MATHS – JEE ADVANCED PAPER – 1 (2019)

SECTION – 1

1. Let M $= \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$,

where $\alpha = \alpha(\theta)$ *and* $\beta = \beta(\theta)$ are real number, and I is the 2 × 2 identity matrix. If

α* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi]\}$ and

β* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi]\}$

then the value of α* + β* is

(a) $-\frac{37}{16}$ (b) $-\frac{29}{16}$ (c) $-\frac{31}{16}$ (d) $-\frac{17}{16}$

Solution:

$$M = \begin{bmatrix} \sin^4\theta & -1-\sin^2\theta \\ 1+\cos^2\theta & \cos^4\theta \end{bmatrix} = \alpha I + \beta M^{-1}$$

$$M^2 = \alpha M + \beta I$$

$$a_{11} : \sin\theta - 1 - \sin^2\theta - \cos^2\theta - \cos^2\theta \sin^2\theta = \beta + \alpha \sin^4\theta$$

$$\sin^8\theta - 2 - \cos^2\theta \sin^2\theta = \beta + \alpha \sin^4\theta$$

$$a_{21} : \sin^4\theta + \cos^2\theta \sin^4\theta + \cos^4\theta + \cos^6\theta = \alpha\left(1+\cos^2\theta\right)$$

$$\left(1+\cos^2\theta\right)\alpha = \sin^4\theta\left(1+\cos^2\theta\right) + \cos^4\theta\left(1+\cos^2\theta\right)$$

$$\alpha = \sin^4\theta + \cos^4\theta = 1 - 2\sin^2\theta\cos^2\theta = 1 - \left(\frac{\sin^2\theta}{2}\right)$$

$$\alpha_{\min} = \frac{1}{2}$$

$$\beta = \cancel{\sin^8\theta} - \cos^2\theta \sin^2\theta - 2 - \cancel{\sin^2\theta} - \sin^4\theta \cos^4\theta$$

$$= -2 - \left(\frac{\sin^2 2\theta}{4}\right) - \left(\frac{\sin^4 2\theta}{16}\right)$$

$$\beta_{\min} = -2 - \frac{1}{16}\left(4t^2 + t^4\right)$$

$$= -2 - \frac{1}{16}\left(t^2 + 2\right)^2 + \frac{1}{4}$$

$$= -\frac{7}{4} - \frac{1}{16}(9) = -\frac{37}{16}$$

$$\alpha + \beta = -\frac{37}{16} + \frac{1}{2} = -\frac{29}{16}$$

2. A line y = mx + 1 intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct?

(a) $6 \le m < 8$ (b) $2 \le m < 4$ (c) $4 \le m < 6$ (d) $-3 \le m < -1$

Solution:

$$y = mx + 1$$

$$(x-3)^2 + ((mx+1)+2)^2 = 25$$

$$\Rightarrow x^2\left(1+m^2\right) + 6(m-1)x - 7 = 0$$

$$\frac{-3}{5} = \frac{\alpha+\beta}{2} = \frac{-6(m-1)}{2\left(1+m^2\right)}$$

$$1 + m^2 = 5m - 5$$

$$m^2 - 5m + 6 = 0 \quad m = 2, 3$$

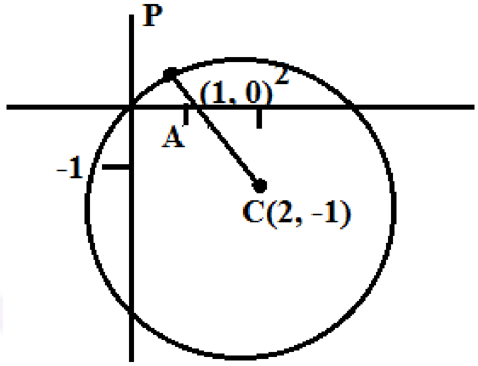
3. Let S be the set of all complex numbers z satisfying $|z-2+i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principle argument of $\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is

(a) $\frac{\pi}{4}$ (b) $-\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) $\frac{\pi}{2}$

Solution:

$$|z-2+i| \geq \sqrt{5}$$

P is along $\overline{AC}$ but at $\sqrt{5}$ distance from C.



$$\overline{CP} = \sqrt{5}\frac{\overrightarrow{AC}}{|\overrightarrow{AC}|} = \sqrt{5}\frac{(\hat{i}+\hat{j})}{\sqrt{2}}$$

$$\vec{P} = (2\hat{i}+\hat{j}) = \frac{-\sqrt{5}}{\sqrt{2}}(\hat{i}-\hat{j})$$

$$\vec{P} = \left(2-\frac{\sqrt{5}}{\sqrt{2}}\right)\hat{i} + \left(\frac{\sqrt{5}}{\sqrt{2}}-1\right)\hat{j}$$

$$z_0 = 2-\sqrt{\frac{5}{2}} + \left(4+\sqrt{\frac{5}{2}}\right)$$

$$\arg\left(\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}\right)$$

$$z_0+\bar{z}_0 = 4-\sqrt{10}$$

$$z_0-\bar{z}_0 = -2i+\sqrt{10}\,\hat{i}$$

$$\arg\left(\frac{\sqrt{10}}{\sqrt{10}\,i}\right)$$

$= -\dfrac{\pi}{2}$

4. The area of the region $\{(x, y): xy \le 8, 1 \le y \le x^2\}$ is

(a) $8\log_e 2 - \dfrac{14}{3}$ (b) $16\log_e 2 - \dfrac{14}{3}$ (c) $16\log_e 2 - 6$ (d) $8\log_e 2 - \dfrac{7}{3}$

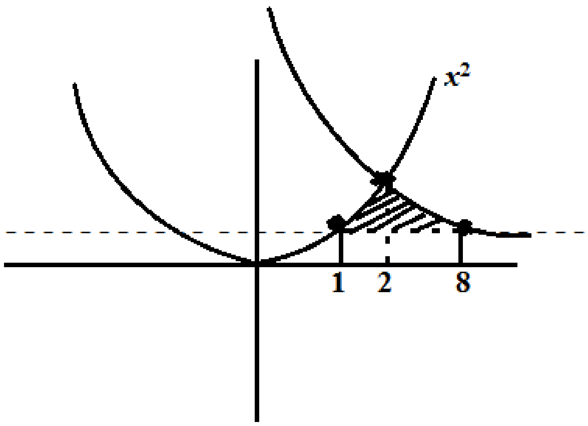
Solution:

$\{(x, y): xy \le 8, 1 \le y \le x^2\}$

$\dfrac{8}{x} = x^2$

$\int_1^2 (x^2 - 1)\,dx + \int_2^8 \left(\dfrac{8}{x} - 1\right)dx$

$= 16\log_e 2 - \dfrac{14}{3}$



SECTION – 2

1. There are three bags B_1, B_2 and B_3. The bag B_1 contains 5 red and 5 green balls, B_2 contains 3 red and 5 green balls, and B_3 contains 5 red and 3 green balls, Bags B_1, B_2 and B_3 have probabilities $\dfrac{3}{10}, \dfrac{3}{10}$ and $\dfrac{4}{10}$ respectively of being chosen. A bag is selected at random and a ball is chosen at random from the bag. Then which of the following options is/are correct?

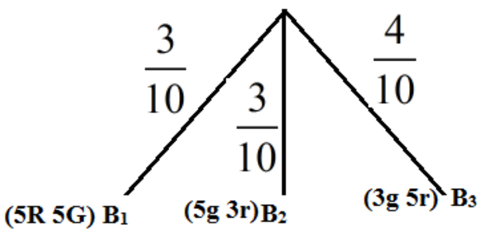
(a) Probability that the selected bag is B_3 and the chosen ball is green equals $\dfrac{3}{10}$

(b) Probability that the chosen ball is green equals $\dfrac{39}{80}$

(c) Probability that the chosen ball is green, given that the selected bag is B_3, equals $\dfrac{3}{8}$

(d) Probability that the selected bag is B_3, given that the chosen balls is green, equals $\dfrac{5}{13}$

Solution:



(i) $P(B_3 \cap G) = P\left(\frac{G}{B_3}\right).P(B_3)$

$$= \frac{3}{8} \times \frac{4}{10} = \frac{3}{20}$$

(ii) $P(G) = \frac{5}{10} \times \frac{3}{10} + \frac{5}{8} \times \frac{3}{10} + \frac{3}{8} \times \frac{4}{10}$

$$= \frac{60+75+60}{400} = \frac{195}{400} = \frac{39}{80}$$

(iii) $P\left(\frac{G}{B_3}\right) = \frac{3}{8}$

(iv) $P\left(\frac{B_3}{G}\right) = \frac{P(G \cap B_3)}{P(G)}$

$$= \frac{\frac{3}{20}}{\frac{39}{80}} = \frac{4}{13}$$

B, C

2. Define the collections $\{E_1, E_2, E_3, ...\}$ of ellipses and $\{R_1, R_2, R_3, …..\}$ of rectangles as follows:

$E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1;$

R_1: rectangle of largest area, with sides parallel to the axes, inscribed in E_1;

E_n: Ellipse $\frac{x^2}{a_n^2} + \frac{y^2}{b_n^2} = 1$ of largest area inscribed in $R_{n-1}, n > 1;$

R_n: rectangle of largest area, with sides parallel to the axes, inscribed in E_n, n > 1.

Then which of the following options is/are correct?

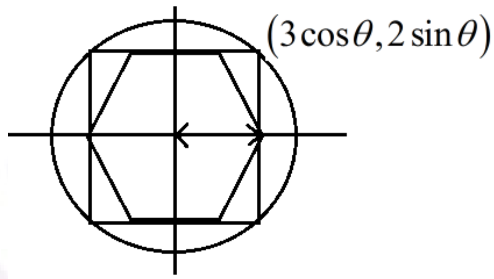
(a) The eccentricities of E_{18} and E_{19} are NOT equal

(b) The distance of a focus from the centre in E_9 is $\frac{\sqrt{5}}{32}$

(c) The length of latus rectum of E_9 is $\frac{1}{6}$

(d) $\sum_{n=1}^{N}\left(area\, of\ R_n\right) < 24,$ for each positive integer N

Solution:



$A = 6\cos\theta . 4\sin\theta$

= 12 sin 2 θ → max

$\theta = \frac{\pi}{4}$

$E_2 = a_2 = 3\ \cos\frac{\pi}{4} = \frac{3}{\sqrt{2}}$

$b_2 = 2\sin\theta = \sqrt{2}$

$r = \frac{1}{\sqrt{2}} \quad a_n = \frac{3}{\left(\sqrt{2}\right)^{n-1}} \quad b_n = \frac{2}{\left(\sqrt{2}\right)^{n-1}}$

$e_{18} = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{\left(\sqrt{2}\right)^{2n-2}}\left(\sqrt{2}^{2n-2}\right) = \frac{\sqrt{5}}{3}$

e of all ellipses → same

Difference of *f* from conic in equation $a_q e$

$E_a = \frac{2b_n^2}{a_n} = \frac{3}{(2)^8} \times \frac{\sqrt{5}}{3} = \frac{\sqrt{5}}{16}$

$= \frac{2 \times 4}{4\left(\sqrt{2}\right)} = \frac{1}{6}$

3. Let $M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix}$ and adjM $= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct?

(a) a + b = 3

(b) $\det(\text{adj}M^2) = 81$

(c) $(\text{adj}M)^{-1} + \text{adj}M^{-1} = -M$

(d) If M $\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

Solution:

$$M = \begin{bmatrix} 0 & 1 & a \\ 1 & 2 & 3 \\ 3 & b & 1 \end{bmatrix} \quad adjM = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$\Rightarrow adjM = \begin{bmatrix} 2-3b & ab-1 & -1 \\ 8 & -6 & 2 \\ b-6 & 3 & -1 \end{bmatrix}$$

$$b - 6 = -5$$

$$b = 1$$

$$ab - 1 = 1$$

$$a = 2$$

$$M = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

$$|M| = -2$$

$$a + b = 2$$

$$|adjM^2| = |M^2|^2 = |M|^4 = 16$$

$$(adjM)^{-1} + adjM^{-1}$$

$$\Rightarrow 2(adjM)^{-1}$$

$$= 2(M^{-1})M$$

$$= 2\times\left(\frac{1}{-2}\right)M = -M$$

$$M\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} \Rightarrow \begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\beta + 2\gamma = 1$$

$$\alpha + 2\beta + 3\gamma = 2$$

$$3\alpha + \beta + \gamma = 1 \qquad \alpha = 1 \quad \beta = -1 \quad \gamma = 1$$

$$\alpha - \beta + \gamma = 3$$

4. Let $f : R \to R$ be given by

$$f(x) = \begin{cases} x^5 + 5x^4 + 10x^3 + 10x^2 + 3x + 1, & x < 0; \\ x^2 - x + 1, & 0 \le x < 1; \\ \frac{2}{3}x^3 - 4x^2 + 7x - \frac{8}{3}, & 1 \le x < 3; \\ (x-2)\log_e(x-2) - x + \frac{10}{3}, & x \ge 3 \end{cases}$$

Then which of the following options is/are correct?

(a) f' has a local maximum at x = 1

(b) f is onto

(c) f is increasing on (-∞, 0)

(d) f' is NOT differentiable at x = 1

Solution:

$$f(x)=\begin{cases}(a+1)^5-2a & x<0\\ x^2-x+1 & 0\le x<1\\ \frac{2}{3}x^3-4x^2+7x-\frac{8}{3} & 1\le x<3\\ (x-2)\log_e(x-2)-x+\frac{10}{3} & x\ge 3\end{cases}$$

When x < 0, $f\to$ unit

$f'(x)=5(x+1)^4-2$ can change sing for x < 0

Range : -∞ + 1

∴ Not monotonic

$$\frac{f(x)=x^2-x+1}{f'(x)=2x-1}$$

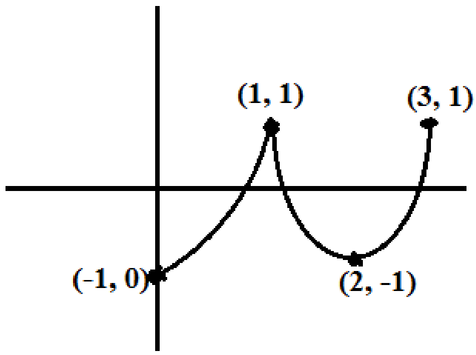
Max at x = 0 & 1

$x\ge 3$

$f(3)=1\log 1-3+\frac{10}{3}=\frac{1}{3}$

$f(\infty)\to\infty$

$$f'(x):\begin{cases}2x-1 & (0\le x<1)\\ 2x^2-8x+7 & |\le x<3\end{cases}$$



Loc. Max at x = 1

5. Let α and β be the roots of $x^2-x-1=0$, with α > β. For all positive integers n, define

$$a_n=\frac{\alpha^n-\beta^n}{\alpha-\beta},n\ge 1$$

$$b_1=1 \text{ and } b_n=a_{n-1}+a_{n+1},n\ge 2.$$

Then which of the following options is/are correct?

(a) $a_1 + a_2 + a_3 + ... + a_n = a_{n+2} - 1$ for all $n \geq 1$

(b) $\sum_{n=1}^{\infty} \frac{a_n}{10^n} = \frac{10}{89}$

(c) $\sum_{n=1}^{\infty} \frac{b_n}{10^n} = \frac{8}{89}$

(d) $b_n = \alpha^n + \beta^n$ for all $n \geq 1$

Solution:

$$x^2 - x - 1 = 0$$

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \beta = \frac{1-\sqrt{5}}{2}$$

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \qquad b_1 = 1$$

$$b_n = a_{n-1} + a_{n+1};\ n \geq 2$$

$$a_{n+2} - a_{n+1} = \left(\frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}\right) - \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)$$

$$= \frac{\alpha^n\left(\alpha^2 - \alpha\right) - \beta^n\left(\beta^2 - \beta\right)}{\alpha - \beta}$$

$$= \frac{\alpha^n(1) - \beta^n(1)}{\alpha - \beta} = a_n$$

$$a_1 + a_2 + ... + a_n \qquad \Rightarrow a_n + a_{n+1} = a_{n+2}$$

$$\sum_{n=1}^{\infty} \frac{a_n}{10^n}$$

$$\Rightarrow \frac{\sum\left(\frac{\alpha}{10}\right)^n - \sum\left(\frac{\beta}{10}\right)^n}{\alpha - \beta}$$

$$\sum_{r=1}^{n} a_r = a_{n+2} - a_2$$

$$= a_{n+2} - \frac{\alpha^2 - \beta^2}{\alpha - \beta}$$

$$= a_{n+2} - (\alpha + \beta)$$

$$= a_{n+2} - 1$$

$$\Rightarrow \frac{\frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} - \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}}}{\alpha - \beta} = \frac{\frac{\alpha}{10-\alpha} - \frac{\beta}{10-\beta}}{\alpha - \beta} = \frac{10}{(10-\alpha)(10-\beta)} = \frac{10}{89}$$

$$\sum \frac{b_n}{10^n} = \sum \frac{a_{n-1} + a_{n+1}}{10^n} = \frac{\frac{\alpha}{10}}{1-\frac{\alpha}{10}} + \frac{\frac{\beta}{10}}{1-\frac{\beta}{10}} = \frac{12}{89}$$

$$b_n = a_{n-1} + a_n$$

$$= \frac{\left(\alpha^{n-1} - \beta^{n-1}\right) + \left(\alpha^{n+1} - \beta^{n+1}\right)}{\alpha - \beta}$$

$$= \alpha\beta = -1 \quad \alpha^{n-1} = -\alpha^n \beta$$

$$\Rightarrow \frac{-\alpha^n \beta + \beta^n \alpha + \alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}$$

$$= \frac{\alpha^n (\alpha - \beta) + \beta^n (\alpha - \beta)}{\alpha - \beta} = \alpha^n + \beta^n$$

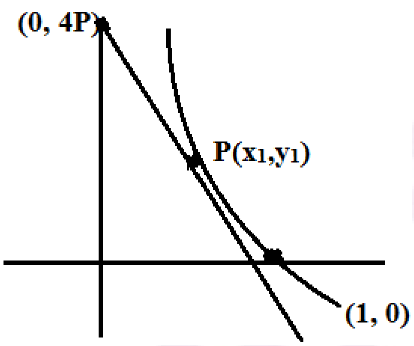
6. Let Γ denote a curve y = y(x) which is in the first quadrant and let the point (1, 0) lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following is options is/are correct?

(a) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$ (b) $xy' - \sqrt{1-x^2} = 0$

(c) $y-\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)+\sqrt{1-x^2}$ (d) $xy'+\sqrt{1-x^2}=0$

Solution:

(1, 0)



$y-y_1=m(x-x_1)$

$y_P-y_1=-mx_1$

$y_P=mx_1+y_1$

$\Rightarrow 1=x_1^2+m^2x_1^2$

$1=x^2\left(1+\left(\frac{dy}{dx}\right)^2\right)$

$\left(\frac{dy}{dx}\right)^2=\frac{1}{x^2}-1$

$\frac{dy}{dx}=\pm\frac{\sqrt{1-x^2}}{x}$

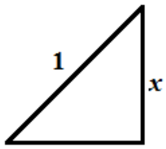
$y=\pm\int\frac{\sqrt{1-x^2}}{x}dx$

$x\sin\theta \quad dx=\cos\theta d\theta$

$=\pm\int\frac{\cos^2\theta\, d\theta}{\sin\theta}$

$=\pm\int(\cos ec\theta-\sin\theta)\,d\theta$

$=\pm\log\left|(\cos ec\theta+\cot\theta)\right|+\cos\theta$



$$= \pm \log\left|\frac{1}{x} + \sqrt{1-x^2}\right| + \left(\sin^{-1} x\right) + \sqrt{1-x^2} + c$$

$$0 = \pm \log(1) + \sqrt{1-1} + C$$

$$C = 0 \qquad \Rightarrow B, D$$

A, B, C, D

7. In a non-right-angle triangle ΔPQR, let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S, the perpendicular from P meets the side QR at E, and RS and PE intersect at 0. If $p = \sqrt{3},\ q = 1,$ and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct?

(a) Area of ΔSOE $= \frac{\sqrt{3}}{12}$

(b) Radius of incircle of ΔPQR $= \frac{\sqrt{3}}{2}\left(2-\sqrt{3}\right)$

(c) Length of RS $= \frac{\sqrt{7}}{2}$

(d) Length of OE $= \frac{1}{6}$

Solution:

$$\frac{\sin P}{\sqrt{3}} = \frac{\sin Q}{1} = \frac{1}{2R} = \frac{1}{2}$$

$$\angle P = \frac{\pi}{3}(or)\frac{2\pi}{3} \quad \angle Q = \frac{\pi}{6}(or)\frac{5\pi}{6}$$

$$p > q \Rightarrow \angle P > \angle Q$$

If $\angle P = \frac{\pi}{3}$ & $\angle Q = \frac{\pi}{6} \Rightarrow R = \frac{\pi}{2}$ (not possible)

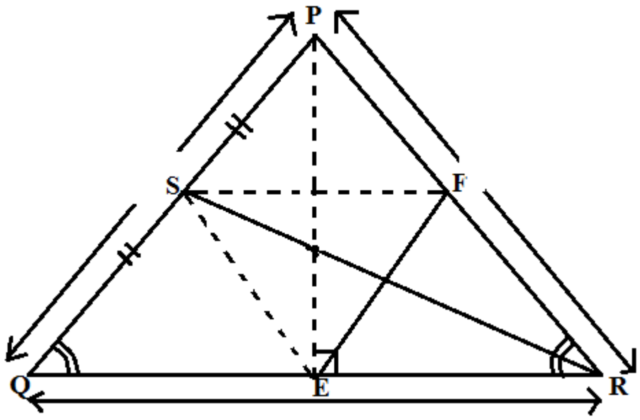
$$\therefore \angle P = \frac{2\pi}{3} \ \& \ \angle Q = \angle R = \frac{\pi}{6}$$

$$r = \frac{\Delta}{S} = \frac{\frac{1}{2}.1.\sqrt{3} = \frac{1}{2}}{\left(\frac{\sqrt{3}+2}{2}\right)} = \frac{\sqrt{3}}{2}\left(2-\sqrt{3}\right)$$

$$\Delta SEF = \frac{1}{4} ar\left(\Delta PQR\right)$$

$$ar\left(\Delta SOE\right) = \frac{1}{3} ar\left(\Delta SEF\right) = \frac{1}{12} ar\left(\Delta PQR\right)$$

$$= \frac{1}{12} \cdot \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{48}$$



$$RS = \frac{1}{2}\sqrt{6+2-1} = \frac{\sqrt{7}}{2}$$

$$OE = \frac{1}{3} PE = \left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\sqrt{2+2-3} = \frac{1}{6}$$

8. Let L_1 and L_2 denotes the lines

$$\vec{r} = \hat{i} + \lambda\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right), \lambda \in \mathbb{R}$$

$$\vec{r} = \mu\left(2\hat{i} - \hat{j} + 2\hat{k}\right), \mu \in \mathbb{R}$$

respectively. If L_3 is a line which is perpendicular to both L_1 and L_2 and cuts both of them, then which of the following options describe(s) L_3?

(a) $\vec{r} = \frac{1}{3}\left(2\hat{i} + \hat{k}\right) + t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in \mathbb{R}$

(b) $\vec{r} = \frac{2}{9}\left(2\hat{i} - \hat{j} + 2\hat{k}\right) + t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in \mathbb{R}$

(c) $\vec{r} = t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in \mathbb{R}$

(d) $\vec{r} = \frac{2}{9}\left(4\hat{i} + \hat{j} + \hat{k}\right) + t\left(2\hat{i} + 2\hat{j} - \hat{k}\right), t \in \mathbb{R}$

Solution:

$L_1 : \vec{r} = \hat{i} + \lambda\left(-\hat{i} + 2\hat{j} + 2\hat{k}\right)$

$L_2 : \vec{r} = \mu\left(2\hat{i} - \hat{j} + 2\hat{k}\right)$

$$L_1 : \frac{x-1}{-1} = \frac{y-0}{2} = \frac{z-0}{2}$$

$L_2 : \frac{x}{2} = \frac{y}{-1} = \frac{z}{2}$

$L_3 : \frac{x}{a} = \frac{y}{b} = \frac{z}{c}$

$L_3 : L_1 \times L_2$

ll^{ly} $6\hat{i} + 6\hat{j} - 3\hat{k}$

A on $L_1 \left(-\lambda + 1, 2\lambda, 2\lambda\right)$

B on $L_2 : \left(2\mu, -\mu, 2\mu\right)$

AB Δrs: $\left(2\mu + \lambda - 1, -\mu - 2\lambda, 2\mu - 2\lambda\right)$

ΔR of AB: (6, 6, -3) or (2, 2, -1)

$\Rightarrow \frac{2\mu + \lambda - 1}{2} = \frac{-\mu - 2\lambda}{2} = \frac{2\mu - 2\lambda}{-1} = k$

$\lambda = \frac{3k+1}{3} \quad \mu = -4k - \frac{2}{3}$

$= k + \frac{1}{3}$

$= 2\mu - 2\lambda + k = 0$

$\Rightarrow 2\left(4k - \frac{2}{3}\right) - 2\left(\frac{3k}{3} + 1\right) + k = 0$

$\Rightarrow k = \frac{-2}{9} \Rightarrow \lambda = \frac{1}{9}, \mu = \frac{2}{9}$

A: $\left(\frac{8}{9}, \frac{2}{9}, \frac{2}{9}\right)$ $B : \left(\frac{4}{9}, \frac{-2}{9}, \frac{4}{9}\right)$

Mid point : $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$

SECTION – 3

1. If $I = \frac{2}{\pi}\int_{-\pi/4}^{\pi/4} \frac{dx}{\left(1+e^{\sin x}\right)\left(2-\cos 2x\right)}$

   Then $27I^2$ equals __________

Solution:

$$I = \frac{2}{\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{\sin x}\right)\left(2-\cos 2x\right)}$$

$$I = \frac{2}{\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{-\sin x}\right)\left(2-\cos 2x\right)}$$

$$\cancel{2}I = \frac{\cancel{2}}{\pi}\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{\cancel{\left(1+e^{\sin x}\right)}dx}{\cancel{\left(1+e^{\sin x}\right)}\left(2-\cos 2a\right)}$$

$$= \frac{2}{\pi}\int_{0}^{\frac{\pi}{4}} \frac{dx}{1+2\sin^2 x} = \frac{2}{\pi}\int_{0}^{\frac{\pi}{4}} \frac{\sec^2 x\, dx}{3\tan^2 x+1} = \frac{2}{3\sqrt{3}}$$

Ans: 4

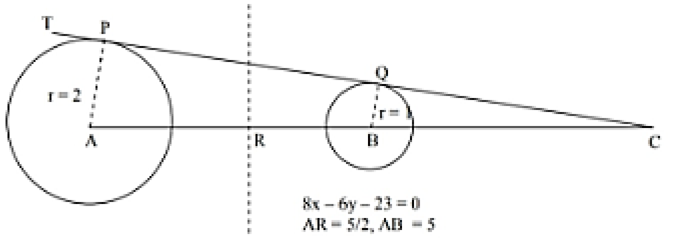
2. Let the point B be the reflection of the point A(2, 3) with respect to the line 8x – 6y – 23 = 0. Let Γ_A and Γ_B be circle of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circle Γ_A and Γ_B such that both the circle are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is ________

Solution:

Now ΔAPC and BQC are similarly

$$\frac{BC}{AC}=\frac{1}{2}\Rightarrow 2\left(AC-AB\right)=AC$$

AC = 2AB = 10



3. Let AP (a; d) denote the set of all the terms of an infinite arithmetic progression with first term a and common difference d > 0. If AP(1; 3) ∩ AP (2; 5) ∩ AP (3; 7) = AP (a; d) then a + d equals)_________

Solution:

$\alpha,\alpha+d,....\alpha>0$

I: 1, 4, 7, 10, 13, 16 ….. 32

II: 2, 7, 12, 17, 22, 27, 32, 37, 42, 47, 52

III: 3, 10, 17, 24, 31, 38, 45, 52

52 ← a + d → LCM of 3, 5, 7 = 105

a + d = 157

4. Let S be the sample space of all 3 × 3 matrices with entries from the set {0, 1}. Let the events E_1 and E_2 be given by

E_1 = {A ϵ S : det A = 0} and

E_2 = {A ϵ S : sum of entries of A is 7}.

If a matrix is chosen at random from S, then the conditional probability $P(E_1|E_2)$ equals …….

Solution:

S : 2^9

$$P\left(\frac{E_1}{E_2}\right)=\frac{P\left(E_1\cap E_2\right)}{P\left(E_2\right)}$$

E_2 : Sum of entries 7

7 1s 2 0's

Total $E_2 = \frac{9!}{7!2!} = 36$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix}$$

Per |A| to be 0, both zeroes should be in the same row/column.

$\therefore\ 3\times3\times2=18$ cases

$$P\left(\frac{E_1}{E_2}\right)=\frac{18}{36}=\frac{1}{2}$$

5. Three lines are given by

$$\vec{r}=\lambda\hat{i}, \lambda\in\mathbb{R}$$

$$\vec{r}=\mu\left(\hat{i}+\hat{j}\right), \mu\in\mathbb{R} \text{ and}$$

$$\vec{r}=v\left(\hat{i}+\hat{j}+\hat{k}\right), v\in\mathbb{R}.$$

Let the lines cut the plane $x+y+z=1$ at the points A, B and C respectively. If the area of the triangle ABC is Δ then the value of $(6\Delta)^2$ equals __________

Solution:

$$\vec{r}=\lambda\hat{i}, \lambda\in\mathbb{R}$$

$\vec{r}=\mu\left(\hat{i}+\hat{j}\right), \mu\in\mathbb{R}$ cuts $x+y+z=1$ at A, B, C graph

$$\vec{r}=v\left(\hat{i}+\hat{j}+\hat{k}\right), v\in\mathbb{R}.$$

1st line : $x=\lambda, y=0, z=0$

$$x+y+z=1 \Rightarrow \lambda=1 \quad A(1,0,0)$$

2nd line: $x=\mu \quad y=\mu \quad z=0$

$$\therefore 2\mu = 1 \quad \mu = \frac{1}{2} \quad B\left(\frac{1}{2}, \frac{1}{2}, 0\right)$$

Parallels C: $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

$$A = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

$$= \frac{1}{2}\left|\frac{\hat{i}}{6} + \frac{\hat{j}}{5} + \frac{\hat{k}}{6}\right| = \frac{\sqrt{3}}{12} \qquad (6\Delta) = \frac{3}{6}$$

6. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\left\{\left|a + b\omega + c\omega^2\right|^2 : \text{a,b,c distinct non-zero integers}\right\}$$

equals ________

Solution:

$$\left|a + b\omega + c\omega^2\right|^2$$

$$= \left(a + b\omega + c\omega^2\right)\left(a + b\omega^2 + c\omega\right)$$

$$= \left(a^2 + b^2 + c^2 = ab - bc - ca\right) \qquad \left[\because a + b\omega + c\omega^2 = \overline{a} + \overline{b\omega} + \overline{c\omega^2} = a + b\omega^2 + c\omega\right]$$

$$\Rightarrow \frac{1}{2}\left((a-b)^2 + (b-c) + (c-a)^2\right)$$

$$= \frac{1}{2}(1+1+4) = 3$$