Exercise 10.4

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1. Find $|\vec a \times \vec b|$, if $\vec a = \hat i - 7\hat j + 7\hat k$ and $\vec b = 3\hat i - 2\hat j + 2\hat k$

Solution:

It is given that,

$\vec a = \hat i - 7\hat j + 7\hat k$ and $\vec b = 3\hat i - 2\hat j + 2\hat k$

$$\vec a \times \vec b = \begin{vmatrix} \hat i & \hat j & \hat k \\ 1 & -7 & 7 \\ 3 & -2 & 2 \end{vmatrix}$$

$$= \hat i(-14+14) - \hat j(2-21) + \hat k(-2+21) = 19\hat j + 19\hat k$$

Therefore,

$$|\vec a \times \vec b| = \sqrt{(19)^2 + (19)^2} = \sqrt{2 \times (19)^2} = 19\sqrt{2}$$

2. Find a unit vector perpendicular to each of the vector $\vec a + \vec b$ and $\vec a - \vec b$, where $\vec a = 3\hat i + 2\hat j + 2\hat k$ and $\vec b = \hat i + 2\hat j - 2\hat k$.

Solution:

It is given that,

$\vec a = 3\hat i + 2\hat j + 2\hat k$ and $\vec b = \hat i + 2\hat j - 2\hat k$

So, we have

$\vec a + \vec b = 4\hat i + 4\hat j,\ \vec a - \vec b = 2\hat i + 4\hat k$

$$(\vec a + \vec b) \times (\vec a - \vec b) = \begin{vmatrix} \hat i & \hat j & \hat k \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix} = \hat i(16) - \hat j(16) + \hat k(-8) = 16\hat i - 16\hat j - 8\hat k$$

Thus,

$$|(\vec a + \vec b) \times (\vec a - \vec b)| = \sqrt{16^2 + (-16)^2 + (-8)^2}$$

$$= \sqrt{2^2 \times 8^2 + 2^2 \times 8^2 + 8^2}$$

$$= 8\sqrt{2^2 + 2^2 + 1} = 8\sqrt{9} = 8 \times 3 = 24$$

Therefore, the unit vector perpendicular to each of the vectors $\vec a + \vec b$ and $\vec a - \vec b$ is given by the relation,

$$= \pm \frac{(\vec a + \vec b) \times (\vec a - \vec b)}{|(\vec a + \vec b) \times (\vec a - \vec b)|} = \pm \frac{16\hat i - 16\hat j - 8\hat k}{24}$$

$$= \pm \frac{2\hat i - 2\hat j - \hat k}{3} = \pm \frac{2}{3}\hat i \mp \frac{2}{3}\hat j \mp \frac{1}{3}\hat k$$

3. If a unit vector $\vec a$ makes an angles $\frac{\pi}{3}$ with $\hat i, \frac{\pi}{4}$ with $\hat j$ and an acute angle θ with $\hat k$, then find θ and hence, the compounds of $\vec a$.

Solution:

Firstly,

Let unit vector $\vec{a}$ have (a_1, a_2, a_3) components.

$\Rightarrow \vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

As $\vec{a}$ is a unit vector, $|\vec{a}| = 1$.

Also given, that $\vec{a}$ makes angles $\frac{\pi}{3}$ with $\hat{i}$, $\frac{\pi}{4}$ with $\hat{j}$, and an acute angle θ with $\hat{k}$.

Then, we have

$$\cos\frac{\pi}{3} = \frac{a_1}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{2} = a_1 \qquad \left[|\vec{a}| = 1\right]$$

$$\cos\frac{\pi}{4} = \frac{a_2}{|\vec{a}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = a_2 \qquad \left[|\vec{a}| = 1\right]$$

Also, $\cos\theta = \frac{a_3}{|\vec{a}|}$.

$\Rightarrow a_3 = \cos\theta$

Now,

$$|a| = 1$$

$$\sqrt{a_1^2 + a_2^2 + a_3^2} = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \cos^2\theta = 1$$

$$\frac{1}{4} + \frac{1}{2} + \cos^2\theta = 1$$

$$\frac{3}{4} + \cos^2\theta = 1$$

$$\cos^2\theta = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\therefore a_3 = \cos\frac{\pi}{3} = \frac{1}{2}$$

Thus, $\theta = \frac{\pi}{3}$ and the components of $\vec{a}$ are $\left(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

4. Show that

$$\left(\vec{a} - \vec{b}\right) \times \left(\vec{a} + \vec{b}\right) = 2\left(\vec{a} \times \vec{b}\right)$$

Solution:

Firstly consider the LHS,
We have,

$$\begin{aligned}
&(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})\\
&=(\vec{a}-\vec{b})\times\vec{a}+(\vec{a}-\vec{b})\times\vec{b} && [\text{By distributivity of vector product over addition}]\\
&=\vec{a}\times\vec{a}-\vec{b}\times\vec{a}+\vec{a}\times\vec{b}-\vec{b}\times\vec{b} && [\text{Again, by distributivity of vector product over addition}]\\
&=\vec{0}+\vec{a}\times\vec{b}+\vec{a}\times\vec{b}-\vec{0}\\
&=2(\vec{a}\times\vec{b})
\end{aligned}$$

5. Find λ and μ if $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$.

Solution:

It is given that,

Given,

$$(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k})=\vec{0}$$

$$\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 6 & 27\\ 1 & \lambda & \mu\end{vmatrix}=0\hat{i}+0\hat{j}+0\hat{k}$$

$$\hat{i}(6\mu-27\lambda)-\hat{j}(2\mu-27)+\hat{k}(2\lambda-6)=0\hat{i}+0\hat{j}+0\hat{k}$$

On comparing the corresponding components, we have

$$6\mu-27\lambda=0$$
$$2\mu-27=0$$
$$2\lambda-6=0$$

Now,

$$2\lambda-6=0\Rightarrow\lambda=3$$
$$2\mu-27=0\Rightarrow\mu=\frac{27}{2}$$

Thus, $\lambda=3$ and $\mu=\frac{27}{2}$.

6. Given that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$. What can you conclude about the vectors $\vec{a}$ and $\vec{b}$?

Solution:

It is given that,

$\vec{a}\cdot\vec{b}=0$

Then,

(i) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\perp\vec{b}$ $\left(\text{in case } \vec{a} \text{ and } \vec{b} \text{ are non-zero}\right)$

$\vec{a}\times\vec{b}=0$

(ii) Either $|\vec{a}|=0$ or $|\vec{b}|=0$, or $\vec{a}\parallel\vec{b}$ $\left(\text{in case } \vec{a} \text{ and } \vec{b} \text{ are non-zero}\right)$

But, $\vec{a}$ and $\vec{b}$ cannot be perpendicular and parallel simultaneously.

Therefore, $|\vec{a}|=0$ or $|\vec{b}|=0$.

7. Let the vectors $\vec{a},\ \vec{b},\ \vec{c}$ given as $a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$. Then show that $\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$

Solution:

It is given that,

$$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k},\ \vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k},\ \vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$$

$$\left(\vec{b}+\vec{c}\right)=(b_1+c_1)\hat{i}+(b_2+c_2)\hat{j}+(b_3+c_3)\hat{k}$$

$$\text{Now, } \vec{a}\times\left(\vec{b}+\vec{c}\right)\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1+c_1 & b_2+c_2 & b_3+c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2\left(b_3+c_3\right)-a_3\left(b_2+c_2\right)\right]-\hat{j}\left[a_1\left(b_3+c_3\right)-a_3\left(b_1+c_1\right)\right]+\hat{k}\left[a_1\left(b_2+c_2\right)-a_2\left(b_1+c_1\right)\right]$$

$$=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[-a_1b_3-a_1c_3+a_3b_1+a_3c_1\right]+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right] \quad ...(1)$$

And,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$=\hat{i}\left[a_2b_3-a_3b_2\right]+\hat{j}\left[b_1a_3-a_1b_3\right]+\hat{k}\left[a_1b_2-a_2b_1\right] \qquad (2)$$

$$\vec{a}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

$$=\hat{i}\left[a_2c_3-a_3c_2\right]+\hat{j}\left[a_3c_1-a_1c_3\right]+\hat{k}\left[a_1c_2-a_2c_1\right] \qquad (3)$$

On adding (2) and (3), we get

$$\left(\vec{a}\times\vec{b}\right)+\left(\vec{a}\times\vec{c}\right)=\hat{i}\left[a_2b_3+a_2c_3-a_3b_2-a_3c_2\right]+\hat{j}\left[b_1a_3+a_3c_1-a_1b_3-a_1c_3\right]+\hat{k}\left[a_1b_2+a_1c_2-a_2b_1-a_2c_1\right] \qquad (4)$$

From (1) and (4), we obtain

$$\vec{a}\times\left(\vec{b}+\vec{c}\right)=\vec{a}\times\vec{b}+\vec{a}\times\vec{c}$$

- Hence proved.

8. If either $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$, then $\vec{a}\times\vec{b}=\vec{0}$. Is the converse true? Justify your answer with an example.

Solution:

Firstly let us consider,

Take any parallel non-zero vectors so that $\vec{a}\times\vec{b}=\vec{0}$.

Let $\vec{a}=2\hat{i}+3\hat{j}+4\hat{k}$, $\vec{b}=4\hat{i}+6\hat{j}+8\hat{k}$.

Then,

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 3 & 4\\ 4 & 6 & 8\end{vmatrix}=\hat{i}(24-24)-\hat{j}(16-16)+\hat{k}(12-12)=0\hat{i}+0\hat{j}+0\hat{k}=\vec{0}$$

Now, it's seen that

$$|\vec{a}|=\sqrt{2^2+3^2+4^2}=\sqrt{29}$$

$$\therefore \vec{a}\neq\vec{0}$$

$$|\vec{b}|=\sqrt{4^2+6^2+8^2}=\sqrt{116}$$

$$\therefore \vec{b}\neq\vec{0}$$

Thus, the converse of the given statement need not be true.

9. Find the area of the triangle with vertices A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5).

Solution:

We know,

Given A (1, 1, 2), B (2, 3, 5) and C (1, 5, 5) are the vertices of triangle ABC.

The adjacent sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of ΔABC are given as:

$$\overrightarrow{AB}=(2-1)\hat{i}+(3-1)\hat{j}+(5-2)\hat{k}=\hat{i}+2\hat{j}+3\hat{k}$$

$$\overrightarrow{BC}=(1-2)\hat{i}+(5-3)\hat{j}+(5-5)\hat{k}=-\hat{i}+2\hat{j}$$

Now,

$$\text{Area of }\Delta ABC=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|$$

$$\overrightarrow{AB}\times\overrightarrow{BC}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & 2 & 3\\ -1 & 2 & 0\end{vmatrix}=\hat{i}(-6)-\hat{j}(3)+\hat{k}(2+2)=-6\hat{i}-3\hat{j}+4\hat{k}$$

$$\therefore\left|\overrightarrow{AB}\times\overrightarrow{BC}\right|=\sqrt{(-6)^2+(-3)^2+4^2}=\sqrt{36+9+16}=\sqrt{61}$$

Therefore, the area of ΔABC is $\frac{\sqrt{61}}{2}$ square units.

10. Find the area of the parallelogram whose adjacent sides are determined by the

vector $\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$ **.**

Solution:

Let us consider,

The area of the parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Now, the adjacent sides are given as:

$\vec{a}=\hat{i}-\hat{j}+3\hat{k}$ and $\vec{b}=2\hat{i}-7\hat{j}+\hat{k}$

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -1 & 3\\ 2 & -7 & 1\end{vmatrix}=\hat{i}(-1+21)-\hat{j}(1-6)+\hat{k}(-7+2)=20\hat{i}+5\hat{j}-5\hat{k}$$

$$|\vec{a}\times\vec{b}|=\sqrt{20^2+5^2+5^2}=\sqrt{400+25+25}=15\sqrt{2}$$

Therefore, the area of the given parallelogram is $15\sqrt{2}$ square units.

11. Let the vectors $\vec{a}$ and $\vec{b}$ be such that $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$, then $\vec{a}\times\vec{b}$ is a unit vector, if the angle between $\vec{a}$ and $\vec{b}$ is

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

Explanation:

Given, $|\vec{a}|=3$ and $|\vec{b}|=\frac{\sqrt{2}}{3}$.

We know that $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ and θ is the angle between $\vec{a}$ and $\vec{b}$

Now, $\vec{a}\times\vec{b}$ is a unit vector if $|\vec{a}\times\vec{b}|=1$.

$$|\vec{a}\times\vec{b}|=1$$

$$\left||\vec{a}||\vec{b}|\sin\theta\,\hat{n}\right|=1$$

$$|\vec{a}||\vec{b}||\sin\theta|=1$$

$$3\times\frac{\sqrt{2}}{3}\times\sin\theta=1$$

$$\sin\theta=\frac{1}{\sqrt{2}}$$

$$\theta=\frac{\pi}{4}$$

Thus, $\vec{a}\times\vec{b}$ is a unit vector if the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{4}$.

So, the correct answer is B.

12. Area of a rectangle having vertices A, B, C, and D with position vectors $-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ and $-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$ respectively is

(A) $\frac{1}{2}$ (B) 1

(C) 2 (D) 4

Solution:

Explanation:

The position vectors of vertices A, B, C, and D of rectangle ABCD are given as:

$$\overline{\text{OA}}=-\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overline{\text{OB}}=\hat{i}+\frac{1}{2}\hat{j}+4\hat{k},\ \overline{\text{OC}}=\hat{i}-\frac{1}{2}\hat{j}+4\hat{k},\ \overline{\text{OD}}=-\hat{i}-\frac{1}{2}\hat{j}+4\hat{k}$$

The adjacent sides $\overrightarrow{\text{AB}}$ and $\overrightarrow{\text{BC}}$ of the given rectangle are given as.

$$\overrightarrow{\text{AB}}=(1+1)\hat{i}+\left(\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=2\hat{i}$$

$$\overrightarrow{\text{BC}}=(1-1)\hat{i}+\left(-\frac{1}{2}-\frac{1}{2}\right)\hat{j}+(4-4)\hat{k}=-\hat{j}$$

$$\therefore \overrightarrow{\text{AB}}\times\overrightarrow{\text{BC}}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 0 & 0\\ 0 & -1 & 0\end{vmatrix}=\hat{k}(-2)=-2\hat{k}$$

$$\Rightarrow \left|\overrightarrow{\text{AB}}\times\overrightarrow{\text{BC}}\right|=2$$

We know that, the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$ is $|\vec{a}\times\vec{b}|$.

Thus, the area of the given rectangle is $\left|\overrightarrow{\text{AB}}\times\overrightarrow{\text{BC}}\right|=2$ square units.
So, the correct answer is C.