					(p ∧ r)
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

Therefore, it is a contingency.

[ii] Find the shortest distance between the lines: x - 1 / 2 = y - 2 / 3 = z - 3 / 4 and x - 2 / 3 = y - 4 / 4 = z - 5 / 5.

Solution:

The lines are x - 1 / 2 = y - 2 / 3 = z - 3 / 4 ---- (1)

x - 2 / 3 = y - 4 / 4 = z - 5 / 5 ---- (2)

Here x1 = 1, y1 = 2, z1 = 3 and a1 = 3, b1 = 3, c1 = 4

x2 = 2, y2 = 4, z2 = 5 and a2 = 3, b2 = 4, c2 = 5

Shortest distance between two lines is

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

= 1 (15 - 16) - 2 (10 - 12) + 2 (8 - 9)

= - 1 + 4 - 2

= 1

Denominator = $(15 - 16)^2 + (12 - 10)^2 + (8 - 9)^2 = 1 + 4 + 1 = 6$

$d = 1 / \sqrt{6}$ units

[iii] Find the general solution of the equation sin 2x + sin 4x + sin 6x = 0.

Solution:

$(\sin 2x + \sin 6x) + \sin 4x = 0$

$2\sin 4x \cdot \cos 2x + \sin 4x = 0$

$\sin 4x (2\cos 2x + 1) = 0$

$\sin 4x = 0$ or $2 \cos 2x + 1 = 0$

$\sin 4x = 0$ or $\cos 2x = -1 / 2$

$= -\cos \pi / 3$

$= \cos (\pi - \pi / 3)$

Using $\sin x = 0 \Rrightarrow x = n\pi$

$\sin 4x = 0$

$4x = n\pi$

The general solution is $x = (n\pi) / 4$

Using $\cos x = \cos \alpha \Rrightarrow x = 2mx \pm \alpha$

$\cos 2x = \cos ((2\pi) / 3)$

$2x = 2m\pi \pm (2\pi) / 3$

The general solution is

$x = m\pi \pm \pi / 3$ where m, n is in z.

[B] Attempt any TWO of the following: (8)

[i] Solve the following equations by method of reduction: x - y + z = 4, 2x + y - 3z = 0, x + y + z = 2.

Solution:

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$R_2 - 2R_1$ and $R_3 - R_1$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -5 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \\ -2 \end{bmatrix}$$

x - y + z = 4 ....(1)
3y - 5z = - 8 ....(2)
2y = - 2 ....(3)
y = - 1
By equation (2)
-3 - 5z = - 8 - 5z = - 5z = 1
By equation (1)
x + 1 + 1 = 4
x = 2
x = 2, y = -1, z = 1

[ii] If θ is the measure of the acute angle between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$, then prove that tan θ = |2√h² − ab / a + b| where a + b ≠ 0 and b ≠ 0. Find the condition for coincident lines.

Solution:
Let m1 and m2 be the slopes of the lines represented by the equation,
$ax^2 + 2hxy + by^2 = 0$ ---- (1)
$y = m_1x$ and $y = m_2x$
The combined equation is $(y - m_1x)(y - m_2x) = 0$
$m_1m_2x^2 - (m_1 + m_2)xy + y^2 = 0$ --- (2)

Since equations (1) and (2) represents the same two lines, on comparing the coefficients,

$m_1m_2 / a = 1 / b = m_1 + m_2 / 2h$

$m_1 + m_2 = - 2h / b$ and $m_1m_2 = a / b$

$(m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1m_2$

$(m_1 - m_2)^2 = (- 2h / b)^2 - 4 (a / b)$

$= 4 (h^2 - ab) / b^2$

The angle between them is given by

$\tan \theta = |m_1 - m_2 / 1 + m_1 - m_2|$

$= |\sqrt{(m_1 - m_2)^2} / 1 + m_1 - m_2|$

$= |4 \sqrt{(h^2 - ab) / b^2} / 1 + (a / b)|$

$= |2\sqrt{h^2 - ab} / a + b|$

$\tan \theta = |2\sqrt{h^2 - ab} / a + b|$

[iii] Using the vector method, find incentre of the triangle whose vertices are P (0, 4, 0), Q (0, 0, 3) and R(0, 4, 3).

Solution:

Let p, q, r be the position vectors of vertices P, Q, R of triangle PQR respectively.

p = 4j, q = 3k, r = 4j + 3k

PQ = q - p = 3k - 4j = - 4j + 3k

QR = r - q = 4j + 3k - 3k = 4j

RP = p - r = 4j - 4j - 3k = - 3k

Let x, y, z be the lengths of opposites of vertices P, Q, R respectively.

x = |QR| = 4

y = |RP| = 3

z = |PQ| = 5

The incentre of the triangle = h = (xp + yq + zr) / x + y + z

= 4 (4j) + 3 (3k) + 5 (4j + 3k) / 4 + 3 + 5

= 16j + 9k + 20j + 15k / 12

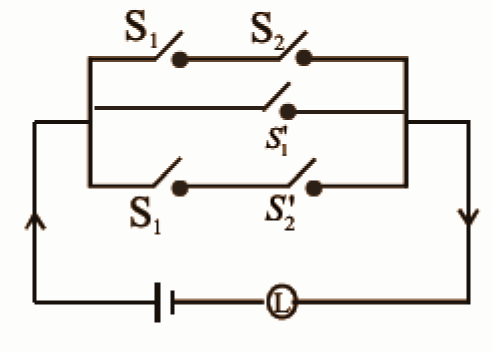
= 3j + 2k

Question 3[A]: Attempt any TWO of the following: (6)

[i] Construct the switching circuit for the statement $(p \wedge q) \vee (\sim p) \vee (p \wedge \sim q)$.

Solution:

Let p = S1 is closed and q = S2 is closed. ~p = switch S_1' and ~q = S_2'



[ii] Find the joint equation of the pair of lines passing through the origin which are perpendicular respectively to the lines represented by $5x^2 + 2xy - 3y^2 = 0$.

Solution:

Given homogeneous equation is $5x^2 + 2xy - 3y^2 = 0$

$5x^2 + 5xy - 3xy - 3y^2 = 0$

$5x (x + y) - 3y (x + y) = 0$

$(x + y) (5x - 3y) = 0$

$x + y = 0$ and $5x - 3y = 0$ are the two lines represented by the given equation.

The slopes are -1 and 5 / 3.

The required two lines are perpendicular to these lines.

The slopes of required lines are 1 and 3 / 5 and the lines pass through the origin.

The individual equations are

$y = 1 . x$ and $y = - 3 / 5 x$

$x - y = 0$ and $3x + 5y = 0$

Their joint equation is

$(x - y) (3x + 5y) = 0$

$3x^2 - 3xy + 5xy - 5y^2 = 0$

$3x^2 + 2xy - 5y^2 = 0$

[iii] Show that $\cos^{-1} (4 / 5) + \cos^{-1} (12 / 13) = \cos^{-1} (33 / 65)$.

Solution:

Let $a = \cos^{-1}(4 / 5)$ and $b = \cos^{-1}(12 / 13)$

$\cos a = 4 / 5$

$\sin^2 a = 1 - \cos^2 a$

$\sin a = \sqrt{1 - \cos^2 a}$

$= \sqrt{1 - (4 / 5)^2}$

$= \sqrt{1 - (16 / 25)}$

$= \sqrt{9 / 25}$

$= 3 / 5$

$\cos b = 12 / 13$

$\sin^2 b = 1 - \cos^2 b$

$\sin b = \sqrt{1 - \cos^2 b}$

$= \sqrt{1 - (12 / 13)^2}$

$= \sqrt{1 - (144 / 169)}$

$= \sqrt{25 / 169}$

$= 5 / 13$

$\cos(a + b) = \cos a \cos b - \sin a \sin b$

$= (4 / 5) * (12 / 13) - (3 / 5) * (5 / 13)$

$= (48 - 15) / 65$

$= 33 / 65$

$\cos(a + b) = 33 / 65$

$a + b = \cos^{-1}(33 / 65)$

$\cos^{-1}(4 / 5) + \cos^{-1}(12 / 13) = \cos^{-1}(33 / 65)$

[B] Attempt any TWO of the following: (8)

[i] If l, m, n are the direction cosines of a line, then prove that $l^2 + m^2 + n^2 = 1$. Hence find the direction angle of the line with the c-axis which makes direction angles of 135^o and 45^o with Y and Z axes respectively.

Solution:

Let α, β, γ be the angles made by the line with X, Y, Z axes respectively.

$l = \cos \alpha$, $m = \cos \beta$ and $n = \cos\gamma$

Let a = a_1i + a_2j + a_3k be any non-zero vector along the line.
Since i is the unit vector along X-axis,
a . i = |a| . |i| cos α = a cosα
Also, a . i = (a_1i + a_2j + a_3k). i
= a_1 x 1 + a_2 x 0 + a_3 x 0 = a_1
a cosα = a_1 ....(1)
Since j is the unit vector along Y-axis,
a . j = |a| . |j| cosβ = a cosβ
a . j = (a_1i + a_2j + a_3k) . j
= a_1 x 0 + a_2 x 1 + a_3 x 0 = a_2
a cos β = a_2 ....(2)
a cos **γ** = a_3 ....(3)
From equations (1), (2) and (3),
$a^2 \cos^2 \alpha + a^2 \cos^2 \beta + a^2 \cos^2 \gamma = a_1^2 + a_2^2 + a_3^2$
$a^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = a^2$ [a = |a| =$\sqrt{(a_1^2 + a_2^2 + a_3^2)}$]
$(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = 1$
$l^2 + m^2 + n^2 = 1$
α = ?, β = 135, **γ** = 45
$(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) = 45$
$\cos^2 \alpha + \cos^2 135 + \cos^2 45 = 1$
$\cos^2 \alpha + 1 / 2 + 1 / 2 = 1$
cosα = 0
α = **π** / 2 or (3**π**) / 2

[ii] Find the vector and cartesian equations of the plane passing through the points A (1, 1, 2), B (1, 2, 1) and C (2, 1, 1).

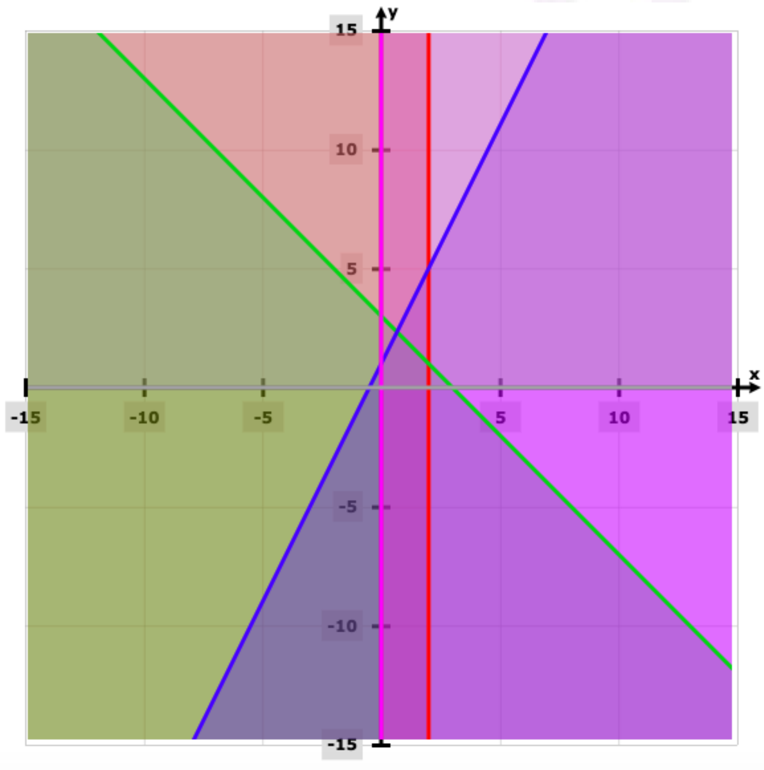
Solution:
The vector equation of the plane passing through the points A (a), B (b) and C (c).
r . ((AB) x (AC)) = a ((AB) x (AC))............(1)
Let a = i + j - 2k, b = i + 2j + k, c = 2i - j + k
(AB) = b - a = (i + 2j + k) - (i + j - 2k) = j + 3k and
(AC) = c - a = (2i - j + k) - (i - j - 2k) = i - 2j + 3k
(AB) x (AC) = |[i, j, k] , [0, 1, 3] , [1, -2, 3]|

= (3 + 6)i - (0 - 3)j + (0 - 1)k
= 9i + 3j - k
a . ((AB) x (AC)) = (i + j - 2k) . (9i + 3j - k)
= 1 (9) + 1 (3) + (-2) (-1)
= 9 + 3 + 2 = 14
From (1), the vector equation of the required plane is r . (9i + 3j - k) = 14
(xi + yj + zk) (9i + 3j- k) = 14
The cartesian equation of the plane is 9x + 3y - z = 14.

[iii] Solve the following L.P.P. by the graphical method:

Maximise : Z = 6x + 4y subject to x ≤ 2, x + y ≤ 3, −2x + y ≤ 1, x ≥ 0, y ≥ 0.

Solution:



Z = 6x + 4y
Z = 6x + 4y
Z at (0, 0) = 6 (0) + 4 (0) = 0
Z at (2, 0) = 6 (2) + 4 (0) = 12
Z at (2, 1) = 6 (2) + 4 (1) = 16

Z at (2 / 3, 7 / 3) = 6 (2 / 3) + (7 / 3) 4 = 40 / 3
Z at (0,1) = 6 (0) + 4 (1) = 4
Thus, Z is maximized at (2, 1) and its maximum value is 16.

SECTION - II

Question 4[A]: Select and write the appropriate answer from the given alternatives in each of the following sub-questions: (6)

[i] Derivatives of $\tan^3 \theta$ with respect to $\sec^3 \theta$ at $\theta = \pi / 3$ is _______.

(A) 3 / 2 (B) √3 / 2 (C) 1 / 2 (D) − √3 / 2

Answer: (b)
$y = \tan^3 \theta$ and $x = \sec^3 \theta$
$dy / d\theta = 3 \tan^2 \theta \sec^2 \theta$
$dx / d\theta = 3 \sec^2 \theta \sec \theta \tan \theta$
$dy / dx = \sin \theta = \sin (\pi / 3) = \sqrt{3} / 2$

[ii] The equation of tangent to the curve $y = 3x^2 - x + 1$ at P(1, 3) is _______.

(A) 5x - y = 2 (B) x + 5y = 16 (C) 5x - y + 2 = 0 (D) 5x = y

Answer: (a)
dy / dx = 6x - 1 at (1, 3)
Slope of the tangent at (1, 3) = (6, 1) = 5
Equation of tangent is $y - y_1 = m (x - x_1)$
y - 3 = 5 (x - 1)
5x - y - 2 = 0
y = 5x - 2

[iii] The expected value of the number of heads obtained when three fair coins are tossed simultaneously is _______.

(A) 1 (B) 1.5 (C) 0 (D) 1

Answer: (b)

p = 1 / 2

q = 1 / 2

n = 3

E (X) = np = 3 * (0.5) = 1.5

[B] Attempt any THREE of the following: (6)

[i] Find dy / dx if x sin y + y sin x = 0.

Solution:

x sin y + y sin x = 0

[x cos y dy / dx + sin y] + [y cos x + sin x dy / dx] = 0

sin y + cos x = dy / dx [- sin x - x cos y]

dy / dx = [- sin x - x cos y] / [sin y + cos x]

[ii] Test whether the function, f (x) = x − (1 / x), x ∈ R, x ≠ 0, is increasing or decreasing.

Solution:

f (x) = x - (1 / x)

f ‘ (x) = 1 - (- 1 / x^2) = 1 + (1 / x^2)

x ≠ 0, for all values of x, $x^2 > 0$

1 / $x^2 > 0$, [1 + (1 / x^2)] is always positive

Thus f ‘ (x) > 0, for all x ∈ R.

It is an increasing function.

[iii] Evaluate: ∫[sin √x / √x] dx.

Solution:

Let I = ∫[sin √x / √x] dx

Let √x = t

$[1 / \sqrt{x}]\ dx = 2dt$

$I = 2 \int \sin t\ dt$

$= - 2 \cos t + c$

$= - 2 \cos \sqrt{x} + c$

[iv] Form the differential equation by eliminating arbitrary constants from the relation $y = Ae^{5x} + Be^{-5x}$.

Solution:

$y = Ae^{5x} + Be^{-5x}$

On differentiating with respect to x,

$dy / dx = Ae^{5x} (5) + Be^{-5x} (-5)$

$dy / dx = 5Ae^{5x} - 5Be^{-5x}$

Again differentiating with respect to x,

$d^2y / dx^2 = 5Ae^{5x} (5) - 5Be^{-5x} (-5)$

$d^2y / dx^2 = 25Ae^{5x} + 25 Be^{-5x}$

$= 25 [Ae^{5x} + Be^{-5x}]$

$= 25y$

$d^2y / dx^2 - 25y = 0$

[v] The probability that a bomb will hit a target is 0.8. Find the probability that out of 10 bombs dropped, exactly 4 will hit the target.

Solution:

Let r be the number of bombs that hit the target.

$p = 0.8$

$q = 1 - p = 1 = 0.8 = 0.2$

$n = 10$

$r = 4$

$P (r = 4) = {}^nC_r\, p^r\, q^{n-r}$

$= {}^{10}C_4 (0.8)^4 (0.2)^{10 - 4}$

$= 210 * (2)^{18} * (1 / 10)^{10}$

$= 55050240 / 10^{10}$

$= \text{anti} (\log 210 + 18 \log 2 - 10)$

= 0.0055

Question 5[A]: Attempt any TWO of the following: (6)

[i] Solve: dy / dx = cos (x + y).

Solution:

dy / dx = cos (x + y) ---- (1)

Put x + y = v ---- (2)

y = v - x

dy / dx = dv / dx - 1 ---- (3)

Subtracting (2) and (3) in (1),

dv / dx - 1 = cos v

dv / dx = 1 + cos v

dv / dx = 2 $\cos^2$ (v / 2)

[1 / $\cos^2$ (v / 2)] dv = 2dx

$\sec^2$ (v / 2) dv = 2 dx

On integrating both sides,

$\int \sec^2 (v / 2)\, dv = 2 \int dx$

2 tan (v / 2) = 2x + c

tan (v / 2) = x + (c / 2)

tan [(x + y) / 2] = x + c

[ii] If u and v are two functions of x, then prove that:

$\int uv\, dx = u \int v\, dx - \int [du / dx \int v\, dx]\, dx$.

Solution:

Let $\int v\, dx = w$ ---- (1)

dw / dx = v ---- (2)

d (u, w) / dx = u . (d [w] / dx] + w . (d [u] / dx]

= u . v + w (du / dx) ---- (from 2)

By the definition of integration,

$u . w = \int [u . v + w (du / dx)]\, dx$

$= \int u . v\, dx + \int w . (du / dx)\, dx$

∫u . v dx = u . w – ∫w . (du / dx) dx
= u ∫v dx – ∫[du / dx ∫ v dx] dx

[iii] If f (x) = $\frac{e^{x^2} - \cos x}{x^2}$ **, for x ≠ 0, is continuous at x = 0, find f (0).**

Solution:
f (x) is continuous at x = 0.
$\lim_{x\to 0} f(x) = f(0)$

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{e^{x^2} - \cos x}{x^2} = \lim_{x\to 0} \frac{\left(e^{x^2} - 1\right) + (1 - \cos x)}{x^2}$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + \frac{1 - \cos x}{x^2}\right)$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + \frac{2\sin^2\left(\frac{x}{2}\right)}{x^2}\right)$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + 2\left(\frac{\sin\left(\frac{x}{2}\right)}{x}\right)^2\right)$$

$$= \lim_{x\to 0} \left(\frac{e^{x^2} - 1}{x^2} + 2\left(\frac{\sin\left(\frac{x}{2}\right)}{\frac{x}{2}} \times \frac{1}{2}\right)^2\right)$$

$$= \lim_{x\to 0} \frac{e^{x^2} - 1}{x^2} + \frac{1}{2}\left(\lim_{x\to 0} \frac{\sin\left(\frac{x}{2}\right)}{x}\right)^2$$

$$= 1 + \frac{1}{2}(1)^2$$

$$= \frac{3}{2}$$

[B] Attempt any TWO of the following: (8)

[i] If y = f (x) is a differentiable function of x such that inverse function $x = f^{-1}(y)$ exists, then prove that x is a differentiable function of y and dx / dy = 1 / (dy / dx) where dy / dx ≠ 0. Hence find d $(\tan^{-1} x)$ / dx.

Solution:

Let δy be the increment in y corresponding to an increment δx in x as δx → 0, δy→ 0.

Now y is a differentiable function of x.

$\lim_{x\to 0} (\delta y / \delta x) = dy / dx$

$(\delta y / \delta x)(\delta x / \delta y) = 1$

$(\delta x / \delta y) = 1 / (\delta y / \delta x)$

Taking limits on both sides as δx → 0,

$\lim_{\delta x \to 0} (\delta x / \delta y) = \lim_{\delta x \to 0} [1 / (\delta y / \delta x)] = 1 / \lim_{\delta x \to 0} (\delta y / \delta x)$

$\lim_{\delta x \to 0} (\delta x / \delta y) = 1 / \lim_{\delta x \to 0} (\delta y / \delta x)$

Since the limit in RHS exists, limits on LHS also exist.

$\lim_{\delta y \to 0} (\delta x / \delta y) = dx / dy$

$dx / dy = 1 / (dy / dx)$, where $dy / dx \neq 0$

Let $y = \tan^{-1} x$

$x = \tan y$

$\cos y = 1 / \sqrt{1 + \tan^2 y} = 1 / \sqrt{1 + x^2}$

$\sec^2 y \, (dy / dx) = 1$

$dx / dy = \sec^2 y$

$dy / dx = 1 / (dx / dy) = 1 / \sec^2 y = \cos^2 y$

$dy / dx = \cos^2 y$

$d(\tan^{-1} x) / dx = \cos^2 y = (\cos y)^2 = (1 / \sqrt{1 + x^2})^2$

$d(\tan^{-1} x) / dx = 1 / (1 + x^2)$

[ii] A telephone company in a town has 5000 subscribers on its list and collects fixed rent charges of 3,000 per year from each subscriber. The company proposes to increase annual rent and it is believed that for every increase of

one rupee in the rent, one subscriber will be discontinued. Find what increased annual rent will bring the maximum annual income to the company.

Solution:

Here, the number of subscribers = 5000 and annual rental charges per subscriber = Rs.3000.

For every increase of 1 rupee in the rent, one subscriber will be discontinued.

Let the rent be increased by Rs. x.

New rental charges per year = 3000 + x and the number of subscribers after the increase in rental charges = 5000 - x.

Let R be the annual income of the company.

Then, R = (3000 + x) (5000 - x)

$= 15000000 - 3000x + 5000x - x^2$

$= 15000000 + 2000x - x^2$ and $(d^2R)/dx^2 = -2$

R is maximum if dR / dx = 0

2000 - 2x = 0

If x = 1000,

$(d^2R) / dx^2)_{(x = 1000)} = -2 < 0$

By the second derivative test, R is maximum when x =1000.

Thus, the annual income of the company is maximum when the annual rental charges are increased by Rs.1000.

[iii] Evaluate: $\int_{-a}^{a} \sqrt{(a - x) / (a + x)}\, dx$.

Solution:

Let $I = \int_{-a}^{a} \sqrt{(a - x) / (a + x)}\, dx$

$= \int_{-a}^{a} \sqrt{(a - x) (a - x) / (a + x) (a - x)}\, dx$

$= \int_{-a}^{a} (a - x) / \sqrt{a^2 - x^2}\, dx$

$= \int_{-a}^{a} a / \sqrt{a^2 - x^2}\, dx - \int_{-a}^{a} x / \sqrt{a^2 - x^2}\, dx$

$= 2a [\sin^{-1} (x / a)]_o^a$

$= 2a [\pi / 2 - 0]$

$= a\pi$

Question 6[A]: Attempt any TWO of the following: (6)

[i] Discuss the continuity of the following function, at $x = 0$.

$f(x) = x / |x|$, for $x \neq 0$

$= 1$, for $x = 0$

Solution:

$f(0) = 1$ ---- (1)

For $x > 0$, $|x| = x$

$\lim_{x\to0+} f(x) = \lim_{x\to0+} x / |x|$

$= \lim_{x\to0+} (x / x)$

$= \lim_{x\to0+} 1$

$= 1$

For $x < 0$, $|x| = -x$

$\lim_{x\to0-} f(x) = \lim_{x\to0-} x / |x|$

$= \lim_{x\to0-} (-x / x)$

$= \lim_{x\to0-} (-1)$

$= -1$

$\lim_{x\to0+} f(x) \neq \lim_{x\to0-} f(x)$

f is discontinuous at $x = 0$.

$\lim_{x\to0} f(x)$ does not exist.

It is discontinuous at $x = 0$.

[ii] If the population of a country doubles in 60 years, in how many years will it triple under the assumption that the rate of increase is proportional to the number of inhabitants? [Given : log 2 = 0.6912 and log 3 = 1.0986.]

Solution:

Let P be the population at time t years. Then dP / dt the rate of increase of population is proportional to P.

$dp / dt \propto p$

$dP / dt = kP$ when k is a constant

$dP / P = kdt$

On integrating we get

$\int dp / P = k \int dt + c$

$\log P = kt + c$

Initially when $t = 0$ let $P = P_0$

$\log P_0 = k \times 0 + c$

$\log P = kt + \log P_0$

$\log P - \log P_0 = kt$

$\log (P / P_0) = kt$

Since the population doubles in 60 years, when $t = 60$ $P = 2P_0$

$\log (2P_0 / P_0) = 60k$,

$\therefore k = [1 / 60] \log 2$

(1) becomes $\log (P / P0) = [t / 60] \log 2$

When population becomes triple when $P = 3P_0$ we get

$\log (3P_0 / P_0) = (t / 60) \log 2$

$\log 3 = [t / 60] \log 2$

$\log 3 = [t / 60] \log 2$

$t = 60 (\log 3 / \log 2)$

$= 60 (1.0986 / 0.6912)$

$= 60 \times 1.5894$

$= 95.364$

= 95.4 years

The population becomes tripled in 95.4 years.

[iii] A fair coin is tossed 8 times. Find the probability that it shows heads

a. exactly 5 times

b. at least once

Solution:

a. Let X = number of heads

P = Probability of getting in the first toss

$p = 0.5$

$q = 1 - p = 1 - 0.5 = 0.5$

$n = 8$

P (exactly 5 heads) = $P [X = 5]$

$P (5) = {}^{8}C_{5} (0.5)^{5} (0.5)^{8-5}$

$= 7 / 32$

$P(X = 5) = 0.21875$

b. P [getting head at least once] = $P[X \geq 1] = 1 - P[X = 0]$

$= 1 - P(0)$

$= 1 - {}^8C_0 (0.5)^0 (0.5)^{8-0}$

$= 255 / 256$

$= 0.996$

[B] Attempt any TWO of the following: (8)

[i] Evaluate: $\int d\theta / \sin\theta + \sin 2\theta$.

Solution:

$I = \int dx / (\sin x + \sin 2x)$

$= \int 1 / (\sin x + 2 \sin x \cos x)\, dx$

$= \int 1 / (\sin x (1 + 2 \cos x))\, dx$

$= \int \sin x / (\sin^2 x (1 + 2 \cos x))\, dx$

Let $u = \cos x$

$du = -\sin x\, dx$

Also,

$\sin^2 x = 1 - \cos^2 x = 1 - u^2$

$I = \int -1 / ((1 - u^2)(1 + 2u))\, du$

$= \int 1 / ((1 + u)(1 - u)(1 + 2u))\, du$

Using partial fractions, we get

$1 / ((1 + u)(1 - u)(1 + 2u)) = A / (1 + u) + B / (1 - u) + C / (1 + 2u)$

$-1 = A(1 - u)(1 + 2u) + B(1 + u)(1 + 2u) + C(1 + u)(1 - u)$

$-1 = A(1 + u - 2u^2) + B(1 + 3u + 2u^2) + C(1 - u^2)$

$-1 = (-2A + 2B - C)u^2 + (A + 3B)u + (A + B + C)$

Equating the respective coefficients on the LHS and the RHS,

$-2A + 2B - C = 0$ ---- (1)

$A + 3B = 0$ ---- (2)

$A + B + C = -1$ ---- (3)

Adding the above equations,

$A = -3B$

$A = 1 / 2$

$C = - 1 - A - B$

$C = - 4 / 3$

$1 / ((1 + u) (1 - u) (1 + 2u)) = 1 / (2 (1 + u)) - 1 / (6 (1 - u)) - 4 / (3 (1 + 2u))$

$= I = \int[1 / (2 (1 + u)) - 1 / (6 (1 - u)) - 4 / (3 (1 + 2u))] du$

$= 1 / 2 \log (1 + u) + 1 / 6 \log (1 - u) - 4 / (2) \log (1 + 2u) + C$

$= 1 / 2 \log (1 + \cos x) + 1 / 6 \log (1 - \cos x) - 2 / 3 \log (1 + 2 \cos x) + C$

[ii] Find the area of the region lying between the parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

Solution:

The equations of the parabolas are

$y^2 = 4ax$.............(1)

$x^2 = 4ay$.............(2)

$[x^2 / (4a)]^2 = 4ax$ by (2)

$x^4 = 64a^3x$

$x [x^3 - (4a)^3] = 0$

$x = 0$ and $x = 4a$

$y = 0$ and $y = 4a$

The points of intersection of curves are O (0, 0), P (4a, 4a).

The required areas is,

A = (Area under parabola $y^2 = 4ax$) - (Area under parabola $x^2 = 4ay$)

$= \int_0^{4a} [\sqrt{(4ax)}dx - \int_0^{4a} x^2 / (4a)] dx$

$= \sqrt{(4a)} . 2 / 3 [x^{(3/2)}]_0^{4a} - (1 / (4a)) (1 / 3) [x^3]_0^{4a}$

$= (4\sqrt{a}) / 3 * 4a\sqrt{(4a)} - (1 / (12a)) * 64a^3$

$= [32 / 3] a^2 - [16 / 3]a^2$

$= [16 / 3]a^2$ sq.units

[iii] Given the probability density function (p.d.f.) of a continuous random variable X as, f (x) = $x^2 / 3$, $-1 < x < 2$

= 0, otherwise.

Determine the cumulative distribution function (c.d.f.) of X and hence find P (X < 1), P (X > 0), P (1 < X < 2).

Solution:

CDF of the continuous random variable is given by

$F(x) = \int_{-1}^{x} [y^2 / 3]\, dx$

$= [y^3 / 9]_{-1}^{x}$

$= (x^3 + 1) / 9$, x in R

Consider P (X < 1) = F (1) = (13 + 1) / 9 = 2 / 9

$P(x \leq -2) = 0$

$P(X > 0) = 1 - P(X \leq 0)$

= 1 - F (0)

= 1 - (0 / 9 + 1 / 9)

= 8 / 9

P (1 < x < 2) = F (2) - F (1)

= 1 - (1 / 9 + 1 / 9)

= 7 / 9