Exercise 8.4 Page: 193

1. Express the trigonometric ratios sin A, sec A and tan A in terms of cot A.

Solution:

To convert the given trigonometric ratios in terms of cot functions, use trigonometric formulas
We know that,
$\text{cosec}^2A - \cot^2A = 1$
$\text{cosec}^2A = 1 + \cot^2A$
Since cosec function is the inverse of sin function, it is written as
$1/\sin^2A = 1 + \cot^2A$
Now, rearrange the terms, it becomes
$\sin^2A = 1/(1+\cot^2A)$
Now, take square roots on both sides, we get
$\sin A = \pm1/(\sqrt{(1+\cot^2A)})$
The above equation defines the sin function in terms of cot function

Now, to express sec function in terms of cot function, use this formula
$\sin^2A = 1/ (1+\cot^2A)$
Now, represent the sin function as cos function
$1 - \cos^2A = 1/ (1+\cot^2A)$
Rearrange the terms,
$\cos^2A = 1 - 1/(1+\cot^2A)$
$\Rightarrow\cos^2A = (1-1+\cot^2A)/(1+\cot^2A)$
Since sec function is the inverse of cos function,
$\Rightarrow 1/\sec^2A = \cot^2A/(1+\cot^2A)$
Take the reciprocal and square roots on both sides, we get
$\Rightarrow \sec A = \pm\sqrt{(1+\cot^2A)}/\cot A$

Now, to express tan function in terms of cot function
$\tan A = \sin A/\cos A$ and $\cot A = \cos A/\sin A$
Since cot function is the inverse of tan function, it is rewritten as
$\tan A = 1/\cot A$

2. Write all the other trigonometric ratios of ∠A in terms of sec A.

Solution:

Cos A function in terms of sec A:

sec A = $1/\cos A$

$\Rightarrow \cos A = 1/\sec A$

sec A function in terms of sec A:

$\cos^2 A + \sin^2 A = 1$

Rearrange the terms

$\sin^2 A = 1 - \cos^2 A$

$\sin^2 A = 1 - (1/\sec^2 A)$

$\sin^2 A = (\sec^2 A - 1)/\sec^2 A$

$\sin A = \pm \sqrt{(\sec^2 A - 1)}/\sec A$

cosec A function in terms of sec A:

$\sin A = 1/\text{cosec } A$

$\Rightarrow \text{cosec } A = 1/\sin A$

$\text{cosec } A = \pm \sec A/\sqrt{(\sec^2 A - 1)}$

Now, tan A function in terms of sec A:

$\sec^2 A - \tan^2 A = 1$

Rearrange the terms

$\Rightarrow \tan^2 A = \sec^2 A + 1$

$\tan A = \sqrt{(\sec^2 A + 1)}$

cot A function in terms of sec A:

$\tan A = 1/\cot A$

$\Rightarrow \cot A = 1/\tan A$

$\cot A = \pm 1/\sqrt{(\sec^2 A + 1)}$

3. Evaluate:

(i) $(\sin^2 63° + \sin^2 27°)/(\cos^2 17° + \cos^2 73°)$

(ii) $\sin 25° \cos 65° + \cos 25° \sin 65°$

Solution:

(i) $(\sin^2 63° + \sin^2 27°)/(\cos^2 17° + \cos^2 73°)$

To simplify this, convert some of the sin functions into cos functions and cos function into sin function and it becomes,

$= [\sin^2(90°-27°) + \sin^2 27°] / [\cos^2(90°-73°) + \cos^2 73°)]$

$= (\cos^2 27° + \sin^2 27°)/(\sin^2 27° + \cos^2 73°)$

$= 1/1 = 1$ (since $\sin^2 A + \cos^2 A = 1$)

Therefore, $(\sin^2 63° + \sin^2 27°)/(\cos^2 17° + \cos^2 73°) = 1$

(ii) sin 25° cos 65° + cos 25° sin 65°
To simplify this, convert some of the sin functions into cos functions and cos function into sin function and it becomes,
= sin(90°-25°) cos 65° + cos (90°-65°) sin 65°
= cos 65° cos 65° + sin 65° sin 65°
= $\cos^2 65° + \sin^2 65° = 1$ (since $\sin^2 A + \cos^2 A = 1$)
Therefore, sin 25° cos 65° + cos 25° sin 65° = 1

4. Choose the correct option. Justify your choice.
(i) $9 \sec^2 A - 9 \tan^2 A =$
(A) 1 (B) 9 (C) 8 (D) 0
(ii) (1 + tan θ + sec θ) (1 + cot θ - cosec θ)
(A) 0 (B) 1 (C) 2 (D) - 1
(iii) (sec A + tan A) (1 – sin A) =
(A) sec A (B) sin A (C) cosec A (D) cos A
(iv) $1+\tan^2 A/1+\cot^2 A =$
(A) $\sec^2 A$ (B) -1 (C) $\cot^2 A$ (D) $\tan^2 A$

Solution:

(i) (B) is correct.
Justification:
Take 9 outside, and it becomes
$9 \sec^2 A - 9 \tan^2 A$
$= 9 (\sec^2 A - \tan^2 A)$
$= 9 \times 1 = 9$ (∵ sec2 A - tan2 A = 1)
Therefore, $9 \sec^2 A - 9 \tan^2 A = 9$

(ii) (C) is correct
Justification:
(1 + tan θ + sec θ) (1 + cot θ - cosec θ)
We know that, tan θ = sin θ/cos θ
sec θ = 1/ cos θ
cot θ = cos θ/sin θ
cosec θ = 1/sin θ
Now, substitute the above values in the given problem, we get
= (1 + sin θ/cos θ + 1/ cos θ) (1 + cos θ/sin θ - 1/sin θ)

Simplify the above equation,
$= (\cos\theta + \sin\theta + 1)/\cos\theta \times (\sin\theta + \cos\theta - 1)/\sin\theta$
$= (\cos\theta + \sin\theta)^2 - 1^2/(\cos\theta \sin\theta)$
$= (\cos^2\theta + \sin^2\theta + 2\cos\theta \sin\theta - 1)/(\cos\theta \sin\theta)$

$= (1+ 2\cos\theta \sin\theta -1)/(\cos\theta \sin\theta)$ (Since $\cos^2\theta + \sin^2\theta = 1$)

$= (2\cos\theta \sin\theta)/(\cos\theta \sin\theta) = 2$

Therefore, $(1 + \tan\theta + \sec\theta)(1 + \cot\theta - \text{cosec}\,\theta) = 2$

(iii) (D) is correct.

Justification:

We know that,

Sec A= 1/cos A

Tan A = sin A / cos A

Now, substitute the above values in the given problem, we get

(secA + tanA) (1 - sinA)

= (1/cos A + sin A/cos A) (1 - sinA)

$= (1+\sin A/\cos A)(1 - \sin A)$

$= (1 - \sin^2 A)/\cos A$

$= \cos^2 A/\cos A = \cos A$

Therefore, (secA + tanA) (1 - sinA) = cos A

(iv) (D) is correct.

Justification:

We know that,

$\tan^2 A = 1/\cot^2 A$

Now, substitute this in the given problem, we get

$1+\tan^2 A/1+\cot^2 A$

$= (1+1/\cot^2 A)/1+\cot^2 A$

$= (\cot^2 A+1/\cot^2 A)\times(1/1+\cot^2 A)$

$= 1/\cot^2 A = \tan^2 A$

So, $1+\tan^2 A/1+\cot^2 A = \tan^2 A$

5. Prove the following identities, where the angles involved are acute angles for which the expressions are defined.

(i) $(\text{cosec}\,\theta - \cot\theta)^2 = (1-\cos\theta)/(1+\cos\theta)$

(ii) $\cos A/(1+\sin A) + (1+\sin A)/\cos A = 2\sec A$

(iii) $\tan\theta/(1-\cot\theta) + \cot\theta/(1-\tan\theta) = 1 + \sec\theta\,\text{cosec}\,\theta$

[Hint : Write the expression in terms of sin θ and cos θ]

(iv) $(1 + \sec A)/\sec A = \sin^2 A/(1-\cos A)$

[Hint : Simplify LHS and RHS separately]

(v) $(\cos A - \sin A+1)/(\cos A + \sin A - 1) = \text{cosec}\,A + \cot A$**, using the identity** $\text{cosec}^2 A = 1+\cot^2 A$**.**

(vi) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

(vii) $(\sin\theta - 2\sin^3\theta)/(2\cos^3\theta - \cos\theta) = \tan\theta$

(viii) $(\sin A + \text{cosec } A)^2 + (\cos A + \sec A)^2 = 7+\tan^2 A+\cot^2 A$

(ix) $(\text{cosec } A - \sin A)(\sec A - \cos A) = 1/(\tan A+\cot A)$

[Hint : Simplify LHS and RHS separately]

(x) $(1+\tan^2 A/1+\cot^2 A) = (1-\tan A/1-\cot A)^2 = \tan^2 A$

Solution:

(i) $(\text{cosec }\theta - \cot\theta)^2 = (1-\cos\theta)/(1+\cos\theta)$

To prove this, first take the Left-Hand side (L.H.S) of the given equation, to prove the Right Hand Side (R.H.S)

L.H.S. = $(\text{cosec }\theta - \cot\theta)^2$

The above equation is in the form of $(a-b)^2$, and expand it

Since $(a-b)^2 = a^2 + b^2 - 2ab$

Here $a = \text{cosec }\theta$ and $b = \cot\theta$

$= (\text{cosec}^2\theta + \cot^2\theta - 2\text{cosec }\theta\cot\theta)$

Now, apply the corresponding inverse functions and equivalent ratios to simplify

$= (1/\sin^2\theta + \cos^2\theta/\sin^2\theta - 2\cos\theta/\sin^2\theta)$

$= (1 + \cos^2\theta - 2\cos\theta)/(1 - \cos^2\theta)$

$= (1-\cos\theta)^2/(1 - \cos\theta)(1+\cos\theta)$

$= (1-\cos\theta)/(1+\cos\theta)$ = R.H.S.

Therefore, $(\text{cosec }\theta - \cot\theta)^2 = (1-\cos\theta)/(1+\cos\theta)$

Hence proved.

(ii) $(\cos A/(1+\sin A)) + ((1+\sin A)/\cos A) = 2\sec A$

Now, take the L.H.S of the given equation.

L.H.S. = $(\cos A/(1+\sin A)) + ((1+\sin A)/\cos A)$

$= [\cos^2 A + (1+\sin A)^2]/(1+\sin A)\cos A$

$= (\cos^2 A + \sin^2 A + 1 + 2\sin A)/(1+\sin A)\cos A$

Since $\cos^2 A + \sin^2 A = 1$, we can write it as

$= (1 + 1 + 2\sin A)/(1+\sin A)\cos A$

$= (2+ 2\sin A)/(1+\sin A)\cos A$

$= 2(1+\sin A)/(1+\sin A)\cos A$

$= 2/\cos A = 2\sec A$ = R.H.S.

L.H.S. = R.H.S.

$(\cos A/(1+\sin A)) + ((1+\sin A)/\cos A) = 2\sec A$

Hence proved.

(iii) $\tan\theta/(1-\cot\theta) + \cot\theta/(1-\tan\theta) = 1 + \sec\theta\ \mathrm{cosec}\ \theta$

L.H.S. = $\tan\theta/(1-\cot\theta) + \cot\theta/(1-\tan\theta)$
We know that $\tan\theta = \sin\theta/\cos\theta$
$\cot\theta = \cos\theta/\sin\theta$
Now, substitute it in the given equation, to convert it in a simplified form

$= [(\sin\theta/\cos\theta)/1-(\cos\theta/\sin\theta)] + [(\cos\theta/\sin\theta)/1-(\sin\theta/\cos\theta)]$

$= [(\sin\theta/\cos\theta)/(\sin\theta-\cos\theta)/\sin\theta] + [(\cos\theta/\sin\theta)/(\cos\theta-\sin\theta)/\cos\theta]$

$= \sin^2\theta/[\cos\theta(\sin\theta-\cos\theta)] + \cos^2\theta/[\sin\theta(\cos\theta-\sin\theta)]$

$= \sin^2\theta/[\cos\theta(\sin\theta-\cos\theta)] - \cos^2\theta/[\sin\theta(\sin\theta-\cos\theta)]$

$= 1/(\sin\theta-\cos\theta)\ [(\sin^2\theta/\cos\theta) - (\cos^2\theta/\sin\theta)]$

$= 1/(\sin\theta-\cos\theta) \times [(\sin^3\theta - \cos^3\theta)/\sin\theta\cos\theta]$

$= [(\sin\theta-\cos\theta)(\sin^2\theta+\cos^2\theta+\sin\theta\cos\theta)]/[(\sin\theta-\cos\theta)\sin\theta\cos\theta]$

$= (1 + \sin\theta\cos\theta)/\sin\theta\cos\theta$

$= 1/\sin\theta\cos\theta + 1$

$= 1 + \sec\theta\ \mathrm{cosec}\ \theta$ = R.H.S.

Therefore, L.H.S. = R.H.S.
Hence proved

(iv) $(1 + \sec A)/\sec A = \sin^2 A/(1-\cos A)$
First find the simplified form of L.H.S

L.H.S. = $(1 + \sec A)/\sec A$
Since secant function is the inverse function of cos function and it is written as

$= (1 + 1/\cos A)/1/\cos A$

$= (\cos A + 1)/\cos A/1/\cos A$

Therefore, $(1 + \sec A)/\sec A = \cos A + 1$

R.H.S. = $\sin^2 A/(1-\cos A)$
We know that $\sin^2 A = (1 - \cos^2 A)$, we get

$= (1 - \cos^2 A)/(1-\cos A)$

$= (1-\cos A)(1+\cos A)/(1-\cos A)$

Therefore, $\sin^2 A/(1-\cos A) = \cos A + 1$
L.H.S. = R.H.S.
Hence proved

(v) $(\cos A–\sin A+1)/(\cos A+\sin A–1) = \mathrm{cosec}\ A + \cot A$, using the identity $\mathrm{cosec}^2 A = 1+\cot^2 A$.

With the help of identity function, $\text{cosec}^2A = 1+\cot^2A$, let us prove the above equation.

L.H.S. = (cos A–sin A+1)/(cos A+sin A–1)

Divide the numerator and denominator by sin A, we get

= (cos A–sin A+1)/sin A/(cos A+sin A–1)/sin A

We know that cos A/sin A = cot A and 1/sin A = cosec A

= (cot A - 1 + cosec A)/(cot A+ 1 – cosec A)

= (cot A - cosec^2A + $\cot^2$A + cosec A)/(cot A+ 1 – cosec A) (using cosec^2A - $\cot^2$A = 1

= [(cot A + cosec A) - (cosec^2A - $\cot^2$A)]/(cot A+ 1 – cosec A)

= [(cot A + cosec A) - (cosec A + cot A)(cosec A - cot A)]/(1 – cosec A + cot A)

= (cot A + cosec A)(1 – cosec A + cot A)/(1 – cosec A + cot A)

= cot A + cosec A = R.H.S.

Therefore, (cos A–sin A+1)/(cos A+sin A–1) = cosec A + cot A

Hence Proved

(vi) $\sqrt{\frac{1+\sin A}{1-\sin A}} = \sec A + \tan A$

L.H.S = $\sqrt{\frac{1+\sin A}{1-\sin A}}$

First divide the numerator and denominator of L.H.S. by cos A,

$$= \sqrt{\frac{\frac{1}{\cos A}+\frac{\sin A}{\cos A}}{\frac{1}{\cos A}-\frac{\sin A}{\cos A}}}$$

We know that 1/cos A = sec A and sin A/ cos A = tan A and it becomes,

= √(sec A+ tan A)/(sec A-tan A)

Now using rationalization, we get

$$= \sqrt{\frac{\sec A+\tan A}{\sec A-\tan A}} \times \sqrt{\frac{\sec A+\tan A}{\sec A+\tan A}}$$

$$= \sqrt{\frac{(\sec A+\tan A)^2}{\sec^2 A-\tan^2 A}}$$

= (sec A + tan A)/1

= sec A + tan A = R.H.S

Hence proved

(vii) $(\sin\theta - 2\sin^3\theta)/(2\cos^3\theta - \cos\theta) = \tan\theta$

L.H.S. = $(\sin\theta - 2\sin^3\theta)/(2\cos^3\theta - \cos\theta)$

Take $\sin\theta$ as in numerator and $\cos\theta$ in denominator as outside, it becomes

$= [\sin\theta(1 - 2\sin^2\theta)]/[\cos\theta(2\cos^2\theta - 1)]$

We know that $\sin^2\theta = 1-\cos^2\theta$

$= \sin\theta[1 - 2(1-\cos^2\theta)]/[\cos\theta(2\cos^2\theta - 1)]$

$= [\sin\theta(2\cos^2\theta - 1)]/[\cos\theta(2\cos^2\theta - 1)]$

$= \tan\theta$ = R.H.S.

Hence proved

(viii) $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7+\tan^2 A+\cot^2 A$

L.H.S. = $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2$

It is of the form $(a+b)^2$, expand it

$(a+b)^2 = a^2 + b^2 + 2ab$

$= (\sin^2 A + \operatorname{cosec}^2 A + 2\sin A \operatorname{cosec} A) + (\cos^2 A + \sec^2 A + 2\cos A \sec A)$

$= (\sin^2 A + \cos^2 A) + 2\sin A(1/\sin A) + 2\cos A(1/\cos A) + 1 + \tan^2 A + 1 + \cot^2 A$

$= 1 + 2 + 2 + 2 + \tan^2 A + \cot^2 A$

$= 7+\tan^2 A+\cot^2 A$ = R.H.S.

Therefore, $(\sin A + \operatorname{cosec} A)^2 + (\cos A + \sec A)^2 = 7+\tan^2 A+\cot^2 A$

Hence proved.

(ix) $(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = 1/(\tan A+\cot A)$

First, find the simplified form of L.H.S

L.H.S. = $(\operatorname{cosec} A - \sin A)(\sec A - \cos A)$

Now, substitute the inverse and equivalent trigonometric ratio forms

$= (1/\sin A - \sin A)(1/\cos A - \cos A)$

$= [(1-\sin^2 A)/\sin A][(1-\cos^2 A)/\cos A]$

$= (\cos^2 A/\sin A)\times(\sin^2 A/\cos A)$

$= \cos A \sin A$

Now, simplify the R.H.S

R.H.S. = $1/(\tan A+\cot A)$

$= 1/(\sin A/\cos A + \cos A/\sin A)$

$= 1/[(\sin^2 A+\cos^2 A)/\sin A\cos A]$

$= \cos A \sin A$

L.H.S. = R.H.S.

$(\operatorname{cosec} A - \sin A)(\sec A - \cos A) = 1/(\tan A+\cot A)$

Hence proved

(x) $(1+\tan^2 A/1+\cot^2 A) = (1-\tan A/1-\cot A)^2 = \tan^2 A$

L.H.S. = $(1+\tan^2 A/1+\cot^2 A)$

Since cot function is the inverse of tan function,

$= (1+\tan^2 A/1+1/\tan^2 A)$

$= 1+\tan^2 A/[(1+\tan^2 A)/\tan^2 A]$

Now cancel the $1+\tan^2 A$ terms, we get

$= \tan^2 A$

$(1+\tan^2 A/1+\cot^2 A) = \tan^2 A$

Similarly,

$(1-\tan A/1-\cot A)^2 = \tan^2 A$

Hence proved