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1. Find the angle between two vectors $\vec{a}$ and $\vec{b}$ with magnitudes √3 and 2, respectively having $\vec{a}\cdot\vec{b}=\sqrt{6}$.

Solution:

Firstly let us consider,

$|\vec{a}|=\sqrt{3},\ |\vec{b}|=2$ and, $\vec{a}\cdot\vec{b}=\sqrt{6}$

Now, we know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.

$$\therefore \sqrt{6}=\sqrt{3}\times 2\times\cos\theta$$

$$\cos\theta=\frac{\sqrt{6}}{\sqrt{3}\times 2}$$

$$\cos\theta=\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta=\frac{\pi}{4}$$

Thus , the angle between the given vectors $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$

2. Find the angle between the vectors $\hat{i}-2\hat{j}+3\hat{k}$ and $3\hat{i}-2\hat{j}+\hat{k}$

Solution:

Let us consider the

Given vectors are: $\vec{a}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}-2\hat{j}+\hat{k}$

$$|\vec{a}|=\sqrt{1^2+(-2)^2+3^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$|\vec{b}|=\sqrt{3^2+(-2)^2+1^2}=\sqrt{9+4+1}=\sqrt{14}$$

$$\text{Now, } \vec{a}\cdot\vec{b}=(\hat{i}-2\hat{j}+3\hat{k})(3\hat{i}-2\hat{j}+\hat{k})$$

$$=1.3+(-2)(-2)+3.1$$

$$=3+4+3$$

$$=10$$

Also, we know that $\vec{a}\cdot b=|\vec{a}||b|\cos\theta$.

$$\therefore 10=\sqrt{14}\sqrt{14}\cos\theta$$

$$\cos\theta=\frac{10}{14}$$

$$\theta=\cos^{-1}\left(\frac{5}{7}\right)$$

Hence, the angle between the vectors is $\cos^{-1}$ (5/7).

3. Find the projection of the vector $\hat{i}-\hat{j}$ on the vector $\hat{i}+\hat{j}$.

Solution:
Firstly,
Let $\vec{a} = \hat{i} - \hat{j}$ and $\vec{b} = \hat{i} + \hat{j}$.
Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,
$$\frac{1}{|\vec{b}|}(\vec{a}.\vec{b}) = \frac{1}{\sqrt{1+1}}\{1.1+(-1)(1)\} = \frac{1}{\sqrt{2}}(1-1) = 0$$
Thus, the projection of vector $\vec{a}$ on $\vec{b}$ is 0.

4. Find the projection of the vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $7\hat{i}-\hat{j}+8\hat{k}$.
Solution:
Firstly,
Let $\vec{a} = \hat{i}+3\hat{j}+7\hat{k}$ and $\hat{b} = 7\hat{i}-\hat{j}+8\hat{k}$.

Now, projection of vector $\vec{a}$ on $\vec{b}$ is given by,

$$\frac{1}{|\vec{b}|}(\vec{a}\cdot\vec{b}) = \frac{1}{\sqrt{7^2+(-1)^2+8^2}}\{1(7)+3(-1)+7(8)\} = \frac{7-3+56}{\sqrt{49+1+64}} = \frac{60}{\sqrt{114}}$$
Hence, the projection is $60/\sqrt{114}$.

5. Show that each of the given three vectors is a unit vector:
$$\frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}), \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}), \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k})$$
Also, show that they are mutually perpendicular to each other.
Solution:
It is given that
$$\text{Let } \vec{a} = \frac{1}{7}(2\hat{i}+3\hat{j}+6\hat{k}) = \frac{2}{7}\hat{i}+\frac{3}{7}\hat{j}+\frac{6}{7}\hat{k},$$
$$\vec{b} = \frac{1}{7}(3\hat{i}-6\hat{j}+2\hat{k}) = \frac{3}{7}\hat{i}-\frac{6}{7}\hat{j}+\frac{2}{7}\hat{k},$$
$$\vec{c} = \frac{1}{7}(6\hat{i}+2\hat{j}-3\hat{k}) = \frac{6}{7}\hat{i}+\frac{2}{7}\hat{j}-\frac{3}{7}\hat{k}.$$
$$|\vec{a}| = \sqrt{\left(\frac{2}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{6}{7}\right)^2} = \sqrt{\frac{4}{49}+\frac{9}{49}+\frac{36}{49}} = 1$$
$$|\vec{b}| = \sqrt{\left(\frac{3}{7}\right)^2+\left(-\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2} = \sqrt{\frac{9}{49}+\frac{36}{49}+\frac{4}{49}} = 1$$
$$|\vec{c}| = \sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{2}{7}\right)^2+\left(-\frac{3}{7}\right)^2} = \sqrt{\frac{36}{49}+\frac{4}{49}+\frac{9}{49}} = 1$$
Hence, each of the given three vectors is a unit vector.

$$\vec{a}\cdot\vec{b}=\frac{2}{7}\times\frac{3}{7}+\frac{3}{7}\times\left(\frac{-6}{7}\right)+\frac{6}{7}\times\frac{2}{7}=\frac{6}{49}-\frac{18}{49}+\frac{12}{49}=0$$
$$\vec{b}\cdot\vec{c}=\frac{3}{7}\times\frac{6}{7}+\left(\frac{-6}{7}\right)\times\frac{2}{7}+\frac{2}{7}\times\left(\frac{-3}{7}\right)=\frac{18}{49}-\frac{12}{49}-\frac{6}{49}=0$$
$$\vec{c}\cdot\vec{a}=\frac{6}{7}\times\frac{2}{7}+\frac{2}{7}\times\frac{3}{7}+\left(\frac{-3}{7}\right)\times\frac{6}{7}=\frac{12}{49}+\frac{6}{49}-\frac{18}{49}=0$$

Therefore, the given threee vectors are mutually perpendicular to each other.

6. Find $|\vec{a}|$ and $|\vec{b}|$, if $(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=8$ and $|\vec{a}|=8|\vec{b}|$

Solution:

Let us consider,

$$\left(\vec{a}\cdot\vec{b}\right).\left(\vec{a}-\vec{b}\right)=8$$
$$\vec{a}\cdot\vec{a}-\vec{a}.\vec{b}+\vec{b}\cdot\vec{a}-\vec{b}\cdot\vec{b}=8$$
$$|\vec{a}|^2-|\vec{b}|^2=8$$
$$\left(8|\vec{b}|\right)^2-|\vec{b}|^2=8 \qquad \left[|\vec{a}|=8|\vec{b}|\right]$$
$$64|\vec{b}|^2-|\vec{b}|^2=8$$
$$63|\vec{b}|^2=8$$
$$|\vec{b}|^2=\frac{8}{63}$$
$$|\vec{b}|=\sqrt{\frac{8}{63}} \qquad [\text{Magnitude of a vector is non-negative}]$$
$$|\vec{b}|=\frac{2\sqrt{2}}{3\sqrt{7}}$$

And,

$$|\vec{a}|=8|\vec{b}|=\frac{8\times2\sqrt{2}}{3\sqrt{7}}=\frac{16\sqrt{2}}{3\sqrt{7}}$$

7. Evaluate the product $\left(3\vec{a}-5\vec{b}\right)\cdot\left(2\vec{a}+7\vec{b}\right)$

Solution:

Let us consider the given expression,

$$(3\vec{a}-5\vec{b})\cdot(2\vec{a}+7\vec{b})$$
$$=3\vec{a}\cdot 2\vec{a}+3\vec{a}\cdot 7\vec{b}-5\vec{b}\cdot 2\vec{a}-5\vec{b}\cdot 7\vec{b}$$
$$=6\vec{a}\cdot\vec{a}+21\vec{a}\cdot\vec{b}-10\vec{a}\cdot\vec{b}-35\vec{b}\cdot\vec{b}$$
$$=6|\vec{a}|^2+11\vec{a}\cdot\vec{b}-35|\vec{b}|^2$$

8. Find the magnitude of two vectors $\vec{a}$ and $\vec{b}$, having the same magnitude and such that the angle between them is 60° and their scalar product is ½.
Solution:
Firstly,
Let θ be the angle between the vectors $\vec{a}$ and $\vec{b}$.
It is given that $|\vec{a}|=|\vec{b}|,\ \vec{a}\cdot\vec{b}=\frac{1}{2}$, and $\theta=60°$. ...(1)
We know that $\vec{a}\cdot\vec{b}=|\vec{a}||\vec{b}|\cos\theta$.
$$\therefore \frac{1}{2}=|\vec{a}||\vec{a}|\cos 60° \qquad [\text{Using (1)}]$$
$$\frac{1}{2}=|\vec{a}|^2\times\frac{1}{2}$$
$$|\vec{a}|^2=1$$
$$|\vec{a}|=|\vec{b}|=1$$
Hence the magnitude of two vectors is 1.

9. Find $|\vec{x}|$, if for a unit vector $\vec{a},\ (\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$
Solution:
Let us consider,
$$(\vec{x}-\vec{a})\cdot(\vec{x}+\vec{a})=12$$
$$\vec{x}\cdot\vec{x}+\vec{x}\cdot\vec{a}-\vec{a}\cdot\vec{x}-\vec{a}\cdot\vec{a}=12$$
$$|\vec{x}|^2-|\vec{a}|^2=12$$
$$|\vec{x}|^2-1=12 \qquad [|\vec{a}|=1 \text{ as } \vec{a} \text{ is a unit vector}]$$
$$|\vec{x}|^2=13$$
$$\therefore |\vec{x}|=\sqrt{13}$$
Hence the value is √13.

10. If $\vec{a}=2\hat{i}+2\hat{j}+3\hat{k},\ \vec{b}=-\hat{i}+2\hat{j}+\hat{k}$ and $\vec{c}=3\hat{i}+\hat{j}$ are such that $\vec{a}+\lambda\vec{b}$ is perpendicular to $\vec{c}$, then find the value of λ.
Solution:

We know that the

Given vectors are $\vec{a} = 2\hat{i} + 2\hat{j} + 3\hat{k},\ \vec{b} = -\hat{i} + 2\hat{j} + \hat{k},$ and $\vec{c} = 3\hat{i} + \hat{j}.$

Now,

$$\vec{a} + \lambda\vec{b} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) + \lambda\left(-\hat{i} + 2\hat{j} + \hat{k}\right) = (2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}$$

If $\left(\vec{a} + \lambda\vec{b}\right)$ is perpendicular to $\vec{c}$, then

$$\left(\vec{a} + \lambda\vec{b}\right)\cdot\vec{c} = 0.$$

$$\left[(2-\lambda)\hat{i} + (2+2\lambda)\hat{j} + (3+\lambda)\hat{k}\right]\cdot\left(3\hat{i} + \hat{j}\right) = 0$$

$$(2-\lambda)3 + (2+2\lambda)1 + (3+\lambda)0 = 0$$

$$6 - 3\lambda + 2 + 2\lambda = 0$$

$$-\lambda + 8 = 0$$

$$\lambda = 8$$

Therefore, the required value of λ is 8.

11. Show that $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ is perpendicular to $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$, for any two nonzero vectors $\vec{a}$ and $\vec{b}$.

Solution:

Let us consider,

$$\left(|\vec{a}|\vec{b} + |\vec{b}|\vec{a}\right)\cdot\left(|\vec{a}|\vec{b} - |\vec{b}|\vec{a}\right)$$

$$= |\vec{a}|^2\,\vec{b}\cdot\vec{b} - |\vec{a}||\vec{b}|\vec{b}\cdot\vec{a} + |\vec{b}||\vec{a}|\vec{a}\cdot\vec{b} - |\vec{b}|^2\,\vec{a}\cdot\vec{a}$$

$$= |\vec{a}|^2|\vec{b}|^2 - |\vec{b}|^2|\vec{a}|^2$$

$$= 0$$

Therefore, $|\vec{a}|\vec{b} + |\vec{b}|\vec{a}$ and $|\vec{a}|\vec{b} - |\vec{b}|\vec{a}$ are perpendicular to each other.

12. If $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$, then what can be concluded about the vector $\vec{b}$?

Solution:

We know

Given, $\vec{a}\cdot\vec{a} = 0$ and $\vec{a}\cdot\vec{b} = 0$.

Now,

$$\vec{a}\cdot\vec{a} = 0 \Rightarrow |\vec{a}|^2 = 0 \Rightarrow |\vec{a}| = 0$$

$\therefore \vec{a}$ is a zero vector.

Thus, vector $\vec{b}$ satisfying $\vec{a}\cdot\vec{b} = 0$ can be any vector.

13. If $\vec{a},\ \vec{b}$ and $\vec{c}$ are unit vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, find the value of $\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{c} + \vec{c}\cdot\vec{a}$.

Solution:

Consider the given vectors,

Given, $\vec{a}+\vec{b}+\vec{c}=\vec{0}$.
So,

$$\vec{a}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{a}\cdot\vec{0}$$

$$\vec{a}\cdot\vec{a}+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=\vec{a}\cdot\vec{0} \qquad [\text{Distributivity of scalar product over addition}]$$

$$1+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}=0 \quad \ldots (1) \qquad \left[\begin{array}{l}\vec{a}\cdot\vec{a}=|\vec{a}|\cdot|\vec{a}|\cos 0^\circ=1\\ (\vec{a}\text{ is unit vector}\Rightarrow|\vec{a}|=1)\end{array}\right]$$

Next,

$$\vec{b}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{b}\cdot\vec{0}$$

$$\vec{b}\cdot\vec{a}+\vec{b}\cdot\vec{b}+\vec{b}\cdot\vec{c}=\vec{b}\cdot\vec{0}$$

$$\vec{b}\cdot\vec{a}+1+\vec{b}\cdot\vec{c}=0 \quad \ldots (2) \qquad [\vec{b}\cdot\vec{b}=1]$$

And,

$$\vec{c}\cdot(\vec{a}+\vec{b}+\vec{c})=\vec{c}\cdot\vec{0}$$

$$\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+\vec{c}\cdot\vec{c}=\vec{c}\cdot\vec{0}$$

$$\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+1=0 \quad \ldots (3) \qquad [\vec{c}\cdot\vec{c}=1]$$

From (1), (2) and (3),

$$(1+\vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c})+(\vec{b}\cdot\vec{a}+1+\vec{b}\cdot\vec{c})+(\vec{c}\cdot\vec{a}+\vec{c}\cdot\vec{b}+1)=0+0+0$$

$$(3+\vec{a}\cdot\vec{b}+\vec{c}\cdot\vec{a})+(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c})+(\vec{c}\cdot\vec{a}+\vec{b}\cdot\vec{c})=0 \qquad [\text{Scalar product is commutative}]$$

$$3+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})=0$$

$$\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a}=\frac{-3}{2}$$

Hence the value is -3/2.

14. If either vector $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\cdot\vec{b}=0$. But the converse need not be true. Justify your answer with an example.

Solution:

Firstly,

Consider $\vec{a}=2\hat{i}+4\hat{j}+3\hat{k}$ and $\vec{b}=3\hat{i}+3\hat{j}-6\hat{k}$.

Then, their dot product is given by:

$$\vec{a}\cdot\vec{b}=2.3+4.3+3(-6)=6+12-18=0$$

Now, it's seen that

$$|\vec{a}|=\sqrt{2^2+4^2+3^2}=\sqrt{29}$$

$$\therefore \vec{a}\neq\vec{0}$$

$$|\vec{b}|=\sqrt{3^2+3^2+(-6)^2}=\sqrt{54}$$

$$\therefore \vec{b}\neq\vec{0}$$

Therefore, the converse of the given statement need not be true.

15. If the vertices A, B, C of a triangle ABC are (1, 2, 3), (–1, 0, 0), (0, 1, 2), respectively, then find ∠ABC. [∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$]

Solution:

We know

The vertices of ΔABC are given as A (1, 2, 3), B (–1, 0, 0), and C (0, 1, 2).

Also given, ∠ABC is the angle between the vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$.

$$\overrightarrow{BA} = \{1-(-1)\}\hat{i} + (2-0)\hat{j} + (3-0)\hat{k} = 2\hat{i} + 2\hat{j} + 3\hat{k}$$

$$\overrightarrow{BC} = \{0-(-1)\}\hat{i} + (1-0)\hat{j} + (2-0)\hat{k} = \hat{i} + \hat{j} + 2\hat{k}$$

$$\therefore \overrightarrow{BA} \cdot \overrightarrow{BC} = \left(2\hat{i} + 2\hat{j} + 3\hat{k}\right) \cdot \left(\hat{i} + \hat{j} + 2\hat{k}\right) = 2\times1 + 2\times1 + 3\times2 = 2+2+6 = 10$$

$$\left|\overrightarrow{BA}\right| = \sqrt{2^2+2^2+3^2} = \sqrt{4+4+9} = \sqrt{17}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1+1+2^2} = \sqrt{6}$$

Now, we know that

$$\overrightarrow{BA} \cdot \overrightarrow{BC} = \left|\overrightarrow{BA}\right|\left|\overrightarrow{BC}\right| \cos(\angle ABC).$$

$$\therefore 10 = \sqrt{17} \times \sqrt{6} \cos(\angle ABC)$$

$$\cos(\angle ABC) = \frac{10}{\sqrt{17} \times \sqrt{6}}$$

$$\angle ABC = \cos^{-1}\left(\frac{10}{\sqrt{102}}\right)$$

Hence, the angle is $\cos^{-1}(10/\sqrt{102})$.

16. Show that the points A (1, 2, 7), B (2, 6, 3) and C (3, 10, –1) are collinear.

Solution:

Let us consider

Given points are A (1, 2, 7), B (2, 6, 3), and C (3, 10, –1).

Now,

$$\overrightarrow{AB} = (2-1)\hat{i} + (6-2)\hat{j} + (3-7)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{BC} = (3-2)\hat{i} + (10-6)\hat{j} + (-1-3)\hat{k} = \hat{i} + 4\hat{j} - 4\hat{k}$$

$$\overrightarrow{AC} = (3-1)\hat{i} + (10-2)\hat{j} + (-1-7)\hat{k} = 2\hat{i} + 8\hat{j} - 8\hat{k}$$

Now,

$$\left|\overrightarrow{AB}\right| = \sqrt{1^2+4^2+(-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{BC}\right| = \sqrt{1^2+4^2+(-4)^2} = \sqrt{1+16+16} = \sqrt{33}$$

$$\left|\overrightarrow{AC}\right| = \sqrt{2^2+8^2+8^2} = \sqrt{4+64+64} = \sqrt{132} = 2\sqrt{33}$$

$$\therefore \left|\overrightarrow{AC}\right| = \left|\overrightarrow{AB}\right| + \left|\overrightarrow{BC}\right|$$

Therefore, the given points A, B, and C are collinear.

17. Show that the vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ **form the vertices of a right angled triangle.**

Solution:

Firstly consider,

Let vectors $2\hat{i}-\hat{j}+\hat{k},\ \hat{i}-3\hat{j}-5\hat{k}$ and $3\hat{i}-4\hat{j}-4\hat{k}$ be position vectors of points A, B, and C respectively.

So,

$\overrightarrow{OA}=2\hat{i}-\hat{j}+\hat{k},\ \overrightarrow{OB}=\hat{i}-3\hat{j}-5\hat{k}$ and $\overrightarrow{OC}=3\hat{i}-4\hat{j}-4\hat{k}$

Now, vectors $\overrightarrow{AB}, \overrightarrow{BC}$, and $\overrightarrow{AC}$ represent the sides of ΔABC.

Hence,

$$\overrightarrow{AB}=(1-2)\hat{i}+(-3+1)\hat{j}+(-5-1)\hat{k}=-\hat{i}-2\hat{j}-6\hat{k}$$

$$\overrightarrow{BC}=(3-1)\hat{i}+(-4+3)\hat{j}+(-4+5)\hat{k}=2\hat{i}-\hat{j}+\hat{k}$$

$$\overrightarrow{AC}=(2-3)\hat{i}+(-1+4)\hat{j}+(1+4)\hat{k}=-\hat{i}+3\hat{j}+5\hat{k}$$

$$|\overrightarrow{AB}|=\sqrt{(-1)^2+(-2)^2+(-6)^2}=\sqrt{1+4+36}=\sqrt{41}$$

$$|\overrightarrow{BC}|=\sqrt{2^2+(-1)^2+1^2}=\sqrt{4+1+1}=\sqrt{6}$$

$$|\overrightarrow{AC}|=\sqrt{(-1)^2+3^2+5^2}=\sqrt{1+9+25}=\sqrt{35}$$

$$\therefore |\overrightarrow{BC}|^2+|\overrightarrow{AC}|^2=6+35=41=|\overrightarrow{AB}|^2$$

Therefore, ΔABC is a right-angled triangle.

18. If $\vec{a}$ **is a nonzero vector of magnitude '*a*' and λ a nonzero scalar, then** $\lambda\vec{a}$ **is unit vector if**

(A) $\lambda = 1$ (B) $\lambda = -1$ (C) $a = |\lambda|$ (D) $a = 1/|\lambda|$

Solution:

Explanation:

Vector $\lambda\vec{a}$ is a unit vector if $|\lambda\vec{a}|=1$.

Now,

$$|\lambda\vec{a}|=1$$

$$|\lambda||\vec{a}|=1$$

$$|\vec{a}|=\frac{1}{|\lambda|} \qquad [\lambda\neq 0]$$

$$a=\frac{1}{|\lambda|} \qquad [|\vec{a}|=a]$$

Therefore, vector $\lambda\vec{a}$ is a unit vector if $a=\dfrac{1}{|\lambda|}$

The correct answer is D.