KSEEB Class 12th Maths Question Paper With Solutions 2018

PART - A

Answer all the ten questions. **[10 * 1 = 10]**

Question 1: Define a bijective function.

Solution:
A function which is both one-one and onto is called a bijective function.
A function f : A → B is a bisection if it is one-one as well as onto.

Question 2: Write the principal value branch of $\cos^{-1} x$.

Solution:
$[0, \pi]$

Question 3: Construct a 2 x 2 matrix $A = [a_{ij}]$, whose elements are given by $a_{ij} = i / j$.

Solution:
$a_{11} = i / j = 1 / 1 = 1$
$a_{12} = i / j = 1 / 2$
$a_{21} = i / j = 2 / 1 = 2$
$a_{22} = i / j = 1 / 1 = 1$

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ 2 & 1 \end{bmatrix}$$

Question 4: If A is an invertible matrix of order 2, then find A^{-1}.

Solution:

A is an invertible matrix of order 2.

$AA^{-1} = I = A^{-1}A$ and $|A| \neq 0$

$|AA^{-1}| = |I|$

$|A|\ |A^{-1}| = 1$

$|A^{-1}| = 1 / |A|$

Question 5: If y = e^{x^3}, then find dy / dx.

Solution:

$y = e^{x^3}$

Taking log on both sides,

$\log y = x^3 \log e$

$\log y = x^3$

$d(\log y) / dx = d(x^3)\ dx$

$(1 / y)(dy / dx) = 3x^2$

$dy / dx = y * 3x^2$

$= 3x^2\ e^{x^3}$

Question 6: Find $\int (x^3 - 1) / x^2\ dx$.

Solution:

$I = \int (x^3 - 1) / x^2\ dx$

$= \int x^3 / x^2\ dx - \int 1 / x^2\ dx$

$= \int x\ dx - \int x^{-2}\ dx$

$= (x^2 / 2) - (x^{-2+1} / (-2 + 1)) + c$

$= (x^2 / 2) + (1 / x) + c$

Question 7: Find the unit vector in the direction of the vector a = i + j + 2k.

Solution:

a [vector] = i + j + 2k

$|a| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$

a = a / |a|

= i + j + 2k / √6

= (1 / √6) i + (1 / √6) j + (2 / √6) k

Question 8: If a line makes an angle 90º, 60º, 30º with the positive direction of x, y and z-axis respectively, find its direction cosines.

Solution:

The direction cosines are given by

l = cos 90º

m = cos 60º

n = cos 30º

0, (1 / 2) and √3 / 2

Question 9: Define an optimal solution in a linear programming problem.

Solution:

A point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an optimal solution.

Question 10: If P (A) = 7 / 13, P (B) = 9 / 13 and P (A ∩ B) = 4 / 13, find P (A / B).

Solution:

P (A) = 7 / 13

P (B) = 9 / 13

P (A ∩ B) = 4 / 13

P (A / B) = P (A ∩ B) / P (B)

= (4 / 13) / (9 / 13)

= 4 / 9

PART - B

Answer any ten questions. [10 * 2 = 20]

Question 11: Let "*" be a binary operation on Q, defined by a * b = ab / 2, ∀ (a, b) ∈ Q. Determine whether "*" is associative or not.

Solution:
(a * b) * c = (ab / 2) * c
= (ab / 2) c / 2
= abc / 4
* is associative.

Question 12: If sin [$\sin^{-1}$ (1 / 5) + $\cos^{-1}$ x] = 1, then find the value of x.

Solution:
sin [$\sin^{-1}$ (1 / 5) + $\cos^{-1}$ x] = 1
[$\sin^{-1}$ (1 / 5) + $\cos^{-1}$ x] = $\sin^{-1}$ (1)
[$\sin^{-1}$ (1 / 5) + $\cos^{-1}$ x] = **π** / 2
$\sin^{-1}$ (1 / 5) = (**π** / 2) - $\cos^{-1}$ x
$\sin^{-1}$ (1 / 5) = $\sin^{-1}$ x + $\cos^{-1}$ x - $\cos^{-1}$ x
$\sin^{-1}$ (1 / 5) = $\sin^{-1}$ x
x = 1 / 5

Question 13: Write the simplest form of $\tan^{-1}$ [(cosx - sinx) / (cosx + sinx)], 0 < x < π / 2.

Solution:
$\tan^{-1}$ [(cosx - sinx) / (cosx + sinx)]
= $\tan^{-1}$ [(cosx / cosx - sinx / cosx) / (cosx / cosx + sinx / cosx)]
= $\tan^{-1}$ ([1 - tanx] / [1 + tanx])
= $\tan^{-1}$ tan [**π** / 4 - x]
= **π** / 4 - x

Question 14: Find the area of the triangle whose vertices are (-2, -3) (3, 2) and (-1, -8) by using the determinant method.

Solution:

Area of triangle = (1 / 2) [x_1 (y_2 - y_3) + x_2 (y_3 - y_1) + x_3 (y_1 - y_2)]

= (1 / 2) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$

= (1 / 2) $\begin{vmatrix} 3 & 2 & 1 \\ -2 & -3 & 1 \\ -1 & -8 & 1 \end{vmatrix}$

= (1 / 2) [3 (-3 x 1 - 1 (-8)) - 2 (-2 x 1 - 1 x (-1)) + 1 [-2 (-8) - (-1) (-3)]

= (1 / 2) [3 (-3 + 8) - 2 (-2 + 1) + 1 (16 - 3)]

= (1 / 2) [15 + 2 + 13]

= 15 square units

Question 15: Differentiate $x^{\sin x}$, x > 0 with respect to x.

Solution:

y = $x^{\sin x}$

Taking log on both sides,

log y = sinx log x

dy / dx = $x^{\sin x}$ d (sinx . logx) / dx

dy / dx = y [sinx * (1 / x) + logx * cosx]

= $x^{\sin x}$ [sinx * (1 / x) + logx * cosx]

Question 16: Find dy / dx if x^2 + xy + y^2 = 100.

Solution:

2x + [x (dy / dx) + y] + 2y (dy / dx) = 0

dy / dx = - [2x + y] / (x + 2y)

Question 17: Find the slope of the tangent to the curve y = x^3 - x at x = 2.

Solution:

dy / dx = $3x^2$ - 1

$(dy / dx)_{x = 2}$ = 11

Question 18: Integrate $\frac{e^{\tan^{-1} x}}{1+x^2}$ with respect to x.

Solution:

Put $\tan^{-1} x = t$

$(1 / (1 + x^2)) = dt / dx$

$dx = (1 + x^2)\, dt$

$I = \int e^t / (1 + x^2) * (1 + x^2)\, dt$

$= \int e^t\, dt$

$= e^t + c$

$= e^{\tan^{-1} x} + c$

Question 19: Evaluate $\int_2^3 x\, dx / x^2 + 1$.

Solution:

Put $t = x^2 + 1$

$x\, dx = dt / 2$

When $x = 2$, $t = 5$ and $x = 3$, $t = 10$

$I = (1 / 2) \int_2^3 2x / (1 + x^2)\, dx$

$= (1 / 2) [\log 10 - \log 5]$

$= (1 / 2) \log 2$

Question 20: Find the order and degree of the differential equation $(d^3y / dx^3)^2 + (d^2y / dx^2)^3 + (dy / dx)^4 + y^5 = 0$.

Solution:

Order = 3

Degree = 2

Question 21: Find the projection of the vector i + 3j + 7k on the vector 7i - j + 8k.

Solution:

a = i + 3j + 7k

b = 7i - j + 8k

The projection of a on b = (a . b) / |b|

= [(i + 3j + 7k) . (7i − j + 8k)] / [$\sqrt{7^2 + (-1)^2 + 8^2}$]

= [1 * 7 + 3 * (−1) + 7 * 8] / $\sqrt{49 + 1 + 64}$

= [7 − 3 + 56] / $\sqrt{144}$

= 60 / $\sqrt{144}$

Question 22: Find the area of the parallelogram whose adjacent sides are determined by the vectors a = 3i + j + 4k and b = i - j + k.

Solution:

a = 3i + j + 4k

b = i - j + k

a x b = $\begin{vmatrix} i & j & k \\ 3 & 1 & 4 \\ 1 & -1 & 1 \end{vmatrix}$

= i (1 * 1 - 4 (-1)) + j (1 * 4 - 1 * 3) + k ((-1) * 3 - 1 * 1)

= 5i + j - 4k

= $\sqrt{5^2 + 1^2 + (-4)^2}$

= $\sqrt{25 + 1 + 16}$

= $\sqrt{42}$

Question 23: Find the angle between the planes whose vector equations are r . (2i + 2j - 3k) = 5 and r . (3i - 3j + 5k) = 3.

Solution:

n_1 = 2i + 2j - 3k

n_2 = 3i - 3j + 5k

$n_1 . n_2$ = 6 - 6 - 15 = - 15

$|n_1|$ = $\sqrt{4 + 4 + 9}$ = $\sqrt{17}$

$|n_2| = \sqrt{9 + 9 + 25} = \sqrt{43}$

$\cos\theta = ||n_1 . n_2| / |n_1| |n_2||$

$= |(-15) / \sqrt{17} . \sqrt{43}|$

$= |(-15) / \sqrt{731}|$

$= 15 / \sqrt{731}$

$\theta = \cos^{-1} [15 / \sqrt{731}]$

Question 24: A random variable X has the following probability distribution.

X	**0**	**1**	**2**	**3**	**4**
P (X)	**0.1**	**k**	**2k**	**2k**	**k**

Determine:
[i] k
[ii] P (X ≥ 2]

Solution:

[i] $\sum p(x) = 1$

$0.1 + k + 2k + 2k + k = 1$

$6k + 0.1 = 1$

$6k = 1 - 0.1$

$6k = 0.9$

$k = 0.9 / 6$

$k = 0.15$

[ii] $P(X \geq 2)$

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= 2k + 2k + k$

$= 5k$

$= 5 * 0.15$

$= 0.75$

PART - C

Answer any ten questions. **[10 * 3 = 30]**

Question 25: Show that the relation R in the set A = {1, 2, 3, 4, 5} given by R = {(a, b) : |a - b| is even}, is an equivalence relation.

Solution:

R = {(1, 1), (1, 3), (1, 5), (2, 2), (2, 4), (3, 1), (3, 3), (3, 5), (4, 2), (4, 4), (5, 1), (5, 3), (5, 5)}

$(a, a) \in R, \forall a \in A$

$\therefore$ R is Reflexive.

$(a, b) \in R$

$\Rightarrow (b, a) \in R$

$\therefore$ R is symmetric.

(a, b) and $(b, c) \in R$

$\Rightarrow (a, c) \in R$

$\therefore$ R is transitive.

$\therefore$ R is an equivalence relation.

Question 26: Prove that 2 $\tan^{-1}$ (1 / 2) + $\tan^{-1}$ (1 / 7) = $\tan^{-1}$ (31 / 17).

Solution:

LHS = 2 $\tan^{-1}$ (1 / 2) + $\tan^{-1}$ (1 / 7)

= $\tan^{-1}$ {2 * (1 / 2) / 1 - $(1 / 2)^2$} + $\tan^{-1}$ (1 / 7)

= $\tan^{-1}$ {1 / (1 - (1 / 4))} + $\tan^{-1}$ (1 / 7)

= $\tan^{-1}$ (4 / 3) + $\tan^{-1}$ (1 / 7)

= $\tan^{-1}$ {(4 / 3) + (1 / 7) / 1 - (4 / 3) (1 / 7)}

= $\tan^{-1}$ {(31 / 21) / (1 - (4 / 21))}

= $\tan^{-1}$ {(31 / 21) / (17 / 21)}

= $\tan^{-1}$ (31 / 17)

Question 27: By using elementary transformation, find the inverse of the matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$.

Solution:

Write the augmented matrix

	A_1	A_2	B_1	B_2
1	1	3	1	0
2	2	7	0	1

Find the pivot in the 1st column in the 1st row

	A_1	A_2	B_1	B_2
1	1	3	1	0
2	2	7	0	1

Eliminate the 1st column

	A_1	A_2	B_1	B_2
1	1	3	1	0
2	0	1	-2	1

Find the pivot in the 2nd column in the 2nd row

	A_1	A_2	B_1	B_2
1	1	3	1	0
2	0	1	-2	1

Eliminate the 2nd column

	A_1	A_2	B_1	B_2
1	1	0	7	-3
2	0	1	-2	1

There is the inverse matrix on the right

	A_1	A_2	B_1	B_2
1	1	0	7	-3
2	0	1	-2	1

Question 28: If x = sint, y = cos2t then prove that dy / dx = - 4 sint.

Solution:

x = sint

dx / dt = cost

y = cos2t

dy / dt = - 2 sin 2t

dy / dx = [dy / dt] / [dx / dt]

= - 2 sin 2t / cos t

= - 2* 2 sint cost / cost

= - 4 sint cost / cost

= - 4 sint

Question 29: Verify Rolle's theorem for the function f (x) = $x^2 + 2$, $x \in [-2, 2]$.

Solution:

f (x) = $x^2 + 2$, $x \in [-2, 2]$

A polynomial function is continuous and differentiable everywhere in a real number.

$\therefore$ f (x) is continuous on [−2, 2] and differentiable on (−2, 2).

f (x) = $x^2 + 2$

f (–2) = $(-2)^2 + 2 = 6$

f (2) = $(2)^2 + 2 = 6$

$\therefore$ f (−2) = f (2) = 6

All the three conditions of Rolle's theorem are satisfied.

f (x) = $x^2 + 2$

f'(x) = 2x

f'(c) = 2c

0 = 2c

$\therefore$ c = 0

$\therefore$ $-2 < c < 2$

Hence, Rolle's theorem is verified.

Question 30: Find two numbers whose sum is 24 and whose product is as large as possible.

Solution:

Let one number be x and another be (24 - x).

Suppose y denotes the product of the two numbers, then $y = x(24 - x) = 24x - x^2$

On differentiating with respect to x,

$dy / dx = 24 - 2x$

$d^2y / dx^2 = -2$

Put $dy / dx = 0$

$24 - 2x = 0$

$24 = 2x$

$24 / 2 = x$

$12 = x$

At x = 12, the second derivative $d^2y / dx^2 < 0$.

The two numbers are 12 and 24 - x = 24 - 12 = 12.

So, 12 and 12 are the two numbers.

Question 31: Find $\int x \, dx / (x + 1)(x + 2)$.

Solution:

$\int x \, dx / (x + 1)(x + 2)$

By using partial fractions method,

$= A / (x + 1) + B / (x + 2)$

$x = A(x + 2) + B(x + 1)$

Put x = - 2,

$-2 = A(-2 + 2) + B(-2 + 1)$

$-2 = B(-1)$

$B = 2$

Put x = - 1,

$-1 = A(-1 + 2) + B(-1 + 1)$

$-1 = A(1) + 0$

$A = -1$

$\int x\, dx / (x + 1)(x + 2) = - 1 / (x + 1) + 2 / (x + 2)$

$= \int - 1 / (x + 1)\, dx + 2 \int dx / (x + 2)$

$= - \log |x + 1| + 2 \log |x + 2| + c$

$= - \log |x + 1| + \log |x + 2|^2 + c$

$= \log |(x + 2)^2 / (x + 1)| + c$

Question 32: Find $\int e^x \sin x\, dx$.

Solution:

$\int e^x \sin x\, dx$

$= \sin x \int e^x\, dx - \int [d(\sin x) / dx \int e^x\, dx]\, dx$

$= e^x \sin x - \int \cos x\, e^x\, dx$

$= e^x \sin x - [\cos x \int e^x\, dx - \int [d(\cos x) / dx \int e^x\, dx]]\, dx$

$= e^x \sin x - [\cos x\, e^x + \int \sin x\, e^x\, dx]$

$= e^x \sin x - e^x \cos x - \int \sin x\, e^x\, dx$

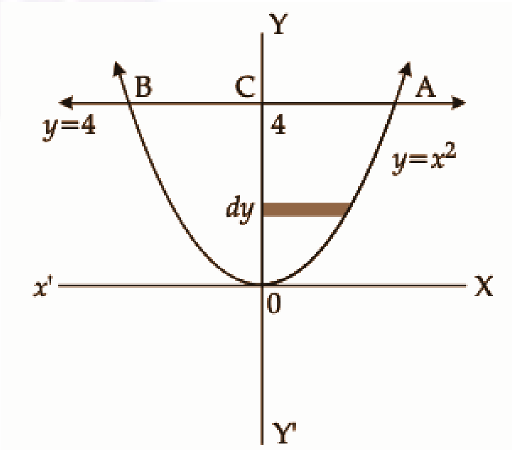
$2 \int \sin x\, e^x\, dx = e^x \sin x - e^x \cos x + c$

$= e^x / 2 (\sin x - \cos x) + c / 2$

$= e^x / 2 (\sin x - \cos x) + c$

Question 33: Find the area of the region bounded by the curve $y = x^2$ and the line $y = 4$.

Solution:



The area enclosed by $y = x^2$ and the line $y = 4$ is given by

Area BOAB = 2 * area of OACO

$= 2 \int_0^4 x\, dy$

$= 2 \int_0^4 \sqrt{y}\, dy$

$= 2 [y^{3/2} / (3 / 2)]_0^4$

$= (4 / 3) [y^{3/2}]_0^4$

$= (4 / 3) [4^{3/2} - 0^{3/2}]$

$= (4 / 3) [8 - 0]$

$= 32 / 3$ square units

Question 34: From the differential equation representing the family of curves y = a sin (x + b), where a, b are arbitrary constants.

Solution:

$y = a \sin (x + b)$

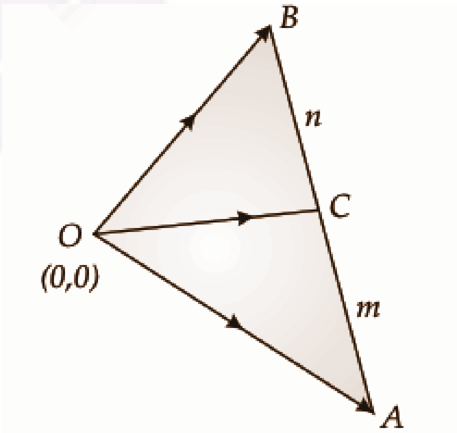
$dy / dx = a \cos (x + b)$

$d^2y / dx^2 = - a \sin (x + b)$

$d^2y / dx^2 + y = 0$

Question 35: Show that the position vector of the point P, which divides the line joining the points A and B having position vectors a and b internally in the ratio m:n is [mb + na] / [m + n].

Solution:



Let O be the origin then OA = a and OB = b.

Let c be the position vector of C which divides AB internally in the ratio m:n then

AC / CB = m / n
n . AC = m . CB
n . (c - a) = m (b - c)
nc - na = mb - mc
nc + mc = mb + na
c (m + n) = mb + na
c = mb + na / (m + n)
The position vector of C is mb + na / (m + n).

Question 36: Find x such that the four points A (3, 2, 1), B (4, X, 5), C (4, 2, -2) and D (6, 5, -1) are coplanar.

Solution:
A (3, 2, 1), B (4, X, 5), C (4, 2, -2) and D (6, 5, -1)
AB = (1, x - 2, 4)
AC = (1, 0, -3)
AD = (3, 3, -2)
[AB, AC, AD] = 0

$$\begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix} = 0$$

x = 5

Question 37: Find the equation of the plane through the intersection of the planes 3x - y + 2z - 4 = 0 and the point (2, 2, 1).

Solution:
The equation of the planes 3x – y + 2z – 4 = 0 and x + y + z – 2 = 0.
Equation of the plane which is passing through intersection of the plane is
3x – y + 2z – 4 + λ (x + y + z – 2) = 0
(3 + λ) x + (λ –1) y + (2 + λ) z – 4 – 2 λ= 0 ....(i)
Plane passes through (2, 2, 1)
∴ (3 + λ) 2 + (λ –1) 2 + (2 + λ) 1 – 4 – 2λ = 0
6 + 2λ + 2λ –2 + 2 + λ – 4 – 2λ = 0

$\lambda = -2 / 3$

∴ Equation of the plane from equ (i)

$(3 - (2 / 3)) x + ((- 2 / 3) - 1) y + (2 - (2 / 3)) z - 4 - 2 (- 2 / 3) = 0$

$(7 / 3) x - (5 / 3) y + (4 / 3) z - (8 / 3) = 0$

$7x - 5y + 4z - 8 = 0$

Question 38: A bag contains 4 red and 4 black balls, another bag contains 2 red and 6 black balls. One of the two bags is selected at random and a ball is drawn from the bag which is found to be red. Find the probability that the ball is drawn from the first bag.

Solution:

$P (E_1) = 1 / 2$

$P (E_2) = 1 / 2$

$P (A / E_1) = 1 / 2$

$P (A / E_2) = 1 / 4$

$P (E_1 / A) = [P (E_1) * P (A / E_1)] / [(P (E_1) * P (A / E_1)) + (P (E_2) * P (A / E_2))]$

$= [(1 / 2) * (1 / 2)] / [(1 / 2) * (1 / 2) + (1 / 2) * (1 / 4)]$

$= 2 / 3$

PART - D

Answer any six questions. **[6 * 5 = 30]**

Question 39: Let R_+ be the set of all non-negative real numbers. Show that the function $f : R_+ \to [4, \infty)$ given by $f (x) = x^2 + 4$ is invertible and write the inverse of f.

Solution:

$f : R_+ \to [4, \infty)$ such that $f (x) = x^2 + 4$

Let $x, y \in R_+$ such that

$f (x) = f (y)$

$x^2 + 4 = y^2 + 4$

$x^2 = y^2$

$x = y$

So, f is an injective.

Let $y \in [4, \infty)$, then $(\therefore x \in R_+)$

$f(x) = x^2 + 4$

$y = x^2 + 4$

$x = \sqrt{y - 4}$

Thus for each $y \in [4, \infty)$ then exists

$x = \sqrt{y - 4}$

$f(x) = f \sqrt{y - 4}$

$f \sqrt{y - 4} = (\sqrt{y - 4})^2 + 2$

$= y - 4 + 4$

$= y$

So $R_+ \to [4, \infty)$ is onto

$\therefore f : R_+ \to [4, \infty)$ is a bijection.

Hence it is invertible.

Let f^{-1} denote the inverse of f (x).

Then $f \circ f^{-1}(x) = x, \ \forall \ x \in [4, \infty)$

$f\{f^{-1}(x)\} = x \ \forall \ x \in [4, \infty)$

$\{f^{-1}(x)\}^2 + 4 = x$

$\{f^{-1}(x)\}^2 = x - 4$

$f^{-1}(x) = \sqrt{x - 4} \ \forall \ x \in [4, \infty)$

Question 40: If $A = \begin{bmatrix} 0 & 6 & 7 \\ -6 & 0 & 8 \\ 7 & -8 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix}$,
calculate AC, BC and (A + B) C. Also, verify that (A + B) C = AC + BC.

Solution:

$$A=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}, B=\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix} \text{ and } C=\begin{bmatrix}2\\-2\\3\end{bmatrix}$$

$$AC=\begin{bmatrix}0&\vec{6}&7\\-6&0&8\\7&-8&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}\downarrow=\begin{bmatrix}0-12+21\\-12+0+24\\14+16+0\end{bmatrix}$$

$$AC=\begin{bmatrix}9\\12\\30\end{bmatrix}, \quad BC=\begin{bmatrix}0&\vec{1}&1\\1&0&2\\1&2&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}\downarrow=\begin{bmatrix}0-2+3\\2+0+6\\2-4+0\end{bmatrix}$$

$$BC=\begin{bmatrix}1\\8\\-2\end{bmatrix} \quad AC+BC=\begin{bmatrix}9\\12\\30\end{bmatrix}+\begin{bmatrix}1\\8\\-2\end{bmatrix}$$

$$AC+BC=\begin{bmatrix}10\\20\\28\end{bmatrix} \quad \text{.............(1)}$$

$$(A+B)=\begin{bmatrix}0&6&7\\-6&0&8\\7&-8&0\end{bmatrix}+\begin{bmatrix}0&1&1\\1&0&2\\1&2&0\end{bmatrix} \qquad (A+B)=\begin{bmatrix}0&7&8\\-5&0&10\\8&-6&0\end{bmatrix}$$

$$(A+B)C=\begin{bmatrix}0&\vec{7}&8\\-5&0&10\\8&-6&0\end{bmatrix}\begin{bmatrix}2\\-2\\3\end{bmatrix}\downarrow=\begin{bmatrix}0+(-14)+24\\-10-0+30\\16+12+0\end{bmatrix}=\begin{bmatrix}10\\20\\28\end{bmatrix} \quad \text{.............(2)}$$

$\therefore$ From equations (1) and (2)

$(A+B)C = AC + BC$

Question 41: Solve the following system of linear equations by matrix method.

x - y + 2z = 7
3x + 4y - 5z = -5
2x - y + 3z = 12

Solution:

$$\begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

AX = B

$AA^{-1}X = A^{-1}B$

$IX = A^{-1}B$

$X = A^{-1}B$

$$|A| = \begin{vmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{vmatrix}$$

$= 1 \cdot 4 \cdot 3 + (-1) \cdot (-5) \cdot 2 + 2 \cdot 3 \cdot (-1) - 2 \cdot 4 \cdot 2 - 1 \cdot (-5) \cdot (-1) - (-1) \cdot 3 \cdot 3$

$= 12 + 10 - 6 - 16 - 5 + 9$

$= 4$

Matrix of cofactors = $\mathbf{C} = \begin{pmatrix} 7 & -19 & -11 \\ 1 & -1 & -1 \\ -3 & 11 & 7 \end{pmatrix}$

Transposed matrix of cofactors = $C^T = \begin{pmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{pmatrix}$

$A^{-1} = C^T / |A| = \begin{pmatrix} \frac{7}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{19}{4} & -\frac{1}{4} & \frac{11}{4} \\ -\frac{11}{4} & -\frac{1}{4} & \frac{7}{4} \end{pmatrix}$

$X = A^{-1}B$

$$= \begin{pmatrix} \frac{7}{4} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{19}{4} & -\frac{1}{4} & \frac{11}{4} \\ -\frac{11}{4} & -\frac{1}{4} & \frac{7}{4} \end{pmatrix} \cdot \begin{pmatrix} 7 \\ -5 \\ 12 \end{pmatrix} = \begin{pmatrix} \frac{7}{4}\cdot 7 + \frac{1}{4}\cdot(-5) + \left(-\frac{3}{4}\right)\cdot 12 \\ \left(-\frac{19}{4}\right)\cdot 7 + \left(-\frac{1}{4}\right)\cdot(-5) + \frac{11}{4}\cdot 12 \\ \left(-\frac{11}{4}\right)\cdot 7 + \left(-\frac{1}{4}\right)\cdot(-5) + \frac{7}{4}\cdot 12 \end{pmatrix} = \begin{pmatrix} \frac{49}{4} - \frac{5}{4} - 9 \\ -\frac{133}{4} + \frac{5}{4} + 33 \\ -\frac{77}{4} + \frac{5}{4} + 21 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

$x = 2, y = 1, z = 3$

Question 42: If $y = (\tan^{-1} x)^2$, show that $(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$.

Solution:

$y = (\tan^{-1} x)^2$

$dy / dx = d (\tan^{-1} x)^2 / dx$

$dy / dx = 2 \tan^{-1} x \; d (\tan^{-1} x) / dx$

$= 2 \tan^{-1} x * (1 / (1 + x^2))$

$(1 / (1 + x^2)) (dy / dx) = 2 \tan^{-1} x$

$(1 / (1 + x^2)) d^2y / dx^2 + (dy / dx) (0 + 2x) = 2 / (1 + x^2)$

$(1 + x^2)^2 d^2y / dx^2 + 2x (1 + x^2) (dy / dx) = 0$

$(x^2 + 1)^2 y_2 + 2x (x^2 + 1) y_1 = 2$

Question 43: Sand is pouring from a pipe at the rate of 12 cm^3/s. The falling sand forms a cone on the ground in such a way that the height of the cone is always one-sixth of the radius of the base. How fast is the height of the sand cone increasing when the height is 4 cm?

Solution:

$dV / dt = 12$ cm^3/sec

Height of the cone = (1 / 6) of the radius of the base of the cone

Volume of the cone = $(1 / 3) \boldsymbol{\pi} r^2 h$

$= (1 / 3) \boldsymbol{\pi} (6h)^2 h$ $\quad\quad$ $[h = r / 6]$

$= 12\boldsymbol{\pi} h^3$

$dV / dt = d (12\boldsymbol{\pi} h^3) / dt$

$12 = 12\boldsymbol{\pi} \,.\, 3h^2 (dh / dt)$

$1 = \boldsymbol{\pi} * 3 (4)^2 (dh / dt)$

$1 / 48\pi = dh / dt$

$dh / dt = 1 / 48\pi$

$= 1 / (48 * (22 / 7))$

$= 7 / (48 * 22)$

$= 0.0066$ cm/sec

Question 44: Find the integral of $1 / x^2 + a^2$ with respect to x and hence find $\int[1 / x^2 - 6x + 13]\, dx$.

Solution:

$I = \int(1 / x^2 + a^2)\, dx$

Let $x = a \tan \theta$

$dx / d\theta = d (a \tan \theta) / d\theta = a \sec^2 \theta$

$dx = a \sec^2 \theta\, d\theta$

$\int(1 / x^2 + a^2)\, dx$

$= \int[1 / a^2 \tan^2 \theta + a^2]\, a \sec^2 \theta\, d\theta$

$= \int a \sec^2 \theta\, d\theta / a^2 [1 + \tan^2 \theta]$

$= \int \sec^2 \theta\, d\theta / a \sec^2 \theta$

$= (1 / a) \int d\theta$

$= (1 / a)\, \theta + c$

$= (1 / a) \tan^{-1} (x / a) + c$

$\int[1 / x^2 - 6x + 13]\, dx$

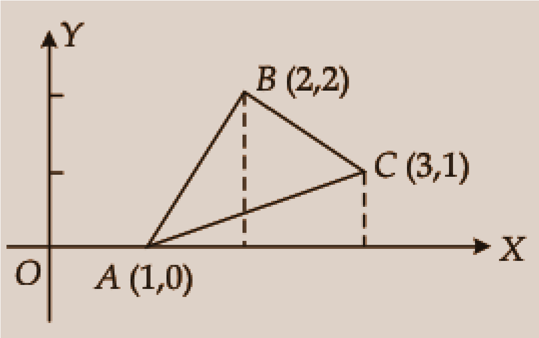
$= \int[1 / x^2 - 6x + 9 + 4]\, dx$

$= \int[1 / (x - 3)^2 + 2^2]\, dx$

$= (1 / 2) \tan^{-1} |(x - 3) / 2| + c$

Question 45: Using integration, find the area of the region bounded by a triangle whose vertices are (1, 0), (2, 2) and (3, 1).

Solution:



Area of the region = Area of triangle ABD + Area of trapezium BDEC - Area of triangle AEC

$y = 2(x - 1)$, $y = 4 - x$, $y = (1 / 2)(x - 1)$

Area of triangle ABC = $\int_1^2 2(x - 1)\, dx + \int_3^2 (4 - x)\, dx - \int_1^3 (x - 1) / 2\, dx$

$= 2[x^2 / 2 - x]_1^2 + [4x - (x^2 / 2)]_3^2 - (1 / 2)[x^2 / 2 - x]_1^3$

$= 2[(2 - 2) - ((1 / 2) - 1)] + [(4 * 3 - (9 / 2)] - (1 / 2)[((9 / 2) - 3) - ((1 / 2) - 1)]$

$= 3 / 2$

Question 46: Find the general solution of the differential equation $x (dy / dx) + 2y = x^2 \log x$.

Solution:

$x (dy / dx) + 2y = x^2 \log x$

$(dy / dx) + (2 / x) y = x \log x$

This is of the form $(dy / dx) + Py = Q$.

$P = 2 / x$

$Q = x \log x$

$IF = e^{\int P\, dx}$

$= e^{\int 2 \log x\, dx}$

$= x^2$

The solution to the differential equation is

$y * IF = \int IF * Q\, dx + c$

$y * x^2 = \int x^2 * x \log x\, dx + c$

$x^2 y = \int x^3 \log x\, dx + c$

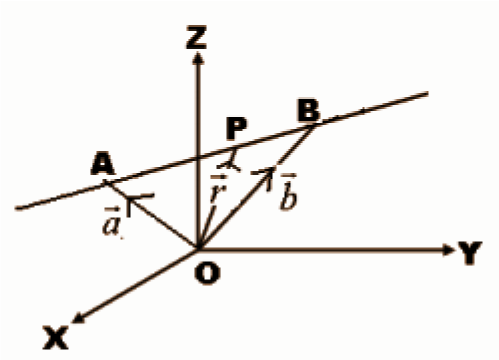
$x^2 y = \log x \int x^3\, dx - \int [d (\log x) / dx \int x^3\, dx]\, dx + c$

$x^2 y = (x^4 / 4) \log x - \int (1 / x) * (x^4 / 4)\, dx + c$

$y = (x^2 / 4) \log x - (x^2 / 16) + c / x^2$
$16y = 4x^2 \log x - x^2 + 16cx^{-2}$

Question 47: Derive the equation of the line in space passing through two given points, both in vector and Cartesian form.

Solution:



Let a, b and r be the position vectors of the two points A (x_1, y_1, z_1) is (x_2, y_2, z_2) and p (x, y, z) respectively.
AP = OP - OA = r - a
AB = OB - OA = b - a
The point p will lie on the line AB if and only if AP and AB are collinear.
AP = λ AB
$(r - a) = \lambda (b - a)$
$r = a + \lambda (b - a)$ is the vector equation of the line passing through two points.
Let $r = xi + yj + zk$, $a = x_1 i + y_1 j + z_1 k$, $b = x_2 i + y_2 j + z_2 k$, $r = a + \lambda (b - a)$
$xi + yj + zk = x_1 i + y_1 j + z_1 k + \lambda ((x_2 - x_1) i + (y_2 - y_1) j + (z_2 - z_1) k)$
$= [x_1 + \lambda (x_2 - x_1)] i + [y_1 + \lambda (y_2 - y_1)] j + [z_1 + \lambda (z_2 - z_1)] k$
$x = x_1 + \lambda (x_2 - x_1)$
$x - x_1 = \lambda (x_2 - x_1)$
$\lambda = (x - x_1) / (x_2 - x_1)$
$y = y_1 + \lambda (y_2 - y_1)$
$y - y_1 = \lambda (y_2 - y_1)$
$\lambda = (y - y_1) / (y_2 - y_1)$
$z = z_1 + \lambda (z_2 - z_1)$
$z - z_1 = \lambda (z_2 - z_1)$
$\lambda = (z - z_1) / (z_2 - z_1)$

Hence the equation of the line passing through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $(x - x_1) / (x_2 - x_1) = (y - y_1) / (y_2 - y_1) = (z - z_1) / (z_2 - z_1)$.

Question 48: If a fair coin is tossed 10 times, find the probability of
(i) Exactly six heads
(ii) At Least six heads.

Solution:
Let X denote the number of heads in an experiment of 10 trials.
$n = 10$
$p = 1 / 2$
$q = 1 - p = 1 - (1 / 2) = 1 / 2$
$P (X = r) = {}^nC_r\, p^r\, q^{n-r}$

[i] $P (X = 6)$
$= {}^{10}C_6\, (1 / 2)^6\, (1 / 2)^{10-6}$
$= {}^{10}C_6\, (1 / 2)^{10}$
$= 105 / 512$

[ii] P (at least 6 heads)
$= P (X \geq 6)$
$= P (X = 6) + P (X = 7) + P (X = 8) + P (X = 9) + P (X = 10)$
$= {}^{10}C_6\, (1 / 2)^{10} + {}^{10}C_7\, (1 / 2)^{10} + {}^{10}C_8\, (1 / 2)^{10} + {}^{10}C_9\, (1 / 2)^{10} + {}^{10}C_{10}\, (1 / 2)^{10}$
$= 193 / 512$

PART - E

Answer any one of the following questions. [1 * 10 = 10]

Question 49: [a] Prove that $\int_0^a f (x)\, dx = \int_0^a f (a - x)\, dx$ and hence evaluate $\int_0^a \sqrt{x} / (\sqrt{x} + \sqrt{a - x})\, dx$.

[b] Prove that $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$ **$= 2\,(x + y + z)^3$.**

Solution:

[a] $\int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx$

Let $a - x = t$

$x = a - t$

$d(a - x) / dx = dt / dx$

$-1 = dt / dx$

$dx = -dt$

When $x = 0$, $a = t$ and $x = a$, $t = 0$

$\int_a^b f(x)\,dx = -\int_0^a f(a - t)\,dt$

$\int_a^b f(x)\,dx = \int_0^a f(a - t)\,dt$

$\int_a^b f(x)\,dx = \int_0^a f(a - x)\,dx$

$I = \int_0^a \sqrt{x} / (\sqrt{x} + \sqrt{a - x})\,dx$

$= \int_0^a \sqrt{a - x} / (\sqrt{a - x} + \sqrt{a - a + x})\,dx$

$2I = \int_0^a \sqrt{x} / (\sqrt{x} + \sqrt{a - x})\,dx + \int_0^a \sqrt{a - x} / (\sqrt{x} + \sqrt{a - x})\,dx$

$= \int_0^a (\sqrt{x} + \sqrt{a - x}) / (\sqrt{x} + \sqrt{a - x})\,dx$

$= \int_0^a dx$

$= [x]_0^a$

$= a - 0$

$2I = a$

$I = a / 2$

[b]

$$\text{L.H.S.} = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$C_1 \to C_1 + C_2 + C_3$$

$$= \begin{vmatrix} 2x+2y+2z & x & y \\ 2x+2y+2z & y+z+2x & y \\ 2x+2y+2z & x & z+x+2y \end{vmatrix}$$

Taking $2x + 2y + 2z$ common from C_1

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 1 & x & z+x+2y \end{vmatrix}$$

Taking $x + y + z$ common from R_2

$$= 2(x+y+z)^2\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 1 & x & z+x+2y \end{vmatrix}$$

Expand along R_2

$= 2\ (x + y + z)^2\ (z + x + 2y – y)$

$= 2\ (x + y + z)^2\ (x + y + z)$

$= 2\ (x + y + z)^3$

= R.H.S.

Question 50: [a] Solve the following problem graphically:

Minimise and Maximize:

Z = 3x + 9y

Subject to constraints

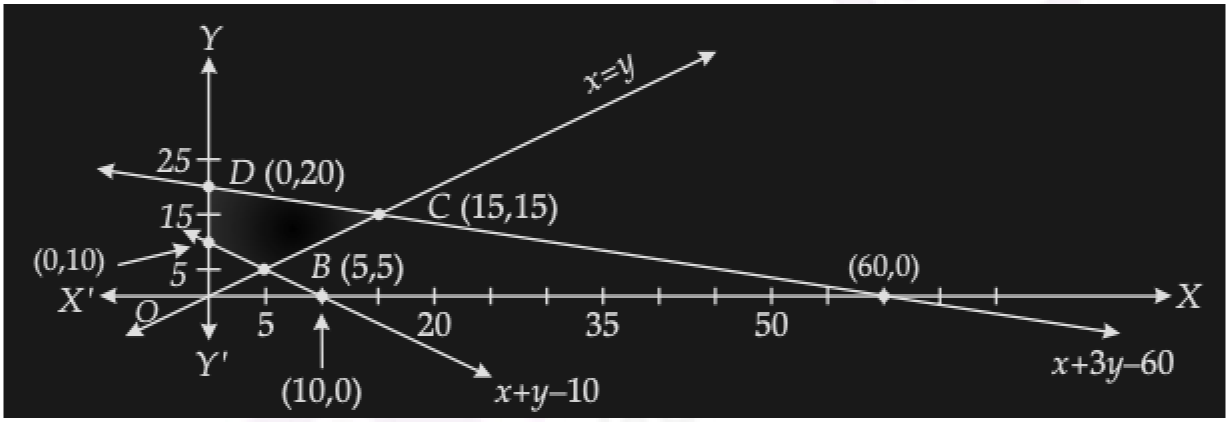
$$x + 3y \le 60$$
$$x + y \ge 10$$
$$x \le y$$
$$x \ge 0,\ y \ge 0$$

[b] Find the relationship between a and b so that the function f defined by

$$f(x) = \begin{cases} ax+1, & \text{if } x \le 3 \\ bx+3, & \text{if } x > 3 \end{cases}$$ **is continuous at x = 3.**

Solution:

[a]



The corner points are A (0, 10), B (5, 5), C (15, 15) and D (0, 20).

Points	Z
A (0, 10)	90
B (5, 5)	60 ← Minimum
C (15, 15)	180 ← Maximum
D (0, 20)	180 ← Maximum

The minimum value of Z = 60 at point B (5, 5).

The maximum value of Z = 180 at the points D (0, 20) and C (15, 15).

[b] $$f(x) = \begin{cases} ax+1, & if \quad x \le 3 \\ bx+3, & if \quad x > 3 \end{cases}$$

f (x) is continuous at x = 3.

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x) = f(3)$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} f(bx + 3)$

$= \lim_{h\to 0} [b(3 + h) + 3]$

$= b(3 + 0) + 3$

$= 3b + 3$

$\lim_{x\to 3^-} f(x) = 3b + 3$

$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (ax + 1)$

$= \lim_{h\to 0} [a(3 - h) + 1]$

$\lim_{x\to 3^-} f(x) = a(3 - 0) + 1$

$= 3a + 1$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^-} f(x)$

$3b + 3 = 3a + 1$

$3 - 1 = 3a - 3b$

$2 = 3(a - b)$

$\therefore a - b = 2/3$