(3x + 4).

Solution:

Given,

f(x) = √(3x + 4)

f(x + h) = √[3(x + h) + 4]

By the first principle,

f'(x) = $\lim_{h \to 0}$ [f(x + h) - f(x)]/ h

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{3x+3h+4} - \sqrt{3x+4}}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{3x+3h+4} - \sqrt{3x+4}}{h} \times \frac{\sqrt{3x+3h+4} + \sqrt{3x+4}}{\sqrt{3x+3h+4} + \sqrt{3x+4}}$$

$$= \lim_{h \to 0} \frac{3x+3h+4 - 3x - 4}{h(\sqrt{3x+3h+4} + \sqrt{3x+4})}$$

$$= \lim_{h \to 0} \frac{3}{\sqrt{3x+3h+4} + \sqrt{3x+4}}$$

$$= \frac{3}{2\sqrt{3x+4}}$$

Question 9

Reduce the equation $x + y + \sqrt{2} = 0$ to the normal form ($x \cos \alpha + y \sin \alpha = p$) and find the values of p and α.

Solution:

Given,

$x + y + \sqrt{2} = 0$

$x + y = -\sqrt{2}$

$\sqrt{[(\text{coefficient of } x)^2 + (\text{coefficient of } y)^2]} = \sqrt{(1^2 + 1^2)}$

$= \sqrt{(1 + 1)}$

$= \sqrt{2}$

$\Rightarrow (1/\sqrt{2})(x + y) = -(1/\sqrt{2}) \times \sqrt{2}$

$\Rightarrow (1/\sqrt{2})x + (1/\sqrt{2})y = -1$

$\Rightarrow x \cos (\pi/4) + y \sin (\pi/4) = -1$

Comparing with the normal form $x \cos \alpha + y \sin \alpha = p$,

$\alpha = \pi/4 = 45°$ and $p = -1$

Therefore, $\alpha = 45°$ and $p = -1$.

Question 10

Write the equation of the circle having radius 5 and tangent as the line $3x - 4y + 5 = 0$ at (1, 2).

Solution:

Let the equation of the circle be:

$(x - h)^2 + (y - k)^2 = r^2$

$(x - h)^2 + (y - k)^2 = 5^2$ (given radius = 5)

Also, $3x - 4y + 5 = 0$ is the tangent to the circle.

$\Rightarrow |3h - 4k + 5|/\sqrt{(3^2 + 4^2)} = 5$

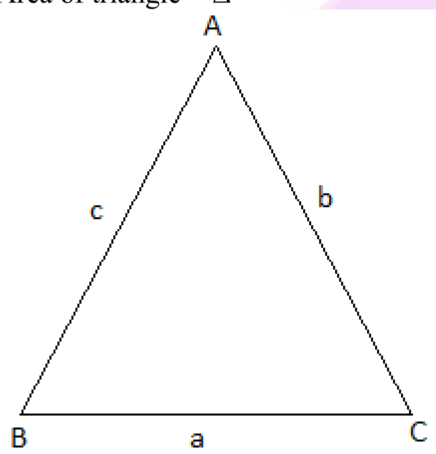
$\Rightarrow |3h - 4k + 5|/ \sqrt{(9 + 16)} = 5$

$\Rightarrow |3h - 4k + 5|/ \sqrt{25} = 5$

⇒ $|3h - 4k + 5| = 25$
⇒ $3h - 4k + 5 = \pm 25$
Now,
$3h - 4k + 5 = 25$
$3h - 4k = 20$....(i)
And
$3h - 4k + 5 = -25$
$3h - 4k = -30$....(ii)
Consider, $3x - 4y + 5 = 0$
$4y = 3x + 5$
$y = (3/4)x + (5/4)$
Slope = (3/4)
Slope of the line passes through radius and the point (1, 2) = -1/slope of the given line
$(k - 2)/ (h - 1) = -1/(3/4) = -4/3$
$3(k - 2) = -4(h - 1)$
$3k - 6 = -4h + 4$
$4h + 3k = 10$....(iii)
By solving (i), (ii), and (iii),
h = 4, k = -2 ad h = -2 and k = 6
Hence, the required equation of circles are:
$(x - 4)^2 + (y + 2)^2 = 25$
$(x + 2)^2 + (y - 6)^2 = 25$

Question 11
In a ΔABC, prove that $\cot A + \cot B + \cot C = (a^2 + b^2 + c^2)/4\Delta$.

Solution:
Area of triangle = Δ



We know that in a triangle ABC,
$\Delta = (1/2)\ bc \sin A$
$bc = 2\Delta/\sin A$....(i)
$\Delta = (1/2)$ ac din B
$ac = 2\Delta/\sin B$....(ii)
$\Delta = (1/2)\ ab \sin C$
$ab = 2\Delta/\sin C$....(iii)
Using cosine rule,

$a^2 = b^2 + c^2 - 2bc \cos A$
$b^2 = a^2 + c^2 - 2ac \cos B$
$c^2 = a^2 + b^2 - 2ab \cos C$
Now,
$a^2 + b^2 + c^2 = 2a^2 + 2b^2 + 2c^2 - 2ab \cos C - 2ac \cos B - 2bc \cos A$
$\Rightarrow a^2 + b^2 + c^2 = 2ab \cos C + 2ac \cos B + 2bc \cos A$....(iv)
From (i), (ii), (iii), and (iv),
$a^2 + b^2 + c^2 = 2 [(2\Delta/\sin C) \cos C + (2\Delta/\sin B) \cos B + (2\Delta/\sin A) \cos A]$
$a^2 + b^2 + c^2 = 2 \times 2\Delta (\cot A + \cot B + \cot C)$
$\Rightarrow \cot A + \cot B + \cot C = (a^2 + b^2 + c^2)/4\Delta$
Hence proved.

Question 12
Find the nth term and deduce the sum to n terms of the series:
4 + 11 + 22 + 37 + 56 + ...

Solution:
Given series is:
4 + 11 + 22 + 37 + 56 + ...
Let $S_n = 4 + 11 + 22 + 37 + 56 + ... + a_{n-1} + a_n$ (n terms)
$\Rightarrow 0 = 4 + 7 + 11 + 15 + 19 + ... + (a_n - a_{n-1}) - a_n$ (n + 1 terms)
$\Rightarrow a_n = 4 + 7 + 11 + 15 + 19 + ... + (a_n - a_{n-1})$
$7 + 11 + 15 + 19 + ... + (a_n - a_{n-1})$ is an AP with a = 7 and d = 4
Sum of these n - 1 terms = $(n - 1)/2 [2 \times 4 + (n - 1 - 1)4]$
$= (n - 1)/2 [14 + (n - 2)4]$
$= (n - 1)[7 + (n - 2)2]$
$a_n = 4 + (n - 1)(7 + 2n - 4)$
$a_n = 4 + (n - 1)(2n + 3)$
$= 4 + 2n^2 + 3n - 2n - 3$
$= 2n^2 + n + 1$

OR

If (p + q)th term and (p - q)th terms of G.P are a and b respectively, prove that the pth term is √(ab).

Solution:
Let t_1 be the first term and r be the common ratio of a GP.
Given,
(p + q)th term and (p - q)th terms of G.P are a and b respectively.
$t_1 \times r(p + q - 1) = a$....(i)
$t_1 \times r(p - q - 1) = b$....(ii)
Multiplying (i) and (ii),
$t_1 \times r^{(p + q - 1)} \times t_1 \times r^{(p - q - 1)} = ab$
$(t_1)^2 \times r^{(p + q - 1 + p - q - 1)} = ab$
$(t_1)^2 \times r^{(2p - 2)} = ab$
$(t_1)^2 \times r^{2(p - 1)} = ab$
$(t_1 \times r^{p - 1})^2 = ab$
$t_1 \times r^{p - 1} = \sqrt{ab}$
Therefore, pth terms = $\sqrt{ab}$

Question 13
If x is real, prove that the value of the expression [(x - 1)(x + 3)] / [(x - 2)(x + 4)] cannot be between 4/9 and 1.

Solution:
Let [(x - 1)(x + 3)] / [(x - 2)(x + 4)] = y
(x - 1)(x + 3) = y(x - 2)(x + 4)
$x^2 + 3x - x - 3 = y(x^2 + 4x - 2x - 8)$
$x^2 + 2x - 3 - x^2y - 2xy + 8y = 0$
$x^2(1 - y) + 2x(1 - y) + (8y - 3) = 0$
Given that x is real.
Therefore, discriminant ≥ 0
$[2(1 - y)]^2 - 4(1 - y)(8y - 3) \geq 0$
$4(1 + y^2 - 2y) - 4(8y - 3 - 8y^2 + 3y) \geq 0$
$4 + 4y^2 - 8y - 44y + 12 + 32y^2 \geq 0$
$36y^2 - 52y + 16 \geq 0$
$4(9y^2 - 13y + 4) \geq 0$
$9y^2 - 13y + 4 \geq 0$
$9y^2 - 9y - 4y + 4 \geq 0$
$9y(y - 1) - 4(y - 1) \geq 0$
$(9y - 4)(y - 1) \geq 0$
$(y - 4/9)(y - 1) \geq 0$

Factor	y < 4/9	y ∈ (4/9, 1)	y > 1
y - (4/9)	-ve	+ve	+ve
y - 1	-ve	-ve	+ve
[y - (4/9)] (y - 1)	+ve	-ve	+ve

[y - (4/9)](y - 1) is positive only when y < 4/9 or y > 1.
Therefore, y cannot lie between 4/9 and 1.

OR

If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is $\frac{(2n)!}{\left[\frac{1}{3}(4n-p)!\right]\left[\frac{1}{3}(2n+p)!\right]}$.

Solution:
Given,
$[x^2 + (1/x)]^{2n}$
General term is:
$T_{r+1} = 2nC_r (x^2)^{2n - r} (1/x)^r$
$= 2nCr\ x^{4n - 2r}\ x^{(-r)}$
$= 2nC_r\ x^{(4n - 2r - r)}$

$= 2nC_r\, x^{(4n - 3r)}$
Let x^p occur in this expansion.
$\Rightarrow 4n - 3r = p$
$\Rightarrow 3r = 4n - p$
$\Rightarrow r = (4n - p)/3$
Coefficient of $x^p = 2nC_r$
$= (2n)!/ [r!(2n - r)!]$
$= (2n)!/ \{[(4n - p)/3]!\ [2n - (4n - p)/3]!\}$
$= (2n)!/ \{[(4n - p)/3]!\ [(6n - 4n + p)/3]!\}$
$= (2n)!/ \{[(4n - p)/3]!\ [(2n + p)/3]!\}$
Hence proved.

Question 14
Calculate the standard deviation of the following distribution:

Age	20 - 25	25 - 30	30 - 35	35 - 40	40 - 45	45 - 50
No. of persons	170	110	80	45	40	35

Solution:

Class	Frequency (f_i)	Class mark (x_i)	f_ix_i	$f_i(x_i)^2$
20 - 25	170	22.5	3825	86062.5
25 - 30	110	27.5	3025	83187.5
30 - 35	80	32.5	2600	84500
35 - 40	45	37.5	1687.5	63281.25
40 - 45	40	42.5	1700	72250
45 - 50	35	47.5	1662.5	78968.75
Total	$\sum f_i = 480$		$\sum f_ix_i = 14500$	$\sum f_i(x_i)^2 = 468250$

Variance $= 1/(N - 1)\ [\sum f_ix_i^2 - (1/N)\ (\sum f_ix_i)^2]$
$= [1/(480 - 1)]\ [468250 - (14500)^2/ 480]$
$= (1/479)\ [468250 - 438020.833]$
$= 30229.167/ 479$
$= 63.109$
Standard deviation $= \sqrt{(63.109)} = 7.944$

SECTION B

Question 15
(a) Find the focus and directrix of the conic represented by the equation $5x^2 = -12y$.

Solution:

$5x^2 = -12y$

$x^2 = (-12/5)y$

$x^2 = 4(-3/5)y$

$(x - 0)^2 = 4(-3/5) (y - 0)$

Comparing with $(x - h)^2 = 4p(y - k)$

h = 0, k = 0, p = -3/5

Thus, the given equation represents a parabola with vertex at (h, k) = (0, 0) and focal length |p| = 3/5.

Here, the parabola is symmetric around the y-axis.

Therefore, the focus lies at a distance p from the centre (0, 0) along the y-axis.

i.e. (0, 0 + p)

= (0, 0 + (-3/5))

= (0, -3/5)

Since the parabola is symmetric around the y-axis, directrix is a line parallel to the x-axis at a distance -p from the centre (0, 0).

Directrix is y = 0 - p

y = 0 - (-3/5)

y = 3/5

Hence, the focus of the given conic is (0, -3/5) and the directrix is y = 3/5.

(b) Construct the truth table (~p ∧ ~q) ∨ (p ∧ ~q)

Solution:

p	q	~p	~q	~p ∧ ~q	p ∧ ~q	(~p ∧ ~q) ∨ (p ∧ ~q)
T	T	F	F	F	F	F
T	F	F	T	F	T	T
F	T	T	F	F	F	F
F	F	T	T	T	F	T

(c) Write the converse, contradiction and contrapositive of the statement.
"If x + 3 = 9, then x = 6."

Solution:

Given statement is:

If x + 3 = 9, then x = 6.

Converse of the given conditional statement is:

If x = 6, then x + 3 = 9.

Contradiction of the given conditional statement is:

If x + 3 ≠ 9, then x ≠ 6.

Contrapositive statement is:

If x ≠ 6, then x + 3 ≠ 9.

Question 16

Show that the point (1, 2, 3) is common to the lines which join A(4, 8, 12) to B(2, 4, 6) and C(3, 5, 4) to D(5, 8, 5).

Solution:
Given,
A(4, 8, 12), B(2, 4, 6), C(3, 5, 4) and D(5, 8, 5).
Let P = (1, 2, 3)
P is a common point to the line AB and CD if ABP and CDP are collinear.
Consider A(4, 8, 12), B(2, 4, 6) and P(1, 2, 3):

$$\det\begin{pmatrix} 4 & 8 & 12 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= 4 \cdot \det\begin{pmatrix} 4 & 6 \\ 2 & 3 \end{pmatrix} - 8 \cdot \det\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} + 12 \cdot \det\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$$

= 4(12 - 12) - 8(6 - 6) + 12(4 - 4)
= 4(0) - 8(0) + 12(0)
= 0
Thus, A, B, and P are collinear.
Consider C(3, 5, 4), D(5, 8, 5) and P(1, 2, 3):

$$\det\begin{pmatrix} 3 & 5 & 4 \\ 5 & 8 & 5 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= 3 \cdot \det\begin{pmatrix} 8 & 5 \\ 2 & 3 \end{pmatrix} - 5 \cdot \det\begin{pmatrix} 5 & 5 \\ 1 & 3 \end{pmatrix} + 4 \cdot \det\begin{pmatrix} 5 & 8 \\ 1 & 2 \end{pmatrix}$$

= 3(24 - 10) - 5(15 - 5) + 4(10 - 8)
= 3(14) - 5(10) + 4(2)
= 42 - 50 + 8
= 0
Thus, C, D, and P are collinear.
Therefore, the point (1, 2, 3) is common to the lines which join A(4, 8, 12) to B(2, 4, 6) and C(3, 5, 4) to D(5, 8, 5).

OR

Calculate the Cosine of the angle A of the triangle with vertices A(1, -1, 2) B (6, 11, 2) and C(1, 2, 6).

Solution:
Given,
Vertices of a triangle are A(1, -1, 2) B (6, 11, 2) and C(1, 2, 6).
Using distance formula,

$$AB = \sqrt{(6-(1))^2 + (11-(-1))^2 + (2-(2))^2}$$
$$= \sqrt{(5)^2 + (12)^2 + (0)^2}$$
$$= \sqrt{25 + 144 + 0}$$
$$= \sqrt{169}$$
$$= 13$$
$$BC = \sqrt{(1-6)^2 + (2-11)^2 + (6-2)^2}$$
$$= \sqrt{(-5)^2 + (-9)^2 + (4)^2}$$
$$= \sqrt{25 + 81 + 16}$$
$$= \sqrt{122}$$
$$AC = \sqrt{(1-(1))^2 + (2-(-1))^2 + (6-(2))^2}$$
$$= \sqrt{(0)^2 + (3)^2 + (4)^2}$$
$$= \sqrt{0 + 9 + 16}$$
$$= \sqrt{25}$$
$$= 5$$

AB = c = 13
BC = a = $\sqrt{122}$
AC = b = 5
cos A = $(b^2 + c^2 - a^2)/2bc$
= (25 + 169 - 122)/ (2 × 5 × 13)
= 72/130
= 36/65

Question 17
Find the equation of the hyperbola whose focus is (1, 1), the corresponding directrix 2x + y - 1 = 0 and e = $\sqrt{3}$.

Solution:
Given,
Focus = S(1, 1)
Directrix is 2x + y - 1 = 0
Eccentricity (e) = $\sqrt{3}$
Let P(x, y) be any point on the hyperbola.
We know that,
The perpendicular distance from the point (x_1, y_1) to the line ax + by + c = 0 = $|ax1 + by1 + c|/ \sqrt{(a^2 + b^2)}$

Here,
$(x_1, y_1) = (x, y)$
$a = 2, b = 1, c = -1$
And
SP = e (perpendicular distance)
$SP^2 = e^2$ (perpendicular distance)^2
$(x - 1)^2 + (y - 1)^2 = (\sqrt{3})^2 [|2x + y - 1|/ \sqrt{(2^2 + 1^2)}]^2$
$x^2 - 2x + 1 + y^2 - 2y + 1 = (3) [|2x + y - 1|^2/ (4 + 1)]$
$x^2 + y^2 - 2x - 2y + 2 = (3/5) (4x^2 + y^2 + 1 + 4xy - 2y - 4x)$
$5x^2 + 5y^2 - 10x - 10y + 10 = 12x^2 + 3y^2 + 3 + 12xy - 6y - 12x$
$12x^2 + 3y^2 + 3 + 12xy - 6y - 12x - 5x^2 - 5y^2 + 10x + 10y - 10 = 0$
$7x^2 - 2y^2 + 12xy - 2x + 4y - 7 = 0$
Therefore, the equation of the hyperbola is $7x^2 - 2y^2 + 12xy - 2x + 4y - 7 = 0$.

OR

Find the equation of tangents to the ellipse $4x^2 + 5y^2 = 20$ which are perpendicular to the line $3x + 2y - 5 = 0$.

Solution:
Given,
$4x^2 + 5y^2 = 20$
$(4x^2/20) + (5y^2/20) = 1$
$(x^2/5) + (y^2/4) = 1$
This is of the form $(x^2/a^2) + (y^2/b^2) = 1$
$a^2 = 5$ and $b^2 = 4$
Equation of the line is $3x + 2y - 5 = 0$
$2y = -3x + 5$
$y = (-3/2)x + (5/2)$
Slope = -3/2
Tangent and the given equation are perpendicular lines.
Thus, the slope of tangent line = $m = -1/(-3/2) = 2/3$
Equation of tangent is:
$y = mx \pm \sqrt{(a^2m^2 + b^2)}$
$y = (2/3)x \pm \sqrt{[5(4/9) + 4]}$
$y = (2/3)x \pm \sqrt{[(20 + 36)/9}$
$y = (1/3) [2x \pm \sqrt{56}]$
$3y = 2x \pm \sqrt{56}$

Question 18
Show that the equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents a hyperbola. Find the lengths of axes and eccentricity.

Solution:
Given,
$16x^2 - 3y^2 - 32x - 12y - 44 = 0$
$(16x^2 - 32x) - (3y^2 + 12y) - 44 = 0$
$16(x^2 - 2x) - 3(y^2 + 4y) - 44 = 0$
$16(x^2 - 2x + 1) - 3(y^2 + 4y + 4) - 44 = 16 - 12$
$16(x - 1)^2 - 3(y + 2)^2 - 44 - 4 = 0$
$16(x - 1)^2 - 3(y + 2)^2 = 48$

⇒ $[(x - 1)^2/ 3] - [(y + 2)^2/ 16] = 1$
This is the equation of parabola in which $a^2 = 3$ and $b^2 = 16$.
Centre = (1, -2)
And
$b^2 = a^2(e^2 - 1)$
$16 = 3(e^2 - 1)$
$e^2 - 1 = 16/3$
$e^2 = (16/3) + 1$
$e^2 = 19/3$
$e = \sqrt{(19/3)}$

SECTION C

Question 19
(i) Two sample sizes of 50 and 100 are given. The mean of these samples respectively are 56 and 50. Find the mean of size 150 by combining the two samples.

Solution:
Let n_1 and n_2 be the sizes of two samples.
X_1 and X_2 are the means of two samples.
According to the given,
$n_1 = 50$ and $n_2 = 100$
$X_1 = 56$ and $X_2 = 50$
New sample size = $n_1 + n_2 = 50 + 100 = 150$
New mean = $(n_1X_1 + n_2X_2)/ (n_1 + n_2)$
= [50(56) + 100(50)]/ 150
= (2800 + 5000)/ 150
= 7800/150
= 52
Therefore, the mean of size 150 by combining the two samples is 52.

(ii) Calculate P95, for the following data:

Marks	0 - 10	10 - 20	20 - 30	30 - 40	40 - 50	50 - 60
Frequency	3	7	11	12	23	4

Solution:

Marks (C.I)	Frequency	Cumulative frequency
0 - 10	3	3
10 - 20	7	10
20 - 30	11	21
30 - 40	12	33

40 - 50	23	56
50 - 60	4	60

N = 60
P at 95 = p = (95/100) × 60 = 57
Cumulative frequency greater than and nearest to 57 is 60 which lies in the interval 50 - 60.
Percentile class = 50 - 60
Lower limit of the percentile class = l = 50
Frequency of the percentile class = f = 4
Cumulative frequency of the class preceding the percentile class = cf = 56
Class height = h = 10
$P_{95} = l + [(p - cf)/ f] \times h$
$= 50 + [(57 - 56)/ 4] \times 10$
$= 50 + (10/4)$
$= 50 + 2.5$
$= 52.5$

OR

Calculate mode for the following data.

C.I.	17 - 19	14 - 16	11 - 13	8 - 10	5 - 7	2 - 4
Frequency	12	4	8	16	11	4

Solution:
Let us arrange the data as shown below:

C.I	Frequency
1.5 - 4.5	4
4.5 - 7.5	11
7.5 - 10.5	16
10.5 - 13.5	8
13.5 - 16.5	4
16.5 - 19.5	12

From the given data,
Maximum frequency = 16
Modal class is 7.5 - 10.5
Frequency of the modal class = $f_1 = 16$
Frequency of the class preceding the modal class = $f_0 = 11$

Frequency of the class succeeding the modal class = f_2 = 8
Lower limit of the modal class = l = 7.5
Class height = h = 3
Mode = $l + [(f_1 - f_0)/ (2f_1 - f_0 - f_2)] \times h$
= 7.5 + [(16 - 11)/ (2 × 16 - 11 - 8)] × 3
= 7.5 + [5/ (32 - 19)] × 3
= 7.5 + (15/ 13)
= 7.5 + 1.154
= 8.654

Question 20
(i) Find the covariance between X and Y when N = 10, ∑X = 50, ∑Y = -30, and ∑XY = 115.

Solution:
Given,
N = 10,
∑X = 50
∑Y = -30
∑XY = 115
Covariance between X and Y = (1/N) ∑XY - (1/N2) ∑X ∑Y
= (1/10) × 115 - (1/100) × 50 × (-30)
= 11.5 + 15
= 26.5

(ii) Calculate Spearman's Rank Correlation for the following data and interpret the result:

Marks in Mathematics	36	48	27	36	29	30	36	39	42	48
Marks in Statistics	27	45	24	27	31	33	35	45	41	45

Solution:

Mathematics	Rank (R_1)	Statistics	Rank (R_2)	$d = R_1 - R_2$	d^2
36	6	27	8.5	-2.5	6.25
48	1.5	45	2	0.5	0.25
27	10	24	10	0	0
36	6	27	8.5	-2.5	6.25
29	9	31	7	2	4
30	8	33	6	2	4
36	6	35	5	1	1
39	4	45	2	2	4

42	3	41	4	-1	1
48	1.5	45	2	-0.5	0.25

n = 10
$\sum d^2 = 27$
The spearman's rank correlation coefficient = $1 - [(6 \times \sum d^2)/ n(n^2 - 1)]$
$= 1 - [(6 \times 27)/ 10(100 - 1)]$
$= 1 - [162/ 10 \times 99]$
$= 1 - (162/990)$
$= (990 - 162)/ 990$
$= 828/990$
$= 0.836$

OR

Find Karl Pearson's Correlation Coefficient from the given data:

x	21	24	26	29	32	43	25	30	35	37
y	120	123	125	128	131	142	124	129	134	136

Solution:

x	(x - mean) (x - 30.2) d_x	d^2_x	y	(y - mean) (y - 129.2) d_y	d^2_y	$d_x d_y$
21	-9.2	84.64	120	-9.2	84.64	84.64
24	-6.2	38.44	123	-6.2	38.44	38.44
26	-4.2	17.64	125	-4.2	17.64	17.64
29	-1.2	1.44	128	-1.2	1.44	1.44
32	1.8	3.24	131	1.8	3.24	3.24
43	12.8	163.84	142	12.8	163.84	163.84
25	-5.2	27.04	124	-5.2	27.04	27.04
30	-0.2	0.04	129	-0.2	0.04	0.04
35	4.8	23.04	134	4.8	23.04	23.04
37	6.8	46.24	136	6.8	46.24	46.24
$\sum x = 302$	$\sum d_x = 0$	$\sum d^2_x = 405.6$	$\sum y = 1292$	$\sum d_y = 0$	$\sum d^2_y = 405.6$	$\sum d_x d_y = 405.6$

Mean of x = ∑x/n = 302/10 = 30.2
Mean of y = ∑y/n = 1292/10 = 129.2

Karl Pearson's Correlation Coefficient = $(\sum d_x d_y) / (\sqrt{\sum d^2_x \sum d^2_y})$
= 405.6 / √(405.6 × 405.6)
= 405.6/405.6
= 1

Question 21
Find the consumer price index for 2007 on the basis of 2005 from the following data using weighted average of price relative method:

Items	Food	Rent	Cloth	Fuel
Prive in 2005 (Rs.)	20	100	150	50
Price in 2007 (Rs.)	280	200	120	100
Weighted	30	20	20	10

Solution:

Items	Weights (w)	Year 2005 (p_0)	Year 2007 (p_1)	Price relative I = $(p_1/p_0) \times 100$	Iw
Food	30	20	280	1400	42000
Rent	20	100	200	200	4000
Cloth	20	150	120	80	1600
Fuel	10	50	100	200	2000
Total	∑w = 80				∑Iw = 49600

Index number of weighted average of price relatives = ∑Iw/ ∑w
= 49600/ 80
= 620

Question 22
Using the following data. Find out the trend using Quarterly moving average and plot them on graph:

Year/ Quarter	Q1	Q2	Q3	Q4
1994	29	37	43	34
1995	90	42	55	43

1996	47	51	63	53

Solution:

Year	Quarter		Quarterly moving total	Quarterly moving average	Centered moving average (trend values)
1994	I	29			
1994	II	37			
			143	35.75	
1994	III	43			43.375
			204	51	
1994	IV	34			51.625
			209	52.25	
1995	I	90			53.75
			221	55.25	
1995	II	42			56.375
			230	57.5	
1995	III	55			52.125
			187	46.75	
1995	IV	43			47.875
			196	49	
1996	I	47			50
			204	51	
1996	II	51			52.25
			214	53.5	
1996	III	63			
1996	IV	53			

Trend: